CHAPTER IX

OCCUPATION TIME PROBLEMS

59. Sojourn Time Problems. In this section we consider stochastic processes $\{\eta(u), 0 \leq u < \infty\}$ with state space $A \cup B$ where A and B are disjoint sets. If $\eta(u) \in A$, then we say that the process is in state A at time u, and if $\eta(u) \in B$, then we say that the process is in state B at time u. We assume that in any finite interval $(0, t)$ the process changes states only a finite number of times with probability 1. Let us suppose that $P\{\eta(0) \in A\} = 1$ and denote by $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ the lengths of the successive intervals spent in states A and B respectively. Denote by $\alpha(t)$ the total time spent in state A in the interval $(0, t)$, and denote by $\beta(t)$ the total time spent in state B in the interval $(0, t)$. Obviously, $\alpha(t)$ and $\beta(t)$ are random variables and $\alpha(t) + \beta(t) = t$ for all $t \geq 0$.

In what follows we shall determine the distribution of $\beta(t)$ and the asymptotic distribution of $\beta(t)$ as $t \to \infty$ for a wide class of stochastic processes $\{\eta(u), 0 \leq u < \infty\}$. The following results were obtained by the author [40], [41], [42], [43], [44], [45].

The distribution of $\beta(t)$. Let us introduce the notation $\gamma_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ for $n = 1,2,\ldots$ and $\gamma_0 = 0$, furthermore $\delta_n = \beta_1 + \beta_2 + \ldots + \beta_n$ for $n = 1,2,\ldots$ and $\delta_0 = 0$.

Theorem 1. If $0 \leq x < t$, then we have

(1) $$\underset{\sim}{P}\{\beta(t) \leq x\} = \sum_{n=0}^{\infty} [\underset{\sim}{P}\{\delta_n \leq x , \gamma_n < t-x\} - \underset{\sim}{P}\{\delta_n \leq x , \gamma_{n+1} < t-x\}].$$

Proof. For $0 \leq x < t$ denote by $\tau = \tau(t-x)$ the smallest $u \in [0, \infty)$ for which $\alpha(u) = t-x$ /. Then $\eta(\tau) \in A$ and we have

(2) $$\{\beta(t) \leq x\} = \{\beta(\tau) \leq x\} .$$

This follows from the following identities

(3) $$\{\beta(t) \leq x\} \equiv \{\alpha(\tau) \leq \alpha (t)\} \equiv \{\tau \leq t\} \equiv \{\alpha(\tau) + \beta(\tau) \leq t\} \equiv \{\beta(\tau) \leq x\} .$$

Since $\alpha(t)$ and $\beta(t)$ are nondecreasing functions of t for $0 \leq t < \infty$ and $\alpha(t) + \beta(t) = t$ for all $t \geq 0$, (3) follows easily.

Since $\beta(\tau) = \delta_n$ $(n = 0,1,\ldots)$ if $\gamma_n < t-x \leq \gamma_{n+1}$, therefore by (2) we obtain that

(4) $$\underset{\sim}{P}\{\beta(t) \leq x\} = \sum_{n=0}^{\infty} \underset{\sim}{P}\{\delta_n \leq x \text{ and } \gamma_n < t-x \leq \gamma_{n+1}\}$$

for $0 \leq x < t$ and this proves (1) .

Now we shall express (1) in an equivalent form which will be useful in finding the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

For each $t \geq 0$ let us define $\rho(t)$ as a discrete random variable taking on nonnegative integers only and satisfying the relation

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/ provided that such a u exists.

$$(5) \qquad \{\rho(t) < n\} \equiv \{\gamma_n \geq t\}$$

for all $t \geq 0$ and $n = 1,2,\ldots$. By using this definition we can write that

$$(6) \qquad P\{\beta(t) \leq x\} = P\{\delta_{\rho(t-x)} \leq x\}$$

for $0 \leq x \leq t$.

If we can determine the asymptotic distribution of $\delta_{\rho(t)}$ as $t \to \infty$, then by (6) we can find the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

<u>Examples</u>. Let us suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences of mutually independent and identically distributed positive random variables. Let $P\{\alpha_n < x\} = G(x)$ and $P\{\beta_n \leq x\} = H(x)$. Then by (1) we have

$$(7) \qquad P\{\beta(t) \leq x\} = \sum_{n=0}^{\infty} [G_n(t-x) - G_{n+1}(t-x)]H_n(x)$$

for $0 \leq x < t$ where $G_n(x)$ $(n = 1,2,\ldots)$ denotes the n-th iterated convolution of $G(x)$ with itself, $H_n(x)$ $(n = 1,2,\ldots)$ denotes the n-th iterated convolution of $H(x)$ with itself, $G_0(x) = H_0(x) = 1$ for $x \geq 0$ and $G_0(x) = H_0(x) = 0$ for $x < 0$.

If, in particular,

$$(8) \qquad G(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x > 0 , \\ 0 & \text{for } x \leq 0 , \end{cases}$$

then (7) reduces to

(9) $$P\{\beta(t) \leq x\} = \sum_{n=0}^{\infty} e^{-\lambda(t-x)} \frac{[\lambda(t-x)]^n}{n!} H_n(x)$$

for $0 \leq x < t$. By (9) we have

(10) $$P\{\beta(a+x) \leq x\} = \sum_{n=0}^{\infty} e^{-\lambda a} \frac{(\lambda a)^n}{n!} H_n(x)$$

for any $a > 0$ and $x \geq 0$. Let

(11) $$\psi(s) = \int_0^{\infty} e^{-sx} \, dH(x)$$

for $\mathrm{Re}(s) \geq 0$. Then by (10) we get

(12) $$\int_0^{\infty} e^{-sx} \, d_x P\{\beta(a+x) \leq x\} = e^{-\lambda a[1-\psi(s)]}$$

for $\mathrm{Re}(s) \geq 0$. If we know $\psi(s)$, then $P\{\beta(a+x) \leq x\}$ can be obtained by inversion from (12).

The asymptotic distribution of $\beta(t)$. If by a suitable normalization the vector variables (γ_n, δ_n) have a limiting distribution as $n \to \infty$, then by a suitable normalization $\beta(t)$ has also a limiting distribution as $t \to \infty$.

In what follows we assume first that $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences of positive random variables for which

(13) $$\lim_{n \to \infty} P\left\{ \frac{\gamma_n - A_1(n)}{A_2(n)} \leq x \right\} = G(x)$$

and

(14) $$\lim_{n\to\infty} P\left\{ \frac{\delta_n - B_1(n)}{B_2(n)} \leq x \right\} = H(x)$$

in the continuity points of the distribution functions $G(x)$ and $H(x)$ and $A_2(n) \to \infty$ and $B_2(n) \to \infty$ as $n \to \infty$.

If either $G(x)$ or $H(x)$ is a nondegenerate distribution function, then there exist a nondegenerate distribution function $R(x)$ and normalizing functions $M_1(t)$ and $M_2(t)$ such that $M_2(t) \to \infty$ and

(15) $$\lim_{t\to\infty} P\left\{ \frac{\beta(t) - M_1(t)}{M_2(t)} \leq x \right\} = R(x)$$

in every continuity point of $R(x)$.

We can prove (15) by using two simple auxiliary theorems.

The first auxiliary theorem is a particular case of a theorem of A. V. Skorokhod [38] .

Lemma 1. Let $F_n(x)$ $(n = 1,2,\dots)$ and $F(x)$ be one-dimensional distribution functions. If

(16) $$\lim_{n\to\infty} F_n(x) = F(x)$$

in every continuity point of $F(x)$, then there exists a probability space $(\Omega,\mathcal{B},P)$ and real random variables ξ_n $(n = 1,2,\dots)$ and ξ such that $P\{\xi_n \leq x\} = F_n(x)$ and $P\{\xi \leq x\} = F(x)$ and

(17)
$$\underset{\sim}{P}\{\lim_{n \to \infty} \xi_n = \xi\} = 1 .$$

Proof. Let Ω be the interval $(0, 1)$, $\mathcal{B}$ the class of Borel subsets of Ω, and $\underset{\sim}{P}$ the Lebesgue measure. Define $\xi_n(\omega) = \inf\{x : \omega \leqq F_n(x)\}$ and $\xi(\omega) = \inf\{x : \omega \leqq F(x)\}$. In this case (16) implies that $\lim_{n \to \infty} \xi_n(\omega) = \xi(\omega)$ for every $\omega \in \Omega$ except possible a countable set of ω values.

In the following discussion we use the symbol $\Longrightarrow$ for denoting convergence in probability.

Lemma 2. Let $\{\delta(n), n = 0,1,2,\ldots\}$ be random variables for which $\underset{\sim}{P}\{\lim_{n \to \infty} \delta(n) = 0\} = 1$. Let $\{\rho(t), 0 \leqq t < \infty\}$ be discrete random variables taking on nonnegative integers only for which

(18)
$$\lim_{t \to \infty} \underset{\sim}{P}\{\rho(t) \geqq m\} = 1$$

for all $m = 0,1,2,\ldots$. Then $\delta(\rho(t))$ converges in probability to 0 as $t \to \infty$, that is,

(19)
$$\lim_{t \to \infty} \underset{\sim}{P}\{|\delta(\rho(t))| \geqq \varepsilon\} = 0$$

for any $\varepsilon > 0$, or briefly $\delta(\rho(t)) \Longrightarrow 0$ as $t \to \infty$.

This lemma is the same as Lemma 4 in Section 45 and we already proved it there.

Our aim is to give methods for finding the limiting distribution (15) if the limiting distributions (13) and (14) are known.

In the following discussion we assume that in (13) $A_1(n) = A_1 n$, $A_2(n) = A_2 n^a$ and in (14) $B_1(n) = B_1 n$, $B_2(n) = B_2 n^b$ where $A_1 \geqq 0$, $A_2 > 0$, $B_1 \geqq 0$, $B_2 > 0$, $a > 0$ and $b > 0$, and if $A_1 > 0$, then $0 < a < 1$, and if $B_1 > 0$, then $0 < b < 1$, that is,

$$\lim_{n \to \infty} P\{ \frac{\gamma_n - A_1 n}{A_2 n^a} \leqq x\} = G(x) \tag{20}$$

and

$$\lim_{n \to \infty} P\{ \frac{\delta_n - B_1 n}{B_2 n^b} \leqq x\} = H(x) \tag{21}$$

in the continuity points of $G(x)$ and $H(x)$.

In the general case, (15) can be obtained in a similar way.

<u>Theorem 2.</u> <u>If</u> $\{\alpha_n\}$ <u>and</u> $\{\beta_n\}$ <u>are independent sequences of positive random variables for which</u> (20) <u>and</u> (21) <u>are satisfied, then there is a distribution function</u> $R(x)$ <u>and there are constants</u> $M_1 \geqq 0$, $M_2 > 0$, $m > 0$ <u>such that</u>

$$\lim_{t \to \infty} P\{ \frac{\beta(t) - M_1 t}{M_2 t^m} \leqq x\} = R(x) \tag{22}$$

<u>in every continuity point of</u> $R(x)$. <u>The constants</u> M_1, M_2, m <u>and the distribution function</u> $R(x)$ <u>are given in Table I where</u> γ <u>and</u> δ <u>are independent real random variables with distribution functions</u> $P\{\gamma \leqq x\} = G(x)$ <u>and</u> $P\{\delta \leqq x\} = H(x)$.

TABLE I.

	A_1	B_1	(a,b)	M_1	M_2	m	R(x)
1.	0	0	$a > b$	0	$B_2A_2^{-b/a}$	b/a	$P\{\delta\gamma^{-b/a} \le x\}$
2.	0	0	$a = b$	0	1	1	$P\{ \frac{B_2\delta}{A_2\gamma+B_2\delta} \le x\}$
3.	0	0	$a < b$	1	$A_2B_2^{-a/b}$	a/b	$P\{-\gamma\delta^{-a/b} \le x\}$
4.	0	>0	$a > 1$	0	$B_1A_2^{-1/a}$	$1/a$	$P\{\gamma^{-1/a} \le x\}$
5.	0	>0	$a = 1$	0	1	1	$P\{ \frac{B_1}{B_1+A_2\gamma} \le x\}$
6.	0	>0	$a < 1$	1	$A_2B_1^{-a}$	a	$P\{-\gamma \le x\}$
7.	>0	0	$b > 1$	1	$A_1B_2^{-1/b}$	$1/b$	$P\{-\delta^{-1/b} \le x\}$
8.	>0	0	$b = 1$	0	1	1	$P\{ \frac{B_2\delta}{A_1+B_2\delta} \le x\}$
9.	>0	0	$b < 1$	0	$B_2A_1^{-b}$	b	$P\{\delta \le x\}$
10.	>0	>0	$a > b$	$\frac{B_1}{A_1+B_1}$	$\frac{B_1A_2}{(A_1+B_1)^{1+a}}$	a	$P\{-\gamma \le x\}$
11.	>0	>0	$a = b$	$\frac{B_1}{A_1+B_1}$	$(\frac{A_1}{A_1+B_1})^{1+a}$	a	$P\{ \frac{A_1B_2\delta - B_1A_2\gamma}{A_1^{1+a}} \le x\}$
12.	>0	>0	$a < b$	$\frac{B_1}{A_1+B_1}$	$\frac{A_1B_2}{(A_1+B_1)^{1+b}}$	b	$P\{\delta \le x\}$

Proof. First, we shall determine the asymptotic distribution of $\delta_{\rho(t)}$ as $t \to \infty$, and then by (6) we shall be able to find the asymptotic distribution of $\beta(t)$ as $t \to \infty$. We can consider $\delta_{\rho(t)}$ as a compound random function and then we can use an idea of R. L. Dobrushin [14] in finding the asymptotic distribution of $\delta_{\rho(t)}$.

If we apply Lemma 1 separately to the distribution functions $P\{\gamma_n \leq A_1 n + A_2 n^a x\}$ $(n = 0,1,2,\ldots)$ and $P\{\delta_n \leq B_1 n + B_2 n^b x\}$ $(n = 0,1,2,\ldots)$, then it follows that we can construct a probability space $(\Omega, \mathcal{B}, P)$ and we can define two independent sets of random variables γ_n^* $(n = 0,1,2,\ldots)$, γ and δ_n^* $(n = 0,1,2,\ldots)$, δ in such a way that $P\{\gamma_n^* \leq x\} = P\{\gamma_n \leq x\}$, $(n = 0,1,\ldots)$, $P\{\gamma \leq x\} = G(x)$, $P\{\delta_n^* \leq x\} = P\{\delta_n \leq x\}$, $(n = 0,1,\ldots)$, $P\{\delta \leq x\} = H(x)$ and

$$(23) \qquad P\{\lim_{n\to\infty} \frac{\gamma_n^* - A_1 n}{A_2 n^a} = \gamma\} = 1,$$

and

$$(24) \qquad P\{\lim_{n\to\infty} \frac{\delta_n^* - B_1 n}{B_2 n^b} = \delta\} = 1.$$

For each $t \geq 0$ let us define $\rho^*(t)$ as a discrete random variable taking on nonnegative integers only and satisfying the relation

$$(25) \qquad \{\rho^*(t) < n\} \equiv \{\gamma_n^* \geq t\}$$

for all $t \geq 0$ and $n = 1,2,\ldots$.

By (6) it is evident that

$$P\{\beta(t) \leq x\} = P\{\delta^*_{\rho^*(t-x)} \leq x\} \tag{26}$$

for $0 \leq x \leq t$. Thus if we determine the asymptotic distribution of $\delta^*_{\rho^*(t)}$ as $t \to \infty$, then by (26) we can obtain also the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

Now let us study the asymptotic behavior of $\delta^*_{\rho^*(t)}$ as $t \to \infty$.

By (23) and (25) we can conclude that

$$\frac{\rho^*(t) - C_1 t}{C_2 t^c} \Longrightarrow \rho \tag{27}$$

as $t \to \infty$ where the constants C_1, C_2, c and the random variable ρ depend on A_1, A_2, a and γ as indicated in Table II.

TABLE II

A_1	C_1	C_2	c	ρ
0	0	$1/A_2^{1/a}$	$1/a$	$\gamma^{-1/a}$
> 0	$1/A_1$	A_2/A_1^{1+a}	a	$-\gamma$

By (27) we can write that

$$\rho^*(t) = C_1 t + C_2 t^c(\rho + \omega(t)) \tag{28}$$

where $\omega(t) \Longrightarrow 0$ as $t \to \infty$.

By (24) it follows that

(29) $$\delta_n^* = B_1 n + B_2 n^b(\delta + \delta(n))$$

where $\delta(n)$ $(n = 0,1,2,\dots)$ is a random variable for which

(30) $$P\{\lim_{n\to\infty} \delta(n) = 0\} = 1 .$$

Thus by (28) and (29) we have

(31) $$\delta^*_{\rho^*(t)} = B_1[C_1 t + C_2 t^c(\rho+\omega(t))] + \\ + B_2[\delta+\delta(\rho^*(t))][C_1 t + C_2 t^c(\rho+\omega(t))]^b .$$

In (31) $\rho^*(t) \Longrightarrow \infty$ as $t \to \infty$. This follows from (28). For if $C_1 = 0$, then ρ is a positive random variable, and if $C_1 > 0$, then $c < 1$. Thus by (30) and by Lemma 2 it follows that in (31) $\delta(\rho^*(t)) \Longrightarrow 0$ as $t \to \infty$. Furthermore $\omega(t) \Longrightarrow 0$ as $t \to \infty$. Taking into consideration these relations we can conclude from (31) that there are constants D_1, D_2, d and a random variable ϑ such that

(32) $$\frac{\delta^*_{\rho^*(t)} - D_1 t}{D_2 t^d} \Longrightarrow \vartheta$$

as $t \to \infty$. The constants D_1, D_2, d and the random variable ϑ depend on B_1, B_2, C_1, C_2, b, c and δ and ρ as indicated in Table III.

TABLE III

B_1	C_1	(b,c)	D_1	D_2	d	ϑ
0	0	–	0	$B_2C_2^b$	bc	$\delta\rho^b$
> 0	0	–	0	B_1C_2	c	ρ
0	> 0	–	0	$B_2C_1^b$	b	δ
> 0	> 0	$b < c$	B_1C_1	B_1C_2	c	ρ
> 0	> 0	$b = c$	B_1C_1	1	b	$B_1C_2\rho + B_2C_1^b\rho$
> 0	> 0	$b > c$	B_1C_1	$B_2C_1^b$	b	δ

Since $\delta_{\rho(t)}$ and $\delta^*_{\rho^*(t)}$ have the same distribution for all $t \geq 0$, it follows from (32) that

$$(33) \qquad \lim_{t \to \infty} \underset{\sim}{P}\{\delta_{\rho(t)} \leq D_1 t + x D_2 t^d\} = \underset{\sim}{P}\{\vartheta \leq x\}$$

in every continuity point of $\underset{\sim}{P}\{\vartheta \leq x\}$.

By (6) we have

$$(34) \qquad \underset{\sim}{P}\{\beta(t) \leq x\} = \underset{\sim}{P}\{\delta_{\rho(t-x)} \leq x\}$$

for $0 \leq x \leq t$.

Finally, by (33) and (34) we can determine the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

Let us define

$$u = t + D_1 t + xD_2 t^d \tag{35}$$

for $t \geq 0$. Then by (34) we can write that

$$\underset{\sim}{P}\{\delta_{\rho(t)} \leq D_1 t + xD_2 t^d\} = \underset{\sim}{P}\{\delta_{\rho(t)} \leq u - t\} = \underset{\sim}{P}\{\beta(u) \leq u - t\} \tag{36}$$

for $0 \leq t \leq u$.

If $d \geq 1$ and $x > 0$, or $d < 1$ and $-\infty < x < \infty$, then there is a $t = t(u)$ which satisfies (35) and for which $0 < t(u) \leq u$ if u is sufficiently large and $t(u) \to \infty$ as $u \to \infty$. If we choose $t = t(u)$ in such a way and let $u \to \infty$ in (36) then by (33) we obtain that

$$\lim_{u \to \infty} \underset{\sim}{P}\{\beta(u) \leq u - t\} = \underset{\sim}{P}\{\vartheta \leq x\} \tag{37}$$

in every continuity point of $\underset{\sim}{P}\{\vartheta \leq x\}$.

If $d > 1$, then $D_1 = 0$, and for $x > 0$ we obtain that

$$t = \left(\frac{u}{xD_2}\right)^{1/d} + o\,(u^{1/d}) \tag{38}$$

as $u \to \infty$.

If $d = 1$, then $D_1 = 0$, and for $x \geq 0$ we obtain that

$$t = \frac{u}{1 + xD_2} \tag{39}$$

for $u \geq 0$.

Finally, if $d < 1$, then we obtain that

$$(40) \qquad t = \frac{u}{1+D_1} - \frac{xD_2}{1+D_1}\left(\frac{u}{1+D_1}\right)^d + o\,(u^d)$$

as $u \to \infty$.

Thus by (37) it follows that if $d > 1$, then

$$(41) \qquad \lim_{u \to \infty} \mathrm{P}\{\beta(u) \leq u - \left(\frac{u}{xD_2}\right)^{1/d}\} = \mathrm{P}\{\vartheta \leq x\}$$

for $x > 0$. If $d = 1$, then

$$(42) \qquad \lim_{u \to \infty} \mathrm{P}\{\beta(u) \leq \frac{uxD_2}{1+xD_2}\} = \mathrm{P}\{\vartheta \leq x\}$$

for $x \geq 0$. If $d < 1$, then

$$(43) \qquad \lim_{u \to \infty} \mathrm{P}\{\beta(u) \leq \frac{D_1 u}{1+D_1} + \frac{xD_2}{1+D_1}\left(\frac{u}{1+D_1}\right)^d\} = \mathrm{P}\{\vartheta \leq x\}$$

for all x. In (41), (42), (43) the limits are valid in the continuity points of $\mathrm{P}\{\vartheta \leq x\}$.

Accordingly, we can conclude that

$$(44) \qquad \lim_{t \to \infty} \mathrm{P}\left\{\frac{\beta(t) - M_1 t}{M_2 t^m} \leq x\right\} = R(x)$$

in every continuity point of $R(x)$ where the constants M_1, M_2, m and the distribution function $R(x)$ are given in Table IV.

TABLE IV

d	M_1	M_2	m	R(x)
$d > 1$	1	$D_2^{-1/d}$	$\frac{1}{d}$	$P\{-\vartheta^{-1/d} \leq x\}$
$d = 1$	0	1	1	$P\{\frac{D_2\vartheta}{1+D_2\vartheta} \leq x\}$
$d < 1$	$\frac{D_1}{1+D_1}$	$\frac{D_2}{(1+D_1)^{1+d}}$	d	$P\{\vartheta \leq x\}$

The entries in Table I can be obtained by Tables II, III and IV. This completes the proof of Theorem 2.

We note that in proving the 7-th, 8-th, 9-th and 12-th statements of Theorem 2 we can replace the assumption (20) by the weaker assumption that

$$\lim_{n\to\infty} \frac{\gamma_n}{n} = A_1 \tag{45}$$

in probability. Similarly, in proving the 4-th, 5-th, 6-th and 10-th statements of Theorem 2 we can replace the assumption (21) by the weaker assumption that

$$\lim_{n\to\infty} \frac{\delta_n}{n} = B_1 \tag{46}$$

in probability.

At the end of this section we shall discuss the problem of finding the asymptotic distribution of $\beta(t)$ as $t\to\infty$ in the case where $(\alpha_1,\beta_1), (\alpha_2,\beta_2), \ldots, (\alpha_n,\beta_n), \ldots$ are mutually independent and identically distributed vector random variables for which

$$\lim_{n\to\infty} P\left\{\frac{\gamma_n - A_1 n}{A_2 n^a} \leqq x,\ \frac{\delta_n - B_1 n}{B_2 n^b} \leqq y\right\} = F(x,y) \tag{47}$$

in every continuity point of the distribution function $F(x,y)$ and $a > 0$, $b > 0$, $A_1 \geqq 0$, $B_1 \geqq 0$, $A_2 > 0$, $B_2 > 0$.

Examples. First, let us suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences of mutually independent and identically distributed positive random variables for which $\mathbf{E}\{\alpha_n\} = \alpha$, $\mathbf{Var}\{\alpha_n\} = \sigma_\alpha^2$ and $\mathbf{E}\{\beta_n\} = \beta$, $\mathbf{Var}\{\beta_n\} = \sigma_\beta^2$ exist and $\sigma_\alpha^2 > 0$ and $\sigma_\beta^2 > 0$. Then the limiting distributions (20) and (21) exist and $A_1 = \alpha$, $A_2 = \sigma_\alpha$, $B_1 = \beta$, $B_2 = \sigma_\beta$, $a = b = 1/2$ and $G(x) = H(x) = \Phi(x)$ where $\Phi(x)$ is the normal distribution function.

In this case by the 11-th statement of Theorem 2 we obtain that

$$(48)\qquad \lim_{t\to\infty} \mathbf{P}\left\{ \frac{\beta(t) - \frac{\beta t}{\alpha+\beta}}{\sqrt{\frac{\alpha^{3/2}\, t^{1/2}}{(\alpha+\beta)^{3/2}}}} \leqq x \right\} = \mathbf{P}\left\{ \frac{\alpha\sigma_\beta\delta - \beta\sigma_\alpha\gamma}{\alpha^{3/2}} \leqq x \right\}$$

where δ and γ are independent random variables for which $\mathbf{P}\{\delta \leqq x\} = \mathbf{P}\{\gamma \leqq x\} = \Phi(x)$. Hence

$$(49)\qquad \lim_{t\to\infty} \mathbf{P}\left\{ \frac{\beta(t) - \frac{\beta t}{\alpha+\beta}}{\sqrt{\frac{(\alpha^2\sigma_\beta^2 + \beta^2\sigma_\alpha^2)t}{(\alpha+\beta)^3}}} \leqq x \right\} = \Phi(x).$$

Second, let us suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences of mutually independent and identically distributed positive random variables for which

$$(50)\qquad \lim_{t\to\infty} \mathbf{P}\{\alpha_n > x\}\, x^{\sigma_1} = A$$

where $0 < \sigma_1 < 1$ and $A > 0$, and $\mathbf{E}\{\beta_n\} = \beta < \infty$ and

$$(51)\qquad \lim_{x\to\infty} \mathbf{P}\{\beta_n > x\}\, x^{\sigma_2} = B$$

where $1 < \sigma_2 \leq 2$ and $B > 0$. Then the limiting distributions (20) and (21) exist and $A_1 = 0$, $A_2 = A^{1/\sigma_1}$, $a = 1/\sigma_1$, $B_1 = \beta$, $B_2 = B^{1/\sigma_2}$, $b = 1/\sigma_2$ and $G(x)$ is a stable distribution function of type $S(\sigma_1, 1, \Gamma(1-\sigma_1)\cos\frac{\pi\sigma_1}{2}, 0)$ and $H(x)$ is a stable distribution function of type $S(\sigma_2, 1, \Gamma(1-\sigma_2)\cos\frac{\pi\sigma_2}{2}, 0)$.

In this case by the 4-th statement of Theorem 2 we obtain that

$$(52) \qquad \lim_{t\to\infty} \mathbf{P}\left\{ \frac{A\beta(t)}{\beta t^{\sigma_1}} \leq x \right\} = \mathbf{P}\{\gamma^{-\sigma_1} \leq x\}$$

where γ is a random variable with distribution function $\mathbf{P}\{\gamma \leq x\} = G(x)$. By (42.177) and (42.181) we can express (52) as

$$(53) \qquad \lim_{t\to\infty} \mathbf{P}\left\{ \frac{A\beta(t)}{\beta t^{\sigma_1}} \leq x \right\} = G_{\sigma_1}(x\Gamma(1-\sigma_1))$$

where the Laplace-Stieltjes transform of $G_{\sigma_1}(x)$ is given by

$$(54) \qquad \int_0^\infty e^{-sx} dG_{\sigma_1}(x) = E_{\sigma_1}(-s) .$$

for $\operatorname{Re}(s) \geq 0$ and $E_{\sigma_1}(z)$ is the Mittag-Leffler function defined by

$$(55) \qquad E_{\sigma_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\sigma_1 + 1)}$$

for $0 \leq \sigma_1 < 1$. If $\frac{1}{2} \leq \sigma_1 < 1$, then we have

(56) $$G_{\sigma_1}(x) = \frac{1}{\sigma_1}\left[R\left(x; \frac{1}{\sigma_1}, -1, -\cos\frac{\pi}{2\sigma_1}, 0\right) - 1 + \sigma_1\right]$$

for $x \geq 0$ where $R(x)$ is a stable distribution function of the indicated type. This follows from (42.184) and (42.192).

We note that if instead of (51) we assume that $\mathrm{Var}\{\beta_n\}$ is a finite positive number, then (52) holds unchangeably.

Third, let us suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences of mutually independent and identically distributed positive random variables for which

(57) $$\lim_{x \to \infty} P\{\alpha_n > x\}x^{\sigma} = A$$

and

(58) $$\lim_{x \to \infty} P\{\beta_n > x\}x^{\sigma} = B$$

where $A > 0$, $B > 0$ and $0 < \sigma < 1$. Then the limiting distributions (20) and (21) exist, and $A_1 = 0$, $A_2 = A^{1/\sigma}$, $a = 1/\sigma$, $B_1 = 0$, $B_2 = B^{1/\sigma}$, $b = 1/\sigma$ and $G(x)$ and $H(x)$ are stable distribution functions of type $S(\sigma, 1, \Gamma(1-\sigma)\cos\frac{\pi\sigma}{2}, 0)$.

In this case by the 2nd statement of Theorem 2 we obtain that

(59) $$P\{\beta(t) \leq tx\} = P\left\{\frac{B^{1/\sigma}\delta}{A^{1/\sigma}\gamma + B^{1/\sigma}\delta} \leq x\right\}$$

where γ and δ are independent random variables having the same stable distribution function of type $S(\sigma, 1, \Gamma(1-\sigma)\cos\frac{\pi\sigma}{2}, 0)$. From (59) it follows that

$$(60)\qquad \underset{\sim}{P}\{\beta(t) \leq tx\} = \underset{\sim}{P}\{ \frac{\delta}{\gamma} \leq (\frac{A}{B})^{1/\sigma} \frac{x}{1-x} \}$$

for $0 \leq x < 1$.

If a random variable ξ has a normal distribution of type $N(0, 1)$, and $c > 0$, then $\eta = c^2/\xi^2$ has a stable distribution of type $S(\frac{1}{2}, 1, c, 0)$. Thus if, in particular, $\sigma = 1/2$ in (57) and (58), then in (60) we can write that $\gamma = \pi/2\gamma^{*2}$ and $\delta = \pi/2\delta^{*2}$ where γ^* and δ^* are independent random variables having the same normal distribution function $\Phi(x)$.

Thus if (57) and (58) hold with $\sigma = \frac{1}{2}$, then by (60) we obtain that

$$(61)\qquad \begin{aligned} &\underset{\sim}{P}\{\beta(t) \leq tx\} = \underset{\sim}{P}\{ \left|\frac{\gamma^*}{\delta^*}\right| \leq \frac{A}{B}\sqrt{\frac{x}{1-x}} \} = \\ &= \frac{2}{\pi} \text{arc tan} \frac{A}{B}\sqrt{\frac{x}{1-x}} = \frac{2}{\pi} \text{arc sin} \sqrt{\frac{A^2x}{A^2x + B^2(1-x)}} \end{aligned}$$

for $0 \leq x \leq 1$.

If $A = B$ and $\sigma = 1/2$ in (57) and (58), then by (61) we obtain that

$$(62)\qquad \underset{\sim}{P}\{\beta(t) \leq tx\} = \frac{2}{\pi} \text{arc sin} \sqrt{x}$$

for $0 \leq x \leq 1$.

By using the theorems of Section 52 we can determine the distribution of the sojourn time for such processes $\{\eta(u), 0 \leq u < \infty\}$ for which Theorem 2 can not be applied directly. We shall illustrate this by an example.

Let $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ be a separable Brownian motion process. (See Definition 1 in Section 50.) Let

$$\beta(t) = \int_0^t \delta(\xi(u))du \tag{63}$$

where

$$\delta(x) = \begin{cases} 1 & \text{for } x > 0\ , \\ 0 & \text{for } x \leq 0\ . \end{cases} \tag{64}$$

Then we have

$$\underset{\sim}{P}\{\beta(t) \leq tx\} = \frac{2}{\pi} \arcsin \sqrt{x} \tag{65}$$

for $0 \leq x \leq 1$. This result is due to P. Lévy [33].

We can prove (65) in the following way: Let $\xi_1, \xi_2, \ldots, \xi_r, \ldots$ be a sequence of mutually independent and identically distributed random variables for which

$$\underset{\sim}{P}\{\xi_r = 1\} = \underset{\sim}{P}\{\xi_r = -1\} = \frac{1}{2}\ . \tag{66}$$

Let $\zeta_r = \xi_1 + \xi_2 + \ldots + \xi_r$ for $r = 1,2,\ldots$ and $\zeta_0 = 0$. Define

$$\xi_n(u) = \frac{\zeta_{[nu]}}{\sqrt{n}} \tag{67}$$

for $u \geq 0$ and $n = 1,2,\ldots$. If

$$\beta_n(t) = \int_0^t \delta(\xi_n(u))du \tag{68}$$

where $\delta(x)$ is defined by (64), then by (37.166) we have

(69) $$\lim_{n \to \infty} P\{\beta_n(t) \leq tx\} = \frac{2}{\pi} \arcsin \sqrt{x}$$

for $t > 0$ and $0 \leq x \leq 1$. The same result can be obtained by (62). See also Problem 61.3.

If we define

(70) $$\xi_n^*(u) = \frac{\zeta_{[nu]} + (nu - [nu])\xi_{[nu+1]}}{\sqrt{n}}$$

for $u \geq 0$ and $n = 1,2,\ldots$ and

(71) $$\beta_n^*(t) = \int_0^t \delta(\xi_n^*(u))du$$

where $\delta(x)$ is defined by (64), then we can easily conclude from (69) that

(72) $$\lim_{n \to \infty} P\{\beta_n^*(t) \leq tx\} = \frac{2}{\pi} \arcsin \sqrt{x}$$

for $t > 0$ and $0 \leq x \leq 1$.

If $n \to \infty$, then the finite dimensional distributions of the process $\{\xi_n^*(u), 0 \leq u < \infty\}$ converge to the finite dimensional distributions of the process $\{\xi(u), 0 \leq u < \infty\}$. Thus by Theorem 45.7 (Theorem 52.2) and by (45.181) we can conclude that (72) implies (65).

Next, we shall study the asymptotic behavior of the moments of $\beta(t)$ in the case when $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences of mutually independent and identically distributed positive random variables. Let

(73) $$P\{\alpha_n \leq x\} = G(x)$$

and

(74) $$P\{\beta_n \leq x\} = H(x)$$

and define the following Laplace-Stieltjes transforms

(75) $$\gamma(s) = \int_0^\infty e^{-sx}\, dG(x)$$

and

(76) $$\psi(s) = \int_0^\infty e^{-sx}\, dH(x)$$

for $\operatorname{Re}(s) \geq 0$.

Let

(77) $$B_r(t) = E\{[\beta(t)]^r\} = \int_0^t x^r\, dP\{\beta(t) \leq x\} = r\int_0^t x^{r-1} P\{\beta(t) > x\}dx$$

for $t \geq 0$ and $r = 1,2,\ldots$.

Theorem 3. *If $\{\alpha_n\}$ and $\{\beta_n\}$ are independent sequences of mutually independent and identically distributed positive random variables for which*

(78) $$E\{e^{-s\alpha_n}\} = \gamma(s)$$

and

(79) $$E\{e^{-s\beta_n}\} = \psi(s)$$

whenever $\operatorname{Re}(s) \geq 0$, then

$$\int_0^\infty e^{-st}\, dB_1(t) = \frac{1}{s}\left[1 - \frac{1-\gamma(s)}{1-\gamma(s)\psi(s)}\right] \tag{80}$$

and

$$\int_0^\infty e^{-st}\, dB_2(t) = \frac{2}{s^2}\left[1 - \frac{1-\gamma(s)}{1-\gamma(s)\psi(s)} + \frac{s[1-\gamma(s)]\gamma(s)\psi'(s)}{[1-\gamma(s)\psi(s)]^2}\right] \tag{81}$$

for $\operatorname{Re}(s) > 0$.

Proof. By (7) and (77) we obtain that

$$\begin{aligned}\int_0^\infty e^{-st} dB_r(t) &= \frac{r!}{s^r} + (-1)^r r[1-\gamma(s)] \sum_{n=0}^\infty [\gamma(s)]^n \frac{d^{r-1}[\psi(s)]^n/s}{ds^{r-1}} = \\ &= \frac{r!}{s^r}\left\{1 - [1-\gamma(s)] \sum_{j=0}^{r-1} \frac{(-1)^j s^j}{j!} \left(\sum_{n=j}^\infty [\gamma(s)]^n \frac{d^j[\psi(s)]^n}{ds^j} \right)\right\}\end{aligned} \tag{82}$$

for $\operatorname{Re}(s) > 0$ and $r = 1,2,\ldots$. In the particular cases where $r = 1$ and $r = 2$ we obtain (80) and (81).

Note. If $P_B(u) = \mathbf{P}\{\eta(u) \in B\}$ for $u \geq 0$, then obviously

$$B_1(t) = \int_0^t P_B(u)du \; . \tag{83}$$

Thus by (80) we have

$$\int_0^\infty e^{-st}\, P_B(t)dt = \frac{\gamma(s)[1-\psi(s)]}{s\,[1-\gamma(s)\psi(s)]} \tag{84}$$

for $\mathrm{Re}(s) > 0$.

There are several examples of processes $\{\eta(u)\ ,\ 0 \leq u < \infty\}$ for which $G(x)$ and $P_B(t)$ can easily be determined. For such processes $\psi(s)$ can be obtained by (84) and $H(x)$ is determined by inversion.

Formula (80) makes it possible to find easily the asymptotic behavior of $B_1(t)$ as $t \to \infty$ if we know the asymptotic behavior of $G(x)$ and $H(x)$ as $x \to \infty$.

We shall consider only the cases where either

(85) $$\alpha = \int_0^\infty x dG(x)$$

is a finite positive number or

(86) $$\lim_{x \to \infty} [1-G(x)]x^{\sigma_1} = A$$

where $0 < \sigma_1 < 1$ and A is a positive number, furthermore where either

(87) $$\beta = \int_0^\infty x dH(x)$$

is a finite positive number or

(88) $$\lim_{x \to \infty} [1-H(x)]x^{\sigma_2} = B$$

where $0 < \sigma_2 < 1$ and B is a positive number.

If $\alpha < \infty$, then $\gamma(s) = 1-\alpha s + o(s)$ as $s \to +0$, and if (86) holds, then

(89) $$\gamma(s) = 1-A\Gamma(1-\sigma_1)s^{\sigma_1} + o(s^{\sigma_1})$$

as $s \to +0$. Furthermore, if $\beta < \infty$, then $\psi(s) = 1-\beta s + o(s)$ as $s \to +0$, and if (88) holds, then

(90) $$\psi(s) = 1-B\Gamma(1-\sigma_2)s^{\sigma_2} + o(s^{\sigma_2})$$

as $s \to +0$. Equations (89) and (90) follow from an Abelian theorem. (See Theorem 9.11 in the Appendix .)

If $G(x)$ satisfies either $\alpha < \infty$ or (86) and if $H(x)$ satisfies either $\beta < \infty$ or (88), then in each case we can determine the asymptotic behavior of (80) as $s \to +0$, and then by a Tauberian theorem (Theorem 9.14 in the Appendix) we obtain the following results. If $\alpha + \beta < \infty$, then

(91) $$\lim_{t \to \infty} \frac{B_1(t)}{t} = \frac{\beta}{\alpha+\beta} .$$

If $G(x)$ satisfies (86) and $\beta < \infty$, then

(92) $$\lim_{t \to \infty} \frac{B_1(t)}{t} = \frac{\beta \sin \pi\sigma_1}{A\pi} .$$

If $\alpha < \infty$ and $H(x)$ satisfies (88), then

(93) $$\lim_{t \to \infty} \frac{t-B_1(t)}{t} = \frac{\alpha \sin \pi\sigma_2}{B\pi} .$$

If $G(x)$ satisfies (86) and $H(x)$ satisfies (88), then

(94) $$\lim_{t\to\infty} \frac{B_1(t)}{t^{1+\sigma_1-\sigma_2}} = \frac{B\Gamma(1-\sigma_2)}{A\Gamma(1-\sigma_1)\Gamma(1+\sigma_1-\sigma_2)}$$

whenever $\sigma_1 < \sigma_2$,

(95) $$\lim_{t\to\infty} \frac{B_1(t)}{t} = \frac{B}{A+B}$$

whenever $\sigma_1 = \sigma_2$, and

(96) $$\lim_{t\to\infty} \frac{t-B_1(t)}{t^{1+\sigma_2-\sigma_1}} = \frac{A\Gamma(1-\sigma_1)}{B\Gamma(1-\sigma_2)\Gamma(1+\sigma_2-\sigma_1)}$$

whenever $\sigma_1 > \sigma_2$.

We note that if $\mathrm{Var}\{\alpha_n\} = \sigma_\alpha^2$ and $\mathrm{Var}\{\beta_n\} = \sigma_\beta^2$ are finite, then we have

(97) $$\lim_{t\to\infty} \frac{\mathrm{Var}\{\beta(t)\}}{t} = \frac{\alpha^2\sigma_\beta^2 + \beta^2\sigma_\alpha^2}{(\alpha+\beta)^3} .$$

Finally, we note that in some cases Theorem 2 remains valid even if we remove the restriction that the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are independent.

In what follows we suppose that (α_1,β_1), $(\alpha_2,\beta_2),\dots,$ (α_n,β_n), ... are mutually independent and identically distributed vector random variables for which

(98) $$\lim_{n \to \infty} P\left\{ \frac{\gamma_n - A_1 n}{A_2 n^a} \leq x \text{ and } \frac{\delta_n - B_1 n}{B_2 n^b} \leq y \right\} = F(x, y)$$

in every continuity point of the distribution function $F(x,y)$, and the normalizing constants satisfy the conditions $\frac{1}{2} \leqq a < 1$, $A_1 > 0$, $A_2 > 0$, or $a \geqq 1$, $A_1 = 0$, $A_2 > 0$, and $\frac{1}{2} \leqq b < 1$, $B_1 > 0$, $B_2 > 0$, or $b \geqq 1$, $B_1 = 0$, $B_2 > 0$.

We shall prove that if (98) is satisfied, then Propositions 4 - 12 in Theorem 2 remain valid with the modification that γ and δ are real random variables with joint distribution function $P\{\gamma \leqq x, \delta \leqq y\} = F(x,y)$. Furthermore, we shall show that Propositions 1 - 3 in Theorem 2 are valid only if $F(x,y) = P\{\gamma \leqq x\} P\{\delta \leqq y\}$, that is, only if γ and δ are independent.

In finding the asymptotic distribution of $\beta(t)$ as $t \to \infty$ we shall use formula (6), that is,

(99) $$P\{\beta(t) \leq x\} = P\{\delta_{\rho(t-x)} \leq x\}$$

for $0 \leqq x \leqq t$, and an analogous formula

(100) $$P\{\alpha(t) < x\} = P\{\gamma_{\omega(t-x)} < x\}$$

for $0 \leqq x \leqq t$, where $\omega(t)$ $(t \geqq 0)$ is a discrete random variable taking on positive integers only and satisfying the relation

(101) $$\{\omega(t) \leq n\} \equiv \{\delta_n > t\}$$

for all $t \geqq 0$ and $n = 0, 1, 2, \ldots$.

We note that if (α_1,β_1), $(\alpha_2,\beta_2),\ldots,(\alpha_n,\beta_n),\ldots$ are mutually independent and identically distributed vector random variables and if

$$\psi(s,q) = \underset{\sim}{E}\{e^{-s\alpha_n - q\beta_n}\} \tag{102}$$

for $\operatorname{Re}(s) \geq 0$ and $\operatorname{Re}(q) \geq 0$, then

$$q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\gamma_{\omega(t)}}\}dt = 1 - \frac{1-\psi(s,0)}{1-\psi(s,q)} \tag{103}$$

for $\operatorname{Re}(s) \geq 0$ and $\operatorname{Re}(q) > 0$, and

$$q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\delta_{\rho(t)}}\}dt = \frac{1-\psi(q,0)}{1-\psi(q,s)} \tag{104}$$

for $\operatorname{Re}(s) \geq 0$ and $\operatorname{Re}(q) > 0$.

If we define $I(A)$ as the indicator variable of the event A, that is, $I(A) = 1$ whenever A occurs and $I(A) = 0$ whenever A does not occur, then we can also write that

$$\underset{\sim}{E}\{e^{-s\gamma_{\omega(t)}}\} = 1 - [1-\psi(s,0)] \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-s\gamma_n} I(\delta_n \leq t)\} \tag{105}$$

for $\operatorname{Re}(s) \geq 0$.

If we know the asymptotic distribution of $\gamma_{\omega(t)}$ as $t \to \infty$, or the asymptotic distribution of $\delta_{\rho(t)}$ as $t \to \infty$, then by (99) and (100) we can determine the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

In what follows if we say that a family of distribution functions converges to a limiting distribution function, then by this we mean that the distribution functions converge in every continuity point of the limiting distribution function.

In finding (44) we have already demonstrated that if

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\delta_{\rho(t)} - D_1 t}{D_2 t^d} \leq x\right\} = \mathbf{P}\{\vartheta \leq x\} \quad , \tag{106}$$

where either $0 < d < 1$, $D_1 > 0$, $D_2 > 0$, or $d \geqq 1, D_1 = 0, D_2 > 0$, then

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\beta(t) - M_1 t}{M_2 t^m} \leq x\right\} = R(x) \quad , \tag{107}$$

and the constants M_1, M_2, m and the distribution function $R(x)$ are given in Table IV.

In exactly the same way we can demonstrate that if

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\gamma_{\omega(t)} - D_1 t}{D_2 t^d} \leq x\right\} = \mathbf{P}\{\vartheta \leq x\} \quad , \tag{108}$$

then

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\alpha(t) - M_1 t}{M_2 t^m} \leq x\right\} = R(x) \quad , \tag{109}$$

and the constants M_1, M_2, m and $R(x)$ have the same meaning as in (107).

The following theorem contains the case $a \geq 1$, $\frac{1}{2} \leq b < 1$ as a particular case.

THEOREM 4. *If* $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ *are mutually independent and identically distributed random variables for which*

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\gamma_n}{A_2 n^a} \leq x\right\} = \mathbf{P}\{\gamma \leq x\} \tag{110}$$

where $a \geq 1$ *and* $A_2 > 0$, *and if*

$$\lim_{n\to\infty} \frac{\delta_n}{n} = B_1 \tag{111}$$

in probability where $B_1 > 0$, *then we have*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\gamma_{\omega(t)} B_1^a}{A_2 t^a} \leq x\right\} = \mathbf{P}\{\gamma \leq x\} \quad . \tag{112}$$

PROOF. By (101) and (111) it follows that

$$\lim_{t\to\infty} \frac{\omega(t)}{t} = \frac{1}{B_1} \tag{113}$$

in probability. Thus (112) immediately follows from Theorem 45.4.

In this case the asymptotic distribution of $\beta(t)$ can be obtained by (109) where now $d = a$, $D_1 = 0$, $D_2 = A_2 B_1^{-a}$ and $\underset{\sim}{P}\{\vartheta \leq x\} = \underset{\sim}{P}\{\gamma \leq x\}$.

The following theorem contains the case $b \geq 1$, $\frac{1}{2} \leq a < 1$ as a particular case.

THEOREM 5. If $\beta_1, \beta_2, \ldots, \beta_n, \ldots$ are mutually independent and identically distributed random variables for which

$$\lim_{n\to\infty} \underset{\sim}{P}\left\{\frac{\delta_n}{B_2 n^b} \leq x\right\} = \underset{\sim}{P}\{\delta \leq x\} \tag{114}$$

where $b \geq 1$ and $B_2 > 0$, and if

$$\lim_{n\to\infty} \frac{\gamma_n}{n} = A_1 \tag{115}$$

in probability where $A_1 > 0$, then we have

$$\lim_{t\to\infty} \underset{\sim}{P}\left\{\frac{\delta_{\rho(t)} A_1^b}{B_2 t^b} \leq x\right\} = \underset{\sim}{P}\{\delta \leq x\} . \tag{116}$$

PROOF. By (5) and (115) it follows that

$$\lim_{t\to\infty} \frac{\rho(t)}{t} = \frac{1}{A_1} \tag{117}$$

in probability. Thus (116) immediately follows by Theorem 45.4.

In this case the asymptotic distribution of $\beta(t)$ is given by (107) where now $d = b$, $D_1 = 0$, $D_2 = B_2 A_1^{-b}$ and $\underset{\sim}{P}\{\vartheta \leq x\} = \underset{\sim}{P}\{\delta \leq x\}$.

THEOREM 6. If (α_n, β_n) $(n=1,2,\ldots)$ are mutually independent and identically distributed vector variables for which (98) holds with $\frac{1}{2} \leq a < 1$ and $\frac{1}{2} \leq b < 1$, then

$$(118) \qquad \lim_{t\to\infty} \underset{\sim}{P}\left\{ \frac{A_1\delta_{\rho(t)} - B_1 t}{A_1^{-d} t^d} \leq x \right\} = Q(x)$$

exists where $d = \max(a,b)$,

$$(119) \qquad Q(x) = \begin{cases} \underset{\sim}{P}\{A_1B_2\delta \leq x\} & \text{for } b > a, \\ \underset{\sim}{P}\{A_1B_2\delta - B_1A_2\gamma \leq x\} & \text{for } b = a, \\ \underset{\sim}{P}\{-B_1A_2\gamma \leq x\} & \text{for } b < a, \end{cases}$$

and $\underset{\sim}{P}\{\gamma \leq x, \ \delta \leq y\} = F(x,y)$.

PROOF. By (98) it follows that

$$(120) \qquad \lim_{n\to\infty} \underset{\sim}{P}\left\{ \frac{A_1\delta_n - B_1\gamma_n}{n^d} \leq x \right\} = Q(x)$$

where $d = \max(a,b)$ and $Q(x)$ is given by (119).

By (5) and (98) it follows that

$$(121) \qquad \lim_{t\to\infty} \underset{\sim}{P}\left\{ \frac{\rho(t) - \frac{t}{A_1}}{A_2A_1^{-(1+a)}t^a} \leq x \right\} = \underset{\sim}{P}\{-\gamma \leq x\},$$

and

$$(122) \qquad \lim_{t\to\infty} \frac{\rho(t)}{t} = \frac{1}{A_1}$$

in probability. If we apply Theorem 45.4 to the random variables $\zeta(n) = A_1\delta_n - B_1\gamma_n$ $(n=0,1,2,\dots)$, and $\{\rho(t),\ 0 \leq t < \infty\}$, then we obtain that

$$(123) \qquad \lim_{t\to\infty} \underset{\sim}{P}\left\{ \frac{A_1\delta_{\rho(t)} - B_1\gamma_{\rho(t)}}{(t/A_1)^d} \leq x \right\} = Q(x).$$

It remains to show that (123) implies (118). This follows from the inequalities

$$(124) \qquad A_1\delta_{\rho(t)} - B_1\gamma_{\rho(t)+1} \leq A_1\delta_{\rho(t)} - B_1 t \leq A_1\delta_{\rho(t)} - B_1\gamma_{\rho(t)}$$

for $t \geq 0$ and from the fact that

$$(125) \qquad \lim_{t\to\infty} \frac{\alpha_{\rho(t)+1}}{t^a} = 0$$

in probability. The relation (125) follows from the inequality

$$(126) \qquad \underset{\sim}{P}\left\{\frac{\alpha_{\rho(t)+1}}{t^a} > \epsilon\right\} \leq \underset{\sim}{P}\{|\rho(t) - \frac{t}{A_1}| > K\, t^a\} + 2Kt^a \underset{\sim}{P}\{\alpha_1 > t^a\epsilon\}$$

which holds for $\epsilon > 0$ and $K > 0$. Since $\underset{\sim}{P}\{\alpha_1 \leq x\}$ belongs to the domain of normal attraction of a stable distribution function with characteristic exponent $1/a$, it follows that

$$(127) \qquad \lim_{t\to\infty} \underset{\sim}{P}\{\alpha_1 > t^a\,\epsilon\}(t^a\epsilon)^{1/a} = c$$

where c is a nonnegative constant. ($c = 0$ if $a = \frac{1}{2}$.) This implies that the second term on the right-hand side of (126) tends to 0 as $t \to \infty$. If $t \to \infty$ and $K \to \infty$, then by (121) the first term on the right-hand side of (126) tends to 0. Since $\epsilon > 0$ is arbitrary, this implies (125). This completes the proof of the theorem.

Now the asymptotic distribution of $\beta(t)$ is given by (107) where $d = \max(a,b)$, $D_1 = B_1$, $D_2 = 1/A_1^d$ and $\underset{\sim}{P}\{\vartheta \leq x\} = Q(x)$ is given by (119).

THEOREM 7. *Let us suppose that* (α_n, β_n) $(n = 1,2,\dots)$ *are mutually independent, and identically distributed vector random variables for which* (98) *holds with* $a \geqq 1$ *and* $b \geqq 1$. *Let*

(128) $$\Phi(s,q) = \int_0^\infty \int_0^\infty e^{-sx-qy} d_x\, d_y\, F(x,y)$$

for $\mathrm{Re}(s) \geq 0$ *and* $\mathrm{Re}(q) \geq 0$. *Then*

(129) $$\lim_{t\to\infty} \underset{\sim}{P}\left\{ \frac{\gamma_{\omega(t)}\, B_2^{a/b}}{A_2 t^{a/b}} \leq x \right\} = Q(x)$$

exists and

(130) $$\int_0^\infty x^s dQ(x) = \frac{1}{\Gamma(1-s)\,\Gamma(1+\frac{as}{b})} \int_0^\infty x^s dV(x)$$

for sufficiently small $|\mathrm{Re}(s)|$ *where*

(131) $$V(s) = 1 - \frac{\log \Phi(\frac{1}{s},0)}{\log \Phi(\frac{1}{s},1)}$$

for $\mathrm{Re}(s) > 0$.

PROOF. In proving this theorem we may assume without loss of generality that $A_2 = B_2 = 1$. Let

(132) $$\psi(s,q) = \underset{\sim}{E}\{e^{-s\alpha_n - q\beta_n}\}$$

for $\mathrm{Re}(s) \geq 0$ and $\mathrm{Re}(q) \geq 0$. Then we have

(133) $$\lim_{n\to\infty}[\psi(\frac{s}{n^a}, \frac{q}{n^b})]^n = \Phi(s, q)$$

and

(134) $$\lim_{n\to\infty} n[\psi(\frac{s}{n^a}, \frac{q}{n^b}) -1] = \log\Phi(s, q)$$

for $\mathrm{Re}(s) \geqq 0$ and $\mathrm{Re}(q) \geqq 0$. We note that necessarily

(135) $$\log\Phi(s, 0) = -As^{1/a}$$

and

(136) $$\log\Phi(0, q) = -Bq^{1/b}$$

where $A > 0$ and $B > 0$ and

(137) $$\log\Phi(s, qs^{b/a}) = s^{1/a}\log\Phi(1, q)$$

for $\mathrm{Re}(s) \geqq 0$ and $\mathrm{Re}(q) \geqq 0$.

For simplicity let us write $\zeta(t) = \gamma_{\omega(t)}$ for $t \geqq 0$. By (105) we have

(138) $$E\{e^{-s\zeta(t)}\} = 1 - [1-\psi(s, 0)]M(t, s)$$

for $\mathrm{Re}(s) \geqq 0$ where

(139) $$M(t, s) = \sum_{n=0}^{\infty} E\{e^{-s\gamma_n} I(\delta_n \leqq t)\}$$

and $I(\delta_n \leqq t)$ is the indicator variable of the event $\{\delta_n \leqq t\}$, that is, $I(\delta_n \leqq t) = 1$ if $\{\delta_n \leqq t\}$ occurs and 0 otherwise. If we express the sum in the above formula in the form of an integral, then we can write that

$$M(t^b, st^{-a}) = t\int_0^\infty \underset{\sim}{E}\{e^{-st^{-a}\gamma_{[ut]}} I(\delta_{[ut]} \leqq t^b)\}du \tag{140}$$

for $\mathrm{Re}(s) \geqq 0$ and $t > 0$. If $\mathrm{Re}(s) \geqq 0$, then

$$\lim_{t\to\infty} \frac{M(t, st^{-a/b})}{t^{1/b}} = \lim_{t\to\infty} \frac{M(t^b, st^{-a})}{t} = \mu(s) \tag{141}$$

exists and

$$\mu(s) = \int_0^\infty \underset{\sim}{E}\{e^{-su^a\gamma} I(\delta \leqq u^{-b})\}du \tag{142}$$

where $\underset{\sim}{P}\{\gamma \leqq x, \delta \leqq y\} = F(x, y)$.

First, let $s = 0$. Since

$$\int_0^\infty e^{-qt} dM(t, 0) = \frac{1}{1-\psi(0, q)} \tag{143}$$

for $\mathrm{Re}(q) > 0$, and since

$$\lim_{q\to+0} [1-\psi(0, q)]q^{-1/b} = B, \tag{144}$$

it follows from a Tauberian theorem (Theorem 9.13 in the Appendix) that

$$\lim_{t\to\infty} \frac{M(t, 0)}{t^{1/b}} = \frac{1}{B\Gamma(1+\frac{1}{b})}. \tag{145}$$

This proves (141) for $s = 0$. For

$$\mu(0) = \int_0^\infty \underset{\sim}{P}\{\delta \leqq u^{-b}\}du = \underset{\sim}{E}\{\delta^{-1/b}\} = \frac{1}{B\Gamma(1+\frac{1}{b})} \tag{146}$$

which follows from $\underset{\sim}{E}\{e^{-q\delta}\} = \Phi(0, q) = \exp\{-Bq^{1/b}\}$. Accordingly, we have

$$\lim_{t\to\infty}\int_0^\infty \underset{\sim}{P}\{\delta_{[ut]} \leqq t^b\}du = \int_0^\infty \underset{\sim}{P}\{\delta \leqq u^{-b}\}du \,. \tag{147}$$

For $u > 0$ and $Re(s) \geqq 0$ the integrand in (140) has absolute value $\leqq 1$ and it tends to the integrand in (142) as $t \to \infty$. On the other hand for any $K > 0$ and $Re(s) \geqq 0$ we have

$$\left|\int_K^\infty \underset{\sim}{E}\{e^{-st^{-a}\gamma_{[ut]}} I(\delta_{[ut]} \leqq t^b)\}du\right| \leqq \int_K^\infty \underset{\sim}{P}\{\delta_{[ut]} \leqq t^b\}du \to \int_K^\infty \underset{\sim}{P}\{\delta \leqq u^{-b}\}du \tag{148}$$

as $t \to \infty$ and the extreme right member is arbitrarily close to 0 if K is sufficiently large. Thus by the dominated convergence theorem we can conclude that in (140) the integral tends to $\mu(s)$ for $Re(s) \geqq 0$ as $t \to \infty$. This proves (141).

Since

$$\lim_{s\to+0}[1 - \psi(s, 0)]s^{-1/a} = A \,, \tag{149}$$

by (141) we obtain that

$$\lim_{t\to\infty} \underset{\sim}{E}\{e^{-s\zeta(t)t^{-a/b}}\} = 1 - As^{1/a}\,\mu(s) \tag{150}$$

for $Re(s) \geqq 0$. Here $|\mu(s)| \leqq \mu(0)$ for $Re(s) \geqq 0$ and if $s \to +0$, then the right-hand side of the above equation tends to 1. Thus by the continuity theorem of Laplace-Stieltjes transforms we can conclude that the limiting distribution

$$\lim_{t\to\infty} \underset{\sim}{P}\left\{\frac{\zeta(t)}{t^{a/b}} \leqq x\right\} = Q(x) \tag{151}$$

exists and

$$\int_0^\infty e^{-sx}\,dQ(x) = 1 - As^{1/a}\,\mu(s) \tag{152}$$

for $\mathrm{Re}(s) \geqq 0$. Hence $Q(x)$ can be obtained by inversion.

We can also determine $Q(x)$ in another way. By (103) we have

(153)
$$q \int_0^\infty e^{-qt} \underset{\sim}{E}\{e^{-s\zeta(t)}\} dt = 1 - \frac{1 - \psi(s, 0)}{1 - \psi(s, q)}$$

for $\mathrm{Re}(s) \geqq 0$ and $\mathrm{Re}(q) > 0$. Now let ν be a positive real random variable which is independent of the process $\{\zeta(t), 0 \leqq t < \infty\}$ and for which $\underset{\sim}{P}\{\nu \leqq x\} = 1 - e^{-x}$ if $x \geqq 0$. Then we can write that

(154)
$$\underset{\sim}{E}\{e^{-s\zeta(\nu/q)}\} = 1 - \frac{1 - \psi(s, 0)}{1 - \psi(s, q)}$$

for $\mathrm{Re}(s) \geqq 0$ and $q > 0$. Hence it follows that

(155)
$$\lim_{q \to 0} \underset{\sim}{E}\{e^{-sq^{a/b}\zeta(\nu/q)}\} = 1 - \lim_{q \to 0} \frac{[1 - \psi(sq^{a/b}, 0)]q^{-1/b}}{[1 - \psi(sq^{a/b}, q)]q^{-1/b}} =$$
$$= 1 - \frac{\log\Phi(s, 0)}{\log\Phi(s, 1)} = V(\tfrac{1}{s})$$

for $\mathrm{Re}(s) \geqq 0$.

If ζ, ν_1, ν_2 are mutually independent random variables for which $\underset{\sim}{P}\{\zeta \leqq x\} = Q(x)$ and $\underset{\sim}{P}\{\nu_1 \leqq x\} = \underset{\sim}{P}\{\nu_2 \leqq x\} = 1 - e^{-x}$ for $x \geqq 0$, then by the last equation we can write that

(156)
$$\underset{\sim}{P}\{\zeta\, \nu_1^{-1}\, \nu_2^{a/b} \leqq x\} = V(x)$$

for $x > 0$. Hence it follows that

(157)
$$\underset{\sim}{E}\{\zeta^s\}\underset{\sim}{E}\{\nu_1^{-s}\}\underset{\sim}{E}\{\nu_2^{as/b}\} = \int_0^\infty x^s dV(x)$$

for sufficiently small $|\mathrm{Re}(s)|$. This proves (130).

The distribution function $Q(x)$ can be obtained by Mellin's inversion formula.

In the particular case when $a = b$ we have

$$\int_0^\infty x^s dQ(x) = \frac{\sin \pi s}{\pi s} \int_0^\infty x^s dV(x) \tag{158}$$

for sufficiently small $|Re(s)|$, and hence it follows by inversion that

$$\frac{dQ(x)}{dx} = \frac{V(xe^{\pi i}) - V(xe^{-\pi i})}{2\pi i x} \tag{159}$$

for $x > 0$ where the definition of $V(x)$ is extended by analytical continuation to the complex plane cut along the negative real axis from 0 to ∞.

In the particular case when $F(x, y) = \underset{\sim}{P}\{\gamma \leq x, \delta \leq y\} = \underset{\sim}{P}\{\gamma \leq x\}\underset{\sim}{P}\{\delta \leq y\}$, that is, when γ and δ are independent random variables, we have

$$Q(x) = \underset{\sim}{P}\{\gamma\, \delta^{-a/b} \leq x\} . \tag{160}$$

Conversely, we can prove that if $Q(x)$ is given by the above formula, then γ and δ are necessarily independent random variables.

To prove this last statement let us suppose that the vector variable (γ, δ) and ν_1 and ν_2 are mutually independent. Let $\underset{\sim}{P}\{\gamma \leq x, \delta \leq y\} = F(x, y)$ with Laplace-Stieltjes transform $\Phi(s, q)$, and $\underset{\sim}{P}\{\nu_1 \leq x\} = \underset{\sim}{P}\{\nu_2 \leq x\} = 1 - e^{-x}$ for $x \geq 0$. Then we have

$$\underset{\sim}{P}\{\gamma\, \nu_1^{-1} \leq x, \delta\, \nu_2^{-1} \leq y\} = \Phi\left(\frac{1}{x}, \frac{1}{y}\right) \tag{161}$$

for $x > 0$ and $y > 0$. Hence we can deduce that

$$\text{(162)} \qquad \underset{\sim}{P}\{\gamma\ \delta^{-a/b}\ \nu_1^{-1}\ \nu_2^{a/b} \leqq x\} = \frac{axV'(x)}{[1-V(x)]}$$

for $x > 0$.

Thus (160) holds if and only if

$$\text{(163)} \qquad \frac{axV'(x)}{1 - V(x)} = V(x)$$

for $x > 0$. The general solution of this differential equation is

$$\text{(164)} \qquad V(x) = \frac{Cx^{1/a}}{1 + Cx^{1/a}}$$

for $x > 0$ where C is a positive constant. Hence

$$\text{(165)} \qquad \Phi(s, q) = e^{-A(s^{1/a}+Cq^{1/b})}$$

for $\mathrm{Re}(s) \geqq 0$ and $\mathrm{Re}(q) \geqq 0$. Finally, it follows that $C = B/A$ and that γ and δ are independent.

In the above case the asymptotic distribution of $\beta(t)$ can be obtained by (109) where now $d = a/b$, $D_1 = 0$, $D_2 = A_2B_2^{-a/b}$, and $\underset{\sim}{P}\{\vartheta \leqq x\} = Q(x)$.

Thus it follows that

$$\text{(166)} \qquad \lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\beta(t) - M_1 t}{M_2 t^m} \leqq x\right\} = R(x)$$

where the constants M_1, M_2, m and the distribution function $R(x)$ are given in the following table. In this table ϑ is a random variable with distribution function $\underset{\sim}{P}\{\vartheta \leqq x\} = Q(x)$ given by (130).

(a, b)	M_1	M_2	m	R(x)
a > b	0	$B_2A_2^{-b/a}$	b/a	$\underset{\sim}{P}\{\vartheta^{-b/a} \leq x\}$
a = b	1	1	1	$\underset{\sim}{P}\{-\frac{A_2\vartheta}{A_2\vartheta + B_2} \leq x\}$
a < b	1	$A_2B_2^{-a/b}$	a/b	$\underset{\sim}{P}\{-\vartheta \leq x\}$

We note that in a similar way we can prove that

$$\lim_{t\to\infty} \underset{\sim}{P}\left\{\frac{\delta_{\rho(t)}A_2^{b/a}}{B_2 t^{b/a}} \leq x\right\} = Q^*(x) \tag{167}$$

exists and

$$\int_0^\infty x^s dQ^*(x) = \frac{1}{\Gamma(1-s)\Gamma(1+\frac{bs}{a})} \int_0^\infty x^s dV^*(x) \tag{168}$$

for sufficiently small $|Re(s)|$ where

$$V^*(s) = \frac{\log \Phi(1,0)}{\log \Phi(1,\frac{1}{s})} \tag{169}$$

for $Re(s) > 0$. The asymptotic distribution of $\beta(t)$ is given by (107) where now $d = b/a$, $D_1 = 0$, $D_2 = B_2A_2^{-b/a}$ and $\underset{\sim}{P}\{\vartheta \leq x\} = Q^*(x)$.

We observe that

$$V^*(x) = 1 - V(x^{-a/b}) \tag{170}$$

for $x > 0$.

60. Sojourn Time Problems for Markov Processes. Let $\{\xi(u), u \in T\}$ be a stochastic process with state space X where X is a metric space and with parameter set T where T is a linear set. We say that $\{\xi(u), u \in T\}$ is a Markov process if for any parameter values $t_1 < t_2 < \dots < t_n$ $(n = 2,3,\dots)$ and for any Borel subset S of X we have

$$(1) \qquad \underset{\sim}{P}\{\xi(t_n) \in S | \xi(t_1),\ldots,\xi(t_{n-1})\} = \underset{\sim}{P}\{\xi(t_n) \in S | \xi(t_{n-1})\}$$

with probability 1 . The probabilities

$$(2) \qquad \underset{\sim}{P}\{\xi(t) \in S | \xi(u) = x\} ,$$

defined for the parameter values $u < t$, for $x \in X$ and for Borel subsets S of X , are called transition probabilites. If (2) depends only on x, S and $t-u$, then we say that the Markov process is homogeneous.

In what follows we suppose that either $T = \{0,1,2,\ldots\}$ or $T = [0, \infty)$ and that $\{\xi(u) , u \in T\}$ is a homogeneous Markov process with state space X where X is a metric space. Let $\delta(x)$ be a nonnegative, measurable function of x defined on the space X .

If $T = \{0,1,2,\ldots\}$, then let

$$(3) \qquad \mu_n = \sum_{r=1}^{n} \delta(\xi(r))$$

for $n = 1,2,\ldots,$ and if $T = [0, \infty)$, then let

$$(4) \qquad \mu(t) = \int_0^t \delta(\xi(u))du$$

for $t \geqq 0$ provided that the integral exists.

We are interested in finding the asymptotic distribution of μ_n as $n \to \infty$, and the asymptotic distribution of $\mu(t)$ as $t \to \infty$.

In the particular case where $\delta(x)$ is the indicator function of a Borel subset S of X, that is,

(5)
$$\delta(x) = \begin{cases} 1 & \text{if } x \in S , \\ 0 & \text{if } x \notin S , \end{cases}$$

then μ_n and $\mu(t)$ can be interpreted as sojourn times spent in the interval $[0, n]$ or in the interval $[0, t]$ in the state S.

In what follows we shall mention a few results for Markov processes $\{\xi(u) , u \in T\}$.

First, let us suppose that $\xi_1, \xi_2, \ldots, \xi_r, \ldots$ are mutually independent and identically distributed random variables for which $E\{\xi_r\} = 0$ and $Var\{\xi_r\} = 1$. Let $\xi(0) = 0$ and $\xi(r) = \xi_1 + \xi_2 + \ldots + \xi_r$ for $r = 1,2,\ldots$. Then $\{\xi(r) , r = 0,1,2,\ldots\}$ is a discrete parameter Markov process. Let us suppose that

(6)
$$\delta(x) = \begin{cases} 1 & \text{if } x > 0 , \\ 0 & \text{if } x \leq 0 \end{cases}$$

and define μ_n by (3). Then by a result of P. Erdős and M. Kac [17] we have

(7)
$$\lim_{n \to \infty} P\{\mu_n \leq nx\} = \frac{2}{\pi} \text{arc} \sin\sqrt{x}$$

for $0 \leq x \leq 1$.

Next, let us suppose that $\xi_1, \xi_2, \ldots, \xi_r, \ldots$ are mutually independent and identically distributed random variables having a stable distribution of type $S(\alpha,0,1,0)$ where $0 < \alpha \leq 2$, that is,

$$(8) \qquad \mathbf{E}\{e^{-s\xi_r}\} = e^{-|s|^\alpha}$$

for $\mathrm{Re}(s) = 0$. For $a > 0$ let us define $\mu_n(a)$ as the number of subscripts $r = 1,2,\ldots,n$ for which $|\xi_1 + \xi_2 + \ldots + \xi_r| < a$.

If $\alpha = 1$, then we have

$$(9) \qquad \lim_{n\to\infty} \mathbf{P}\left\{\frac{\mu_n(a)}{\log n} \leq \frac{2ax}{\pi}\right\} = 1 - e^{-x}$$

for $x \geq 0$, and if $1 < \alpha \leq 2$, then

$$(10) \qquad \lim_{n\to\infty} \mathbf{P}\left\{\frac{\mu_n(a)}{n^{1-\frac{1}{\alpha}}} \leq \frac{2ax}{\alpha \sin\frac{\pi}{\alpha}}\right\} = G_{1-\frac{1}{\alpha}}(x)$$

where $G_\sigma(x)$ is defined by (59.54) and (59.55) for $0 < \sigma < 1$. If $0 < \alpha < 1$, then $\mathbf{P}\{\lim_{n\to\infty} \mu_n(a) < \infty\} = 1$. These results were found in 1951 by K. L. Chung and M. Kac [6], [7].

In 1954 G. Kallianpur and H. Robbins [25] studied the asymptotic distribution of (3) in the case where $\xi(r) = \xi_1 + \ldots + \xi_r$ $(r = 1,2,\ldots)$ and $\xi_1, \xi_2, \ldots, \xi_r, \ldots$ are mutually independent and identically distributed random variables belonging to the domain of attraction of a symmetric stable distribution function, and furthermore $\delta(x)$ is Riemann integrable on some finite interval (a, b) and 0 elsewhere.

In 1957 D. A. Darling and M. Kac [9] found the asymptotic distribution of μ_n as $n \to \infty$ for a general class of discrete parameter Markov processes $\{\xi(r), r = 0,1,2,\dots\}$ and the asymptotic distribution of $\mu(t)$ as $t \to \infty$ for a general class of continuous parameter Markov processes $\{\xi(t), 0 \leq t < \infty\}$. They proved the following results.

Theorem 1. Let $\{\xi(r), r = 0,1,2,\dots\}$ be a homogeneous discrete parameter Markov process. Let us suppose that there exists a function $g(z)$ and a positive constant C such that

(11) $$\lim_{z \to 1} g(z) = \infty$$

and

(12) $$\lim_{z \to 1} \frac{1}{g(z)} \sum_{n=0}^{\infty} E\{\delta(\xi(n)) | \xi(0) = x\} z^n = C$$

where the convergence is uniform in x on the set $\{x : \delta(x) > 0\}$.

In order that for some normalizing sequence m_n $(n = 1,2,\dots)$ the random variables

(13) $$\frac{\mu_n}{m_n} = \frac{1}{m_n} \sum_{r=1}^{n} \delta(\xi(r))$$

have a nondegenerate limiting distribution it is necessary and sufficient that

(14) $$g(z) = \frac{1}{(1-z)^{\alpha}} L\left(\frac{1}{1-z}\right)$$

for some α $(0 \leq \alpha < 1)$ and for some slowly varying function $L(x)$.

If (14) is satisfied, then

(15)
$$\lim_{n\to\infty} \underset{\sim}{P}\left\{ \frac{\mu_n}{Cg(1-\frac{1}{n})} \leq x \right\} = G_\alpha(x)$$

<u>where</u> $G_\alpha(x)$ <u>is defined by</u> (59.54) <u>and</u> (59.55).

<u>Theorem 2.</u> <u>Let</u> $\{\xi(u), 0 \leq u < \infty\}$ <u>be a homogeneous continuous parameter Markov process.</u> <u>Let us suppose that there exists a function</u> $h(s)$ <u>and a positive constant</u> C <u>such that</u>

(16)
$$\lim_{s\to 0} h(s) = \infty$$

<u>and</u>

(17)
$$\lim_{s\to 0} \frac{1}{h(s)} \int_0^\infty e^{-st} \underset{\sim}{E}\{\delta(\xi(t)) \mid \xi(0) = x\}\, dt = C$$

<u>where the convergence is uniform in</u> x <u>on the set</u> $\{x : \delta(x) > 0\}$.

<u>In order that for some normalizing function</u> $m(t)$ $(0 \leq t < \infty)$ <u>the random variables</u>

(18)
$$\frac{\mu(t)}{m(t)} = \frac{1}{m(t)} \int_0^t \delta(\xi(u))du$$

<u>have a nondegenerate limiting distribution it is necessary and sufficient that</u>

(19)
$$h(s) = \frac{L(1/s)}{s^\alpha}$$

<u>for some</u> α $(0 \leq \alpha < 1)$ <u>and for some slowly varying function</u> $L(x)$.

<u>If</u> (19) <u>is satisfied, then</u>

(20) $$\lim_{t \to \infty} P\{ \frac{\mu(t)}{Ch(\frac{1}{t})} \leq x\} = G_\alpha(x)$$

where $G_\alpha(x)$ is defined by (59.54) and (59.55).

In Theorem 1 and in Theorem 2 the function $L(x)$ defined for $0 < x < \infty$ is slowly varying if

(21) $$\lim_{x \to \infty} \frac{L(\omega x)}{L(x)} = 1$$

for any $\omega > 0$.

By using Karamata's Tauberian theorem (Theorem 9.14 in the Appendix) D. A. Darling and M. Kac [9] demonstrated that

(22) $$\lim_{n \to \infty} E\{(\frac{\mu_n}{C\ g(1-\frac{1}{n})})^r\} = \frac{r!}{\Gamma(r\alpha+1)}$$

for $r = 0,1,2,\ldots$. Since

(23) $$\int_0^\infty x^r d\ G_\alpha(x) = \frac{r!}{\Gamma(r\alpha+1)}$$

for $r = 0,1,2,\ldots$ and since $G_\alpha(x)$ is uniquely determined by its moments, by Theorem 41.11 it follows that (15) is true. In a similar way (20) follows from the relations

(24) $$\lim_{t \to \infty} E\{(\frac{\mu(t)}{C\ h(\frac{1}{t})})^r\} = \frac{r!}{\Gamma(r\alpha+1)}$$

for $r = 0,1,2,\ldots$.

We mention that S. Karlin and J. McGregor [26] determined the asymptotic distribution of $\mu(t)$ for some birth and death processes by using Theorem 2.

Finally we mention a related result which was found by E. B. Dynkin [56]. Let us suppose that $\{\xi(u), 0 \leq u < \infty\}$ is a separable stable process of type $S(\alpha,1,c,0)$ where $0 < \alpha < 1$ and $c > 1$. Then

$$\mathrm{E}\{e^{-s\xi(u)}\} = e^{-cs^{\alpha}} \tag{25}$$

for $\mathrm{Re}(s) \geq 0$ and $u \geq 0$.

Let $R_\alpha(x)$ be the stable distribution function of type $S(\alpha,1,1,0)$ and let

$$V_a(x) = \frac{\sin\pi\alpha}{\pi}\int_0^a \frac{R_\alpha((a-u)x^{-1/\alpha})}{u^\alpha(1+u)}\,du \tag{26}$$

for $x > 0$, and $a > 0$ and

$$V_a(0) = \frac{\sin\pi\alpha}{\pi}\int_0^a \frac{du}{u^\alpha(1+u)} . \tag{27}$$

If $\theta(a)$ denotes the first passage time of the process $\{\xi(u), 0 \leq u < \infty\}$ through a where $a > 0$, and if $0 < a < b$, then we have

$$\mathrm{P}\{\theta(b) - \theta(a) > x\} = V_{(b-a)/a}\left(\frac{cx}{\Gamma(1-\alpha)a^\alpha}\right) \tag{28}$$

for $x \geq 0$ and

$$\mathrm{E}\{\theta(b) - \theta(a)\} = \frac{c(b^\alpha - a^\alpha)}{\Gamma(1-\alpha)} . \tag{29}$$

61. Problems

61.1. Find the probability $\underset{\sim}{P}\{\beta(t) \leq x\}$ defined by (59.9) in the case where $H(x) = 1-e^{-\mu x}$ for $x \geq 0$. (See R. P. Dobrushin [12 p. 102].)

61.2. Let us suppose that a particle performs a random walk on the x - axis. It starts at $x = 0$ and at times $u = 1,2,\ldots$ it moves either a unit distance to the right with probability 1/2 or a unit distance to the left with probability 1/2. Let us suppose that the successive displacements are mutually independent random variables. Denote by $\xi(u)$ the position of the particle at time u $(0 \leq u < \infty)$. We say that at time u the process is in state A if $\xi(u) \leq 0$ and in state B if $\xi(u) \geq 1$. Denote by $\beta(t)$ the total time spent in state B in the interval $(0, t)$. Find the asymptotic distribution of $\beta(t)$ as $t \to \infty$. (See P. Lévy [33], P. Erdős and M. Kac [17], and E. S. Andersen [1].)

61.3. Let $\{\xi(u), 0 \leq u < \infty\}$ be a separable stable process of type $S(\alpha,0,1,0)$ where $0 < \alpha \leq 2$. Let $\delta(x) = 1$ for $x > 0$ and $\delta(x) = 0$ for $x \leq 0$. Determine the distribution function of the random variable

$$\beta(t) = \int_0^t \delta(\xi(u))du$$

for $t > 0$. (See M. Kac [22].)

61.4. Let $\{\xi(u), 0 \leq u < \infty\}$ be the random walk process defined in Problem 61.2. Let m be a given positive integer. If $\xi(u) = 1,2,\ldots, m$, then we say that the process is in state B at time u, otherwise, the process is in state A at time u. Denote by $\beta(t)$ the total time spent in state B in the interval $(0, t)$. Find the asymptotic distribution of $\beta(t)$ as $t \to \infty$. (See R. L. Dobrushin [12], and K.L. Chung and M. Kac [6], [7].)

61.5. Let us suppose that in Theorem 59.2 $(\alpha_1 + \alpha_2 + \dots + \alpha_n)/n \Rightarrow A$ as $n \to \infty$ where A is a positive constant and $\beta_1, \beta_2, \dots, \beta_n, \dots$ are mutually independent and identically distributed positive random variables for which $x^\alpha \mathrm{P}\{\beta_n > x\} = h(x)$ where $0 < \alpha < 1$ and $\lim_{x \to \infty} h(\omega x)/h(x) = 1$ for any $\omega > 0$. Find the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

61.6. Let us suppose that in Theorem 59.2 $(\beta_1 + \beta_2 + \dots + \beta_n)/n \Rightarrow B$ as $n \to \infty$ where B is a positive constant and $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ are mutually independent and identically distributed positive random variables for which $x^\alpha \mathrm{P}\{\alpha_n > x\} = h(x)$ where $0 < \alpha < 1$ and $\lim_{x \to \infty} h(\omega x)/h(x) = 1$ for any $\omega > 0$. Find the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

61.7. Let us suppose that (59.98) holds with $a = b = 1/2$, $A_1 > 0$, $B_1 > 0$, $A_2 > 0$, $B_2 > 0$ and that $F(x,y)$ is a two-dimensional normal distribution function of type

$$N\left(\begin{Vmatrix} 0 \\ 0 \end{Vmatrix}, \begin{Vmatrix} 1 & r \\ r & 1 \end{Vmatrix}\right)$$

where $-1 < r < 1$. Find the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

61.8. Let us suppose that in Theorem 59.7 $\Phi(s,q)$ is given either by (i) $\Phi(s,q) = e^{-s^\alpha - q^\alpha}$ or by (ii) $\Phi(s,q) = e^{-(s+q)^\alpha}$ where $0 < \alpha < 1$. Find the asymptotic distribution of $\beta(t)$ as $t \to \infty$.

REFERENCES

[1] Andersen, E. S., "On the number of positive sums of random variables," Skandinavisk Aktuarietidskrift 32 (1949) 27-36.

[2] Berman, S. M., "Occupation times of stationary Gaussian processes," Journal of Applied Probability 7 (1970) 721-733.

[3] Bingham, N. H., "Limit theorems for occupation times of Markov processes," Zeitschr. f. Wahrscheinlichkeitstheorie 17 (1971) 1-22.

[4] Boiko, R. V., "Asymptotic distribution of the sojourn time of the simplest random walk on the positive semi-axis," (Russian) Ukrain. Mat. Zhur. 24 (1972) 80-83. [English translation: Ukrainian Mathematical Journal 24 (1972) 63-65.]

[5] Chung, K. L., "Contributions to the theory of Markov chains. II," Transactions of the American Mathematical Society 76 (1954) 397-419.

[6] Chung, K. L., and M. Kac, "Remarks on fluctuations of sums of independent random variables," Four Papers on Probability. Memoirs of the American Mathematical Society. No. 6 (1951) 11 pp.

[7] Chung, K. L., and M. Kac, "Corrections to the paper 'Remarks on fluctuations of sums of independent random variables'," Proceedings of the American Mathematical Society 4 (1953) 560-563.

[8] Ciesielski, Z., and S. J. Taylor, "First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path," Transactions of the American Mathematical Society 103 (1962) 434-450.

[9] Darling, D. A., and M. Kac, "On occupation times for Markoff processes," Trans. Amer. Math. Soc. 84 (1957) 444-458.

[10] Davydov, Ju. A., "On some analogs of the arcsine law for random walks," (Russian) Dokl. Akad. Nauk SSSR 198 (1971) 513-515. [English translation: Soviet Mathematics-Doklady 12 (1971) 822-825.]

[11] Davydov, Yu. A., and I. A. Ibragimov, "On asymptotical behavior of some functionals of processes with independent increments," Theory of Probability and its Applications 16 (1971) 162-167.

[12] Dobrushin, R. L., "Limit theorems for a Markov chain of two states," (Russian) Izvestija Akademii Nauk SSSR. Ser. Mat. 17 (1953) 291-330. [Enlgish translation: Selected Translations in Mathematical Statistics and Probability, IMS and AMS, 1 (1961) 97-134.]

[13] Dobrushin, R. L., "Two limit theorems for the simplest random walk on a line," (Russian) Uspekhi Matem. Nauk (N.S.) 10 (1955) 139-146.

[14] Dobrushin, R. L., "Lemma on the limit of compound random functions," (Russian) Uspekhi Mat. Nauk 10 (1955) 157-159.

[15] Doeblin, W., "Sur deux problèmes de M. Kolmogoroff concernant les chaines dénombrables," Bulletin de la Societé Mathématique de France 66 (1938) 210-220.

[16] Dwass, M., and S. Karlin, "Conditioned limit theorems," The Annals of Mathematical Statistics 24 (1963) 1147-1167.

[17] Erdös, P., and M. Kac, "On the number of positive sums of indepedent random variables," Bulletin of the American Mathematical Society 53 (1947) 1011-1020.

[18] Gibson, A. E., and B. W. Conolly, "On a three-state sojourn time problem," Journal of Applied Probability 8 (1971) 716-723.

[19] Good, I. J., "The frequency count of a Markov chain and the transition to continuous time," The Annals of Mathematical Statistics 32 (1961) 41-48.

[20] Kac, M., "On the average of certain Wiener functional and a related limit theorem in calculus of probability," Transactions of the American Mathematical Society 59 (1946) 401-414.

[21] Kac, M., "On distributions of certain Wiener functionals," Transactions of the American Mathematical Society 65 (1949) 1-13.

[22] Kac, M., "On some connections between probability theory and differential and integral equations," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (1950), University of California Press, 1951, pp. 189-215.

[23] Kac, M., "A class of limit theorems," Transactions of the American Mathematical Society 84 (1957) 459-471.

[24] Kallianpur, G., and H. Robbins, "Ergodic property of the Brownian motion process," Proceedings of the National Academy of Sciences U.S.A. 39 (1953) 525-533.

[25] Kallianpur, G., and H. Robbins, "The sequence of sums of independent random variables," Duke Mathematical Journal 21 (1954) 285-307.

[26] Karlin, S., and J. McGregor, "Occupation time laws for birth and death processes," Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (1960), University of California Press, 1961, Vol. II, pp. 249-272.

[27] Kemperman, J. H. B., The Passage Problem for a Stationary Markov Chain. The University of Chicago Press, 1961.

[28] Kendall, D. G., "A note on Doeblin's central limit theorem," Proceedings of the American Mathematical Society 8 (1957) 1037-1039.

[29] Kesten, H., "Occupation times for Markov and semi-Markov chains," Trans. Amer. Math. Soc. 103 (1962) 82-112.

[30] Koopman, B. O., "A generalization of Poisson's distribution for Markoff chains," Proceedings of the National Academy of Sciences, U.S.A. 36 (1950) 202-207.

[31] Lamperti, J., "An occupation time theorem for a class of stochastic processes," Trans. Amer. Math. Soc. 88 (1958) 380-387.

[32] Lamperti, J., "Some limit theorems for stochastic processes," Jour. Math. and Mech. 7 (1958) 433-448.

[33] Lévy, P., "Sur certains processus stochastiques homogènes," Compositio Mathematica 7 (1940) 283-339.

[34] Pepper, E. D., "Asymptotic expression for the probability of trials connected in a chain," Annals of Mathematics 28 (1927) 318-326.

[35] Puri, P. S., "A method for studying the integral functionals of stochastic processes with applications. II. Sojourn time distributions for Markov chains," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 23 (1972) 85-96.

[36] Ray, D., "Sojourn times and the exact Hausdorff measure of the sample path for planar Brownian motion," Transactions of the American Mathematical Society 106 (1963) 436-444.

[37] Ray, D., "Sojourn times of diffusion processes," Illinois Journal of Mathematics 7 (1963) 615-630.

[38] Skorokhod, A. V., "Limit theorems for stochastic processes," Theory of Probability and its Applications 1 (1956) 261-290.

[39] Skorokhod, A. V., "On positive additive functionals of a homogeneous process with indpendent increments," (Russian) Teor. Verojatnost. i Mat. Statist. Vyp 4 (1971) 171-178.

[40] Takács, L., "On certain sojourn time problems in the theory of stochastic processes," Acta Mathematica Acad. Sci. Hungaricae 8 (1957) 169-191.

[41] Takács, L., "On limiting distributions concerning a sojourn time problem," Acta Mathematica Acad. Sci. Hungaricae 8 (1957) 279-294.

[42] Takács, L., "On sojourn time problems," (Hungarian) Magyar Tud. Akad. III. (Mat.-Fiz.) Oszt. Közl. 7 (1957) 371-395.

[43] Takács, L., "On a sojourn time problem," Theory of Probability and its Applications 3 (1958) 58-65.

[44] Takács, L., "On a sojourn time problem in the theory of stochastic processes," Transactions of the American Mathematical Society 93 (1959) 531-540.

[45] Takács, L., "Sojourn time problems," Annals of Probability 2 (1974).

[46] Taga, Y., "On the limiting distributions in Markov renewal processes with finitely many states," Annals of the Institute of Statistical Mathematics 15 (1963) 1-10.

[47] Tomkó, J., "On the estimation of the remainder term in the central limit theorem for sums of random number of summands," Theory of Probability and its Applications 16 (1971) 167-175.

[48] Turbin, A. F., "On the asymptotic behavior of the sojourn time of a semi-Markov process in a reducible subset of the states. The linear case." (Russian) Teor. Verojatnost. i Mat. Statist. Vyp. 4 (1971) 179-194.

[49] Volkonskii, V. A., and Yu. A. Rozanov, "Some limit theorems for random functions. I," Theory of Probability and its Applications 4 (1959) 178-197.

[50] Weiss, N. A., "The occupation time of a set by countably many recurrent random walks," The Annals of Mathematical Statistics 43 (1972) 293-302.

[51] Pyke, R., "Markov renewal processes: Definitions and preliminary properties," The Annals of Mathematical Statistics 32 (1961) 1231-1242.

[52] Pyke, R., "Markov renewal processes with finitely many states," The Annals of Mathematical Statistics 32 (1961) 1243-1259.

[53] Volkonskii, V. A., and Yu. A. Rozanov, "Some limit theorems for random functions. II," Theory of Probability and its Applications 6 (1961) 186-198.

[54] Waugh, W. A. O'N., "Uses of the sojourn time series for the Markovian birth process," Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. Vol. III. Probability Theory. University of California Press, 1972, pp. 501-514.

Additional References

[57] Whittle, P., "Some distribution and moment formulae for the Markov chain," Journal of the Royal Statistical Society. Ser.B 17 (1955) 235-242.

[56] Dynkin, E.B., "Some limit theorems for sums of independent random variables with infinite mathematical expectation," (Russian) Izvestya Akad. Nauk SSSR. Ser. Mat. 19 (1955) 247-266. [English translation: Selected Translations in Mathematical Statistics and Probability, IMS and AMS, 1 (1961) 171-189.]

[55] Bartlett, M.S., "The frequency goodness of fit test for probability chains," Proceedings of the Cambridge Philosophical Society 47 (1951) 86-95.