CHAPTER VIII

THE DISTRIBUTION OF THE SUPREMUM FOR STOCHASTIC PROCESSES

54. Compound Recurrent Processes. We have already defined the notion of a compound recurrent process in Section 49 (Definition 2). In this section we shall use a slightly more general definition.

Let us suppose that $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots, \tau_0 = 0)$ is a sequence of mutually independent and identically distributed positive random variables with distribution function

(1) $$P\{\tau_n - \tau_{n-1} \leq x\} = F(x)$$

and χ_n $(n = 1,2,\ldots)$ is a sequence of mutually independent and identically distributed real random variables with distribution function

(2) $$P\{\chi_n \leq x\} = H(x) .$$

Furthermore, let us assume that the two sequences $\{\tau_n\}$ and $\{\chi_n\}$ are also independent.

Let us define

(3) $$\xi(u) = \sum_{0<\tau_n\leq u} \chi_n - cu$$

for $u \geq 0$ where c is a real constant.

We say that $\{\xi(u), 0 \leq u < \infty\}$ is a (general) compound recurrent process. If $c = 0$, then this definition reduces to Definition 2 in Section 49.

Let us introduce the following notation

$$(4) \qquad \phi(s) = \int_0^\infty e^{-sx} dF(x)$$

for $\operatorname{Re}(s) \geq 0$ and

$$(5) \qquad \psi(s) = \int_{-\infty}^\infty e^{-sx} dH(x)$$

for $\operatorname{Re}(s) = 0$.

Denote by $F_n(x)$ $(n = 1,2,\ldots)$ the n-th iterated convolution of $F(x)$ with itself and by $H_n(x)$ $(n = 1,2,\ldots)$ the n-th iterated convolution of $H(x)$ with itself. Let $F_0(x) = H_0(x) = 1$ for $x \geq 0$ and $F_0(x) = H_0(x) = 0$ for $x < 0$.

Our aim is to give mathematical methods for finding the distribution function of the random variable

$$(6) \qquad \eta(t) = \sup_{0 \leq u \leq t} \xi(u)$$

for $t \geq 0$ where $\{\xi(u) , 0 \leq u < \infty\}$ is a separable compound recurrent process defined by (3) .

To solve this problem we shall deduce first some basic relations for the process $\{\xi(u) , 0 \leq u < \infty\}$.

In what follows we assume that τ_n $(n = 0,1,2,\ldots)$ and χ_n $(n = 1,2,\ldots)$

are numerical (non-random) quantities and that $\tau_0 = 0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ and $\tau_n \to \infty$ as $n \to \infty$. Let us define $\xi(u)$ for $u \geqq 0$ by (3) where c is a real constant and let us define $\eta(t)$ for $t \geqq 0$ by (6). Furthermore, let us introduce the following notation

(7) $$\delta(q,s,v,t) = e^{-qt-s[\eta(t)-\xi(t)]-v\xi(t)}$$

for $t \geqq 0$ where q,s,v are complex or real numbers.

We note that

(8) $$\xi(t) = \xi(\tau_n+0) - c(t-\tau_n)$$

for $\tau_n < t < \tau_{n+1}$ $(n = 0,1,2,\ldots)$ and

(9) $$\eta(t) = \max(\eta(\tau_n+0)\ ,\ \xi(\tau_n+0) - c(t-\tau_n))$$

for $\tau_n < t < \tau_{n+1}$ $(n = 0,1,2,\ldots)$. By (8) and (9) we can also write that

(10) $$\eta(t)-\xi(t) = [\eta(\tau_n+0) - \xi(\tau_n+0) + c(t-\tau_n)]^+$$

for $\tau_n < t < \tau_{n+1}$ $(n = 0,1,2,\ldots)$. Here $[x]^+ = \max(0, x)$.

Now we shall prove two auxiliary theorems which express certain relations between the functions $\xi(t)$, $\eta(t)$ $(0 \leqq t < \infty)$, and the sequences τ_n, $\xi(\tau_n \pm 0)$, $\eta(\tau_n \pm 0)$ $(n = 0,1,2,\ldots)$.

<u>Lemma 1.</u> <u>If</u> $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ <u>is a deterministic process as defined above, and if</u> $c \geq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$, <u>then we have</u>

(11) $$(q+cs-cv)\int_0^\infty \delta(q,s,v,t)dt = \sum_{n=0}^\infty \delta(q,s,v,\tau_n+0)[1-e^{-(q+cs-cv)(\tau_{n+1}-\tau_n)}] .$$

<u>Proof.</u> If $c \geq 0$, and $\tau_n < t < \tau_{n+1}$, then $\eta(t) = \eta(\tau_n+0)$, $\xi(t) = \xi(\tau_n+0) - c(t-\tau_n)$ and

(12) $$\delta(q,s,v,t) = \delta(q,s,v,\tau_n+0)e^{-(q+cs-cv)(t-\tau_n)} .$$

If we integrate (12) from τ_n to τ_{n+1} and add for $n = 0,1,2,\ldots,$ then we obtain (11) which was to be proved.

We note that if $c \geq 0$, then we have the following relations:

(13) $$\xi(\tau_{n+1} + 0) = \xi(\tau_n + 0) + \chi_{n+1} - c(\tau_{n+1} - \tau_n)$$

and

(14) $$\eta(\tau_{n+1} + 0) = \max(\eta(\tau_n + 0)\ ,\ \xi(\tau_{n+1} + 0))$$

for $n = 0,1,2,\ldots$. By (13) and (14) we obtain that

(15) $$\eta(\tau_{n+1}+ 0)-\xi(\tau_{n+1}+ 0) = [\eta(\tau_n+ 0)-\xi(\tau_n+ 0)-\chi_{n+1}+ c(\tau_{n+1}- \tau_n)]^+$$

for $n = 0,1,2,\ldots$ and evidently $\eta(\tau_0+ 0) = \xi(\tau_0+ 0) = 0$.

<u>Lemma 2.</u> <u>If</u> $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ <u>is a deterministic process as defined above, and if</u> $c \leq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$, <u>then we have</u>

$$(16)\qquad \begin{aligned}(q+cs-cv)\int_0^\infty \delta(q,s,v,t)dt &= \sum_{n=0}^\infty \delta(q,s,v,\tau_n+0) + \frac{cs}{q-cv}\sum_{n=0}^\infty \delta(q,v-\tfrac{q}{c},v,\tau_n+0)\\ &- \sum_{n=1}^\infty \delta(q,s,v,\tau_n-0) - \frac{cs}{q-cv}\sum_{n=1}^\infty \delta(q,v-\tfrac{q}{c},v,\tau_n-0)\ .\end{aligned}$$

<u>Proof.</u> The proof of (16) is based on the following identity: If $\alpha < \beta$, a and b are real numbers, s and $w \neq 0$ are complex or real numbers, then

$$(17)\qquad \begin{aligned}(w+bs)\int_\alpha^\beta e^{-wt-s[a+bt]^+}dt &= \{e^{-w\alpha-s[a+b\alpha]^+} - e^{-w\beta-s[a+b\beta]^+}\}\\ &+ \frac{bs}{w}\{e^{-w\alpha+w[a+b\alpha]^+/b} - e^{-w\beta+w[a+b\beta]^+/b}\}.\end{aligned}$$

If $b = 0$, then the second expression on the right-hand side of (17) is 0.

If $\tau_n < t < \tau_{n+1}$, then $\xi(t)$ and $\eta(t) - \xi(t)$ are given by (8) and (10) respectively. If we integrate (7) from τ_n to τ_{n+1}, then by (17) we obtain that

$$(18)\qquad \begin{aligned}(q+cs-cv)\int_{\tau_n}^{\tau_{n+1}} \delta(q,s,v,t)dt &= [\delta(q,s,v,\tau_n+0)-\delta(q,s,v,\tau_{n+1}-0)]\\ &+ \frac{cs}{q-cv}[\delta(q,v-\tfrac{q}{c},v,\tau_n+0)-\delta(q,v-\tfrac{q}{c},v,\tau_{n+1}-0)]\end{aligned}$$

for $n = 0,1,2,\ldots$ and $cv \neq q$. If $c = 0$, then the second expression on the right-hand side of (18) is 0. If we add (18) for $n = 0,1,2,\ldots$, then we obtain (16) which was to be proved.

We note that if $c \leq 0$, then we have the following relations:

(19) $$\xi(\tau_n + 0) = \xi(\tau_n - 0) + \chi_n$$

and

(20) $$\eta(\tau_n + 0) = \max(\eta(\tau_n - 0)\ ,\ \ \xi(\tau_n - 0) + \chi_n)$$

for $n = 1,2,\ldots$. By (19) and (20) we obtain that

(21) $$\eta(\tau_n + 0) - \xi(\tau_n + 0) = [\eta(\tau_n - 0) - \xi(\tau_n - 0) - \chi_n]^+$$

for $n = 1,2,\ldots,$ and evidently $\eta(\tau_0 + 0) = \xi(\tau_0 + 0) = 0$.

Furthermore, if $c \leq 0$ we have

(22) $$\xi(\tau_{n+1} - 0) = \xi(\tau_n - 0) + \chi_n - c(\tau_{n+1} - \tau_n)$$

and

(23) $$\eta(\tau_{n+1} - 0) = \max(\eta(\tau_n - 0)\ ,\ \xi(\tau_{n+1} - 0))$$

for $n = 1,2,\ldots$. By (22) and (23) we obtain that

(24) $$\eta(\tau_{n+1} - 0) - \xi(\tau_{n+1} - 0) = [\eta(\tau_n - 0) - \xi(\tau_n - 0) - \chi_n + c(\tau_{n+1} - \tau_n)]^+$$

for $n = 1,2,\ldots,$ and evidently $\eta(\tau_1 - 0) = \xi(\tau_1 - 0) = -c\tau_1$.

If we suppose that $\{\tau_n\}$ and $\{\chi_n\}$ are random variables and $P\{\lim_{n\to\infty} \tau_n = \infty\} = 1$, then the identities (11) and (16) hold for almost all realizations of the process $\{\xi(u)\ ,\ 0 \leq u < \infty\}$. These identities make it possible to study the time dependent behavior of the process $\{\xi(u)\ ,$

$0 \leq u < \infty\}$ if we know the behavior of the sequence $\{\xi(\tau_n \pm 0)\ ,\ n = 0,1,2,\ldots\}$.

Now let us assume that $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a separable compound recurrent process defined by (3). In this case $\eta(t)$, defined by (6) for $t \geq 0$, is a random variable and the distribution function of $\eta(t)$ is uniquely determined by the Laplace-Stieltjes transform $\mathbf{E}\{e^{-s\eta(t)}\}$ for $\mathrm{Re}(s) \geq 0$. This transform can be determined by the next two theorems.

In what follows we shall make use of the transformation $\mathbf{T}$ which we introduced in Section 3.

In Section 3 we assumed that $\Phi(s) = \mathbf{E}\{\zeta e^{-s\eta}\}$ belongs to $\mathbf{R}$ and defined $\mathbf{T}\{\Phi(s)\} = \mathbf{E}\{\zeta e^{-s\eta^{+}}\}$ for $\mathrm{Re}(s) \geq 0$. Now if $\Phi(s) \in \mathbf{R}$ and v is a given complex or real number, then $\Phi(s-v)$ does not necessarily belong to $\mathbf{R}$; however, we can define $\mathbf{T}\{\Phi(s-v)\} = \mathbf{E}\{\zeta e^{v\eta}\, e^{-s\eta^{+}}\}$ for $\mathrm{Re}(s) \geq \mathrm{Re}(v)$. The function $\mathbf{T}\{\Phi(s-v)\}$ is uniquely determined for $\mathrm{Re}(s) \geq \mathrm{Re}(v)$ by $\Phi(s)$ given for $\mathrm{Re}(s) = 0$. The function $\mathbf{T}\{\Phi(s-v)\}$ is regular in the domain $\mathrm{Re}(s) > \mathrm{Re}(v)$ and continuous for $\mathrm{Re}(s) \geq \mathrm{Re}(v)$. If $\mathrm{Re}(v) = 0$, then we can use formula (5.1) for finding $\mathbf{T}\{\Phi(s-v)\}$ for $\mathrm{Re}(s) > 0$. If $\mathrm{Re}(v) > 0$, then we have

$$(25) \qquad \mathbf{T}\{\Phi(s-v)\} = \frac{s}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Phi(z)}{(z+v)(s-v-z)}\, dz$$

for $\mathrm{Re}(s) > \mathrm{Re}(v)$.

We note that Theorem 6.1 can be formulated in the following more general form: If $\gamma(s) \in \mathbf{R}$, $|\rho|\, \|\gamma\| < 1$ and

(26) $$1-\rho\gamma(s) = \Gamma^+(s,\rho)\Gamma^-(s,\rho)$$

for $\mathrm{Re}(s) = 0$ where $\Gamma^+(s,\rho)$ and $\Gamma^-(s,\rho)$ satisfy the requirements A_1, A_2, A_3 and B_1, B_2, B_3 respectively, then

(27) $$\underset{\sim}{T}\{\log[1-\rho\gamma(s-v)]\} = \log\Gamma^+(s-v,\rho) + \log\Gamma^-(-v,\rho)$$

for $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$.

Theorem 1. *Let* $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ *be a separable compound recurrent process defined by* (3). *If* $c \geq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$, *then we have*

(28) $$(q+cs-cv)\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\eta(t)-(v-s)\xi(t)}\}dt = [1-\phi(q+cs-cv)]e^{-\underset{\sim}{T}\{\log[1-\phi(q+cs-cv)\psi(v-s)]\}}$$

where $\phi(s)$ *and* $\psi(s)$ *are defined by* (4) *and* (5) *respectively and* $\underset{\sim}{T}$ *operates on the variable* **s.**

Proof. Let us introduce the notation

(29) $$U_n(s,v,q) = \underset{\sim}{E}\{e^{-q\tau_n-s\eta(\tau_n+0)-(v-s)\xi(\tau_n+0)}\}$$

for $n = 0,1,2,\ldots$ and $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$.

Let us define $\delta(q,s,v,t)$ by (7) . Since $c \geq 0$, the identitiy (11) holds for almost all realizations of the process $\{\xi(u)\ ,\ 0 \leq u < \infty\}$. If we form the expectation of (11), then we obtain that

$$(30)\qquad (q+cs-cv)\int_0^\infty e^{-qt}\underset{\sim}{E}\{e^{-s\eta(t)-(v-s)\xi(t)}\}dt = [1-\phi(q+cs-cv)]\sum_{n=0}^{\infty} U_n(s,v,q)\,.$$

Starting from $U_0(s,v,q) \equiv 1$ and by using the relations (13) and (15) we can determine $U_n(s,v,q)$ recursively for $n = 1,2,\ldots$. If we introduce the linear transformation $\underset{\sim}{T}$ defined in Section 3 , then by (13) and (15) we obtain that

$$(31)\qquad U_{n+1}(s,v,q) = \underset{\sim}{T}\{\phi(q+cs-cv)\psi(v-s)U_n(s,v,q)\}$$

for $n = 0,1,2,\ldots$ and $Re(q) > 0$, $Re(s) \geqq Re(v) \geqq 0$, and $\underset{\sim}{T}$ operates on the variable s . Hence by Theorem 4.1 it follows that

$$(32)\qquad \sum_{n=0}^{\infty} U_n(s,v,q)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(q+cs-cv)\psi(v-s)]\}}$$

for $|\rho| \leqq 1$, $Re(q) > 0$ and $Re(s) \geqq Re(v) \geqq 0$.

If we put $\rho = 1$ in (32), then by (30) we obtain (28) which was to be proved. If $v = s$ in (28), then we obtain the Laplace transform of $\underset{\sim}{E}\{e^{-s\eta(t)}\}$, and $\underset{\sim}{E}\{e^{-s\eta(t)}\}$ can be obtained by inversion.

We can also express (32) in the following equivalent form

$$(33)\qquad \sum_{n=0}^{\infty} U_n(s,v,q)\rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n}\underset{\sim}{T}\{[\phi(q+cs-cv)\psi(v-s)]^n\}\}\,.$$

Since

$$(34)\quad \begin{aligned} \underset{\sim}{T}\{[\phi(q+cs-cv)\psi(v-s)]^n\} &= \underset{\sim}{T}\{\int_0^\infty e^{-(q+cs-cv)u}\, dF_n(u) \int_{-\infty}^\infty e^{-(v-s)x}\, dH_n(x)\} \\ &= \int_0^\infty e^{-(q-cv)u} [\int_{-\infty}^{cu+0} e^{-csu-(v-s)x}\, dH_n(x) + \int_{cu+0}^\infty e^{-vx}\, dH_n(x)]dF_n(u) \ , \end{aligned}$$

it follows that

$$(35)\quad \begin{aligned} \sum_{n=0}^\infty U_n(s,v,q)\rho^n = \exp\{ \sum_{n=1}^\infty \frac{\rho^n}{n} \int_0^\infty e^{-(q-cv)u} [\int_{-\infty}^{cu+0} e^{-csu-(v-s)x}\, dH_n(x) \, + \\ + \int_{cu+0}^\infty e^{-vx}\, dH_n(x)]dF_n(u)\} \end{aligned}$$

for $|\rho| \leq 1$, $Re(q) > 0$ and $Re(s) \geq Re(v) \geq 0$.

Theorem 2. *Let* $\{\xi(u) \ , 0 \leq u < \infty\}$ *be a separable compound recurrent process defined by* (3). *If* $c \leq 0$, $Re(q) > 0$, $Re(s) \geq Re(v) \geq 0$, *then we have*

$$(36)\quad (q+cs-cv)\int_0^\infty e^{-qt}\underset{\sim}{E}\{e^{-s\eta(t)-(v-s)\xi(t)}\}dt = Q(s,v,q) + \frac{cs}{q-cv}\, Q(v-\frac{q}{c},v,q)$$

where

$$(37)\quad Q(s,v,q) = 1-\phi(q-cv)\underset{\sim}{T}\{[1-\psi(v-s)]e^{-\underset{\sim}{T}\{\log[1-\phi(q+cs-cv)\psi(v-s)]\}}\}$$

and $\underset{\sim}{T}$ *operates on the variable* s .

Proof. Let us introduce the notation

(38) $$U_n(s,v,q) = \underset{\sim}{E}\{e^{-q\tau_n - s\eta(\tau_n+0)-(v-s)\xi(\tau_n+0)}\}$$

for $n = 0,1,2,...$ and

(39) $$V_n(s,v,q) = \underset{\sim}{E}\{e^{-q\tau_n - s\eta(\tau_n-0)-(v-s)\xi(\tau_n-0)}\}$$

for $n = 1,2,...$ where $Re(s) \geq Re(v) \geq 0$ and $Re(q) > 0$. Furthermore, let

(40) $$U(s,v,q) = \sum_{n=0}^{\infty} U_n(s,v,q) ,$$

(41) $$V(s,v,q) = \sum_{n=1}^{\infty} V_n(s,v,q) ,$$

and

(42) $$Q(s,v,q) = U(s,v,q) - V(s,v,q) .$$

Let us define $\delta(q,s,v,t)$ by (7). Since $c \leq 0$, the identity (16) holds for almost all realizations of the process $\{\xi(u), 0 \leq u < \infty\}$. If we use the above notation, and if we form the expectation of (16), then we obtain (36). It remains to find $U_n(s,v,q)$ and $V_n(s,v,q)$ for $n = 1,2,...$.

In this case by (19) and (21) we obtain that

(43) $$U_n(s,v,q) = \underset{\sim}{T}\{\psi(v-s)V_n(s,v,q)\}$$

for $n = 1,2,...$ and $Re(s) \geq Re(v) \geq 0$ and $Re(q) > 0$, and evidently $U_0(s,v,q) \equiv 1$. If we add (43) for $n = 1,2,...,$ then we get

(44) $$U(s,v,q)-1 = \underset{\sim}{T}\{\psi(v-s)V(s,v,q)\} .$$

On the other hand, by (22) and (24) we obtain that

(45) $$V_{n+1}(s,v,q) = \underset{\sim}{T}\{\phi(q+cs-cv)\psi(v-s)V_n(s,v,q)\}$$

for $n = 1,2,\ldots$ and $Re(s) \geqq Re(v) \geqq 0$ and $Re(q) > 0$, and evidently $V_1(s,v,q) = \phi(q-cv)$. Thus by Theorem 4.1 it follows that

(46) $$\sum_{n=1}^{\infty} V_n(s,v,q)\rho^n = \rho\phi(q-cv)e^{-\underset{\sim}{T}\{\log[1-\rho\phi(q+cs-cv)\psi(v-s)]\}}$$

for $|\rho| \leqq 1$ and $Re(s) \geqq Re(v) \geqq 0$ and $Re(q) > 0$. If we put $\rho = 1$ in (46), then we get $V(s,v,q)$. Finally, by (44) we have

(47) $$Q(s,v,q) = 1 - \underset{\sim}{T}\{[1-\psi(v-s)]V(s,v,q)\} .$$

This completes the proof of the theorem.

Since

(48) $$1-\psi(v-s) = e^{\log[1-\psi(v-s)]} = e^{-\sum_{n=1}^{\infty}\frac{[\psi(v-s)]^n}{n}} ,$$

we can also express $Q(s,v,q)$ as follows:

(49) $$Q(s,v,q) = 1-\phi(q-cv)\underset{\sim}{T}\Big\{\exp\Big[\sum_{n=1}^{\infty}\frac{\underset{\sim}{T}\{[\phi(q+cs-cv)\psi(v-s)]^n\}-[\psi(v-s)]^n}{n}\Big]\Big\}$$

for $Re(q) > 0$ and $Re(s) \geqq Re(v) \geqq 0$.

In both cases, if either $c \geqq 0$ or $c \leqq 0$, we can use the method of factorization to obtain

(50)
$$\int_0^\infty e^{-qt} \underset{\sim}{E}\{e^{-s\eta(t)-(v-s)\xi(t)}\}dt \ .$$

Since $\|\phi(q-cs)\| \leq \phi(\mathrm{Re}(q)) < 1$ for $\mathrm{Re}(q) > 0$ and $\|\psi(s)\| = 1$, therefore by the results of Section 6 we can write that

(51)
$$1-\phi(q-cs)\psi(s) = \Phi^+(s,q,c)\Phi^-(s,q,c)$$

for $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(q) > 0$ where $\Phi^+(s,q,c)$ is a regular functions of s in the domain $R(s) > 0$, continuous and free from zeros in $\mathrm{Re}(s) \geq 0$ and satisfies $\lim_{|s|\to\infty}[\log\Phi^+(s,q,c)]/s = 0$ $(\mathrm{Re}(s) \geq 0)$, furthermore $\Phi^-(s,q,c)$ is a regular function of s in the domain $\mathrm{Re}(s) < 0$, continuous and free from zeros in $\mathrm{Re}(s) \leq 0$ and satisfies $\lim_{|s|\to\infty}[\log\Phi^-(s,q,c)]/s = 0$ $(\mathrm{Re}(s) \leq 0)$. Such a factorization always exists.

By Theorem 6.1 and by (27) it follows from (51) that

(52)
$$\underset{\sim}{T}\{\log[1-\phi(q+cs-cv)\psi(v-s)]\} = \log\Phi^+(v,q,c) + \log\Phi^-(v-s,q,c)$$

for $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$ and $\mathrm{Re}(q) > 0$. We can use (52) both in (28) and in (37).

Finally, we shall determine the distribution function of

(53)
$$\eta(\infty) = \sup_{0\leq u<\infty} \xi(u)$$

which is a nonnegative random variable (possibly ∞). Let

(54)
$$W(x) = \underset{\sim}{P}\{\sup_{0\leq u<\infty} \xi(u) \leq x\} \ .$$

If $x < 0$, then $W(x) = 0$. In the interval $[0, \infty)$ the function $W(x)$ is nondecreasing and $\lim_{x \to \infty} W(x) = W(\infty) \leq 1$.

In the following theorem we determine

$$(55) \qquad \Omega(s) = \int_{-0}^{\infty} e^{-sx}\, dW(x)$$

for $\mathrm{Re}(s) > 0$ and thus $W(x)$ can be obtained by inversion.

<u>Theorem 3</u>. <u>If</u> $c \geq 0$, <u>then we have</u>

$$(56) \qquad \Omega(s) = \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\infty} \left[\int_{cu}^{\infty} (e^{-s(x-cu)}-1)\, dH_n(x) \right] dF_n(u) \right\}$$

<u>for</u> $\mathrm{Re}(s) > 0$, <u>and if</u> $c \leq 0$, <u>then we have</u>

$$(57) \qquad \Omega(s) = \phi(-cs) \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\infty} \left[\int_{cu}^{\infty} (e^{-s(x-cu)}-1)\, dH_n(x) \right] dF_n(u) \right\}$$

<u>for</u> $\mathrm{Re}(s) > 0$.

<u>Proof</u>. By the continuity theorem for probabilities (see (41.6)) we have

$$(58) \qquad \lim_{t \to \infty} \underset{\sim}{P}\{\eta(t) \leq x\} = W(x)$$

for every x. Hence by an Abelian theorem for Laplace transforms (Theorem 9.10 in the Appendix) we obtain that

$$(59) \qquad \Omega(s) = \lim_{q \to +0} q \int_{0}^{\infty} e^{-qt}\, \underset{\sim}{E}\{e^{-s\eta(t)}\}\, dt$$

for $\text{Re}(s) > 0$. The right-hand side of (59) can be obtained by (28) for $c \geqq 0$ and by (36) for $c \leqq 0$. Thus we can get (56) and (57). However, the following proof is somewhat simpler.

We can easily see that if $c \geqq 0$, then

(60) $$W(x) = \lim_{n \to \infty} \underset{\sim}{P}\{\eta(\tau_n + 0) \leqq x\} ,$$

and therefore

(61) $$\Omega(s) = \lim_{n \to \infty} U_n(s,s,0)$$

for $\text{Re}(s) > 0$ where $U_n(s,s,0)$ is defined by (29). Thus by the Abel theorem for power series we obtain that

(62) $$\Omega(s) = \lim_{\rho \to 1-0} (1-\rho) \sum_{n=0}^{\infty} U_n(s,s,0)\rho^n$$

for $\text{Re}(s) > 0$. If in (62) we write

(63) $$1-\rho = \exp\{-\sum_{n=1}^{\infty} \frac{\rho^n}{n}\}$$

for $|\rho| < 1$, and if we use the representation (35) with $v = 0$ and $q = 0$, then we get (56).

If $c \leqq 0$, then we can easily see that

(64) $$W(x) = \lim_{n \to \infty} \underset{\sim}{P}\{\eta(\tau_n - 0) \leqq x\}$$

and therefore

(65) $$\Omega(s) = \lim_{n \to \infty} V_n(s,s,0)$$

for $\mathrm{Re}(s) > 0$ where $V_n(s,s,0)$ is defined by (39). Thus by the Abel theorem for power series we obtain that

$$\Omega(s) = \lim_{\rho\to 1-0} (1-\rho) \sum_{n=1}^{\infty} V_n(s,s,0)\rho^n \tag{66}$$

for $\mathrm{Re}(s) > 0$. If we write $v = s$ and $q = 0$ in (46) and if we use (63), then by (66) we get (57).

<u>Theorem 4.</u> <u>Let</u> $W(x)$ <u>be defined by</u> (54). <u>The function</u> $W(x)$ <u>is a proper distribution function if and only if</u>

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} [1-H_n(cu)]dF_n(u) < \infty . \tag{67}$$

<u>Proof.</u> If $c \geqq 0$, then we can write that

$$\eta(\infty) = \sup(0, \chi_1 - c\tau_1, \chi_1+\chi_2-c\tau_2, \ldots) , \tag{68}$$

and if $c \leqq 0$, then we have

$$\eta(\infty) = -c\tau_1 + \sup(0, \chi_1 - c(\tau_2 - \tau_1), \chi_1 + \chi_2 - c(\tau_3 - \tau_1), \ldots) . \tag{69}$$

Thus by Theorem 43.12 we can conclude that

$$\underset{\sim}{P}\{\eta(\infty) < \infty\} = 1 \tag{70}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \underset{\sim}{P}\{\chi_1 + \ldots + \chi_n > c\tau_n\} < \infty . \tag{71}$$

If $c \geqq 0$, then we define $\xi_r = \chi_r - c(\tau_r - \tau_{r-1})$ for $r = 1,2,\ldots$ in

Theorem 43.12 and if $c \leqq 0$, then we define $\xi_r = \chi_r - c(\tau_{r+1} - \tau_r)$ for $r = 1,2,\ldots$ in Theorem 43.12 .

If the series (67) is convergent, then $W(x)$ is a proper distribution function and $\Omega(s)$ is given by (56) or by (57) for $Re(s) \geqq 0$. If the series (67) is divergent, then $W(x) = 0$ for every x, and $\Omega(s) = 0$ for $Re(s) \geqq 0$.

In the case where $E\{\chi_r - c(\tau_r - \tau_{r-1})\}$ exists, the function $W(x)$ is a distribution function if and only if either $E\{\chi_r - c(\tau_r - \tau_{r-1})\} < 0$ or $P\{\chi_r - c(\tau_r - \tau_{r-1}) = 0\} = 1$. If $E\{\chi_r - c(\tau_r - \tau_{r-1})\} \geqq 0$ and $P\{\chi_r - c(\tau_r - \tau_{r-1}) = 0\} < 1$, then $W(x) = 0$ for all x. This follows from Corollary 43.1 of Theorem 43.12 .

We note that by using the representations (68) and (69) we can also obtain Theorem 3 from Theorem 43.13 .

The results of this section have been obtained by the author in his paper [211] .

55. <u>Compound Poisson Processes.</u> We have already defined the notion of a compound Poisson process in Section 48 (Definition 2). In this section we shall use a slightly more general definition.

Let us suppose that $\{\nu(u), 0 \leqq u < \infty\}$ is a Poisson process of density λ. Let $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ be mutually independent and identically distributed real random variables which are independent of the process $\{\nu(t), 0 \leqq t < \infty\}$. Let

$$P\{\chi_n \leqq x\} = H(x) \tag{1}$$

and let us define

$$\xi(u) = \sum_{1 \leqq n \leqq \nu(u)} \chi_n - cu \tag{2}$$

for $u \geqq 0$ where c is a real constant.

We say that $\{\xi(u)\,,\ 0 \leqq u < \infty\}$ is a (general) compound Poisson process. If $c = 0$, then this definition reduces to Definition 2 in Section 48.

A compound Poisson process is a particular case of a compound recurrent process. If we suppose that

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqq 0\,, \\ 0 & \text{for } x < 0\,, \end{cases} \tag{3}$$

in the process $\{\xi(u)\,,\ 0 \leqq u < \infty\}$ defined in Section 54, then $\{\xi(u)\,,\ 0 \leqq u < \infty\}$ reduces to a compound Poisson process.

For a separable compound Poisson process $\{\xi(u)\,,\ 0 \leqq u < \infty\}$, the distribution function of the random variable

$$\eta(t) = \sup_{0 \leqq u \leqq t} \xi(u) \tag{4}$$

for $t \geqq 0$ can be obtained by Theorem 54.1 and by Theorem 54.2.

<u>Theorem 1.</u> <u>If</u> $\{\xi(u)\,,\ 0 \leqq u < \infty\}$ <u>is a separable compound Poisson process defined by</u> (2), <u>then we have</u>

$$\begin{aligned} & q \int_0^\infty e^{-qt}\, \mathbf{E}\{e^{-s\eta(t)-(v-s)\xi(t)}\}dt = \\ & \exp\{\int_0^\infty \frac{e^{-qu}}{u} [\int_{-\infty}^{+0} e^{-(v-s)x}\, d\mathbf{P}\{\xi(u) \leqq x\} + \int_{+0}^\infty e^{-vx}\, d\mathbf{P}\{\xi(u) \leqq x\} - 1]du\} \end{aligned} \tag{5}$$

for $\operatorname{Re}(q) > 0$ *and* $\operatorname{Re}(s) \geqq \operatorname{Re}(v) \geqq 0$. *If* $v = s$, *then* (5) *reduces to*

$$(6)\qquad q\int_0^\infty e^{-qt}\underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \exp\{\int_0^\infty \frac{e^{-qu}}{u}[\int_{-0}^\infty e^{-sx}d\underset{\sim}{P}\{\xi(u) \leqq x\} - 1]du\}$$

for $\operatorname{Re}(q) > 0$ *and* $\operatorname{Re}(s) \geqq 0$.

Proof. If $c \geqq 0$, then (5) is a particular case of (54.28) . If we put $\phi(s) = \lambda/(\lambda+s)$ in (54.28) and if $\psi(s)$ is the Laplace-Stieltjes transform of $H(x)$, then (54.28) reduces to (5) . Actually, it is more convenient to use (54.30) with (54.35). By (54.30) we have

$$(7)\qquad q\int_0^\infty e^{-qt}\underset{\sim}{E}\{e^{-s\eta(t)-(v-s)\xi(t)}\}dt = \frac{q}{\lambda+q+cs-cv}\sum_{n=0}^\infty U_n(s,v,q)$$

and by (54.35)

$$(8)\qquad \begin{aligned}\sum_{n=0}^\infty U_n(s,v,q) = \exp\{\sum_{n=1}^\infty \frac{1}{n!}\int_0^\infty \frac{e^{-\lambda u-(q-cv)u}(\lambda u)^n}{u}\ .\\ [\int_{-\infty}^{cu+0} e^{-csu-(v-s)x}\,dH_n(x) + \int_{cu+0}^\infty e^{-vx}\,dH_n(x)]du\}\end{aligned}$$

for $\operatorname{Re}(q) > 0$ and $\operatorname{Re}(s) \geqq \operatorname{Re}(v) \geqq 0$.

If we take into consideration that

$$(9)\qquad \underset{\sim}{P}\{\xi(u) \leqq x\} = \sum_{n=0}^\infty e^{-\lambda u}\frac{(\lambda u)^n}{n!}H_n(cu+x)$$

where $H_n(x)$ denotes the n-th iterated convolution of $H(x)$ with itself and $H_0(x) = 1$ for $x \geq 0$ and $H_0(x) = 0$ for $x < 0$, and that

$$(10)\qquad \exp\{-\int_0^\infty \frac{e^{-qu}}{u}(1-e^{-\lambda u+c(v-s)u})du\} = \frac{q}{\lambda+q+cs-cv}$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$, then by (7), (8), (9) and (10) we obtain (5) for $c \geq 0$.

If $c \leq 0$, then (5) is a particular case of (54.36). However, it is simpler to reduce the case of $c \leq 0$ to the case of $c \geq 0$ for which the theorem just has been proved. Since the two processes $\{\xi(t)-\xi(u)$ for $0 \leq u \leq t\}$ and $\{\xi(t-u)$ for $0 \leq u \leq t\}$ have identical finite dimensional distribution functions, we can conclude that $\eta(t)-\xi(t)$ and $-\xi(t)$ have exactly the same joint distribution as $\sup_{0\leq u\leq t}[-\xi(u)]$ and $-\xi(t)$. Furthermore, if $c \leq 0$, then for the process $\{-\xi(u)\ ,\ 0 \leq u < \infty\}$ we can apply (5). By replacing $\xi(u)$ by $-\xi(u)$ in (5) we obtain that if $c \leq 0$, then

$$(11)\qquad \begin{aligned} &q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s[\eta(t)-\xi(t)]+(v-s)\xi(t)}\}dt = \\ &\exp\{\int_0^\infty \frac{e^{-qu}}{u}[\int_{-\infty}^{+0} e^{-(v-s)x}\,d\underset{\sim}{P}\{-\xi(u)\leq x\}+\int_{+0}^{\infty} e^{-vx}\,d\underset{\sim}{P}\{-\xi(u)\leq x\}-1]du\} \end{aligned}$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$. If we replace v by $s-v$ in (11), then we obtain (5) for $c \leq 0$. This completes the proof of (5).

If $v = s$ in(5), then we get (6).

Formula (5) makes it possible to determine the joint distribution of

$\eta(t)$ and $\xi(t)$ for all $t \geq 0$. By (6) we can determine $\mathbf{P}\{\eta(t) \leq x\}$ for all $t \geq 0$ and x.

Let

(12) $$W(x) = \mathbf{P}\{\sup_{0 \leq u < \infty} \xi(u) \leq x\}$$

and

(13) $$\Omega(s) = \int_{-0}^{\infty} e^{-sx} dW(x)$$

for $\mathrm{Re}(s) \geq 0$.

<u>Theorem 2.</u> <u>If</u> $\{\xi(u), 0 \leq u < \infty\}$ <u>is a separable compound Poisson process defined by</u> (2), <u>then we have</u>

(14) $$\Omega(s) = \exp\{\int_0^{\infty} \frac{1}{u} [\int_{-0}^{\infty} e^{-sx} d\mathbf{P}\{\xi(u) \leq x\} - 1] du\}$$

<u>for</u> $\mathrm{Re}(s) > 0$.

<u>Proof.</u> In exactly the same way as in the proof of Theorem 54.3 we have

(15) $$\Omega(s) = \lim_{q \to +0} q \int_0^{\infty} e^{-qt} \mathbf{E}\{e^{-s\eta(t)}\} dt$$

for $\mathrm{Re}(s) > 0$. The right-hand side of (15) can be obtained by (6) and thus we get (14). Of course Theorem 2 is a particular case of Theorem 54.3.

<u>Theorem 3.</u> <u>The function</u> $W(x)$ <u>defined by</u> (12) <u>is a proper distribution function if and only if</u>

$$(16) \qquad \int_{\varepsilon}^{\infty} \frac{\mathbf{P}\{\xi(u) > 0\}}{u}\, du < \infty$$

<u>where</u> ε <u>is some positive number.</u>

<u>Proof</u>. By Theorem 54.4 it follows that $\lim_{x \to \infty} W(x) = W(\infty) = 1$ if and only if

$$(17) \qquad \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\infty} [1-H_n(cu)]e^{-\lambda u} (\lambda u)^{n-1} du < \infty .$$

If $c \geqq 0$, then (17) can be expressed in the form of

$$(18) \qquad \int_0^{\infty} \frac{\mathbf{P}\{\xi(u) > 0\}}{u}\, du < \infty$$

and if $c < 0$, then (17) can be expressed in the form of

$$(19) \qquad \int_{+0}^{\infty} \frac{\mathbf{P}\{\xi(u) > 0\} - e^{-\lambda u}}{u}\, du < \infty .$$

This follows from (9). The conditions (18) and (19) are equivalent to (16).

If (16) is not satisfied, then $W(x) = 0$ for every x .

In the case when $\mathbf{E}\{\chi_n\}$ exists we have $W(\infty) = 1$ if and only if $\lambda\mathbf{E}\{\chi_n\} < c$. If $\lambda\mathbf{E}\{\chi_n\} \geqq c$, then $W(x) = 0$ for every x .

We can also determine the distribution and the limiting distribution of $\eta(t)$ by using the method of factorization.

Let us define

(20) $$\psi(s) = \int_{-\infty}^{\infty} e^{-sx}\, dH(x)$$

for $\mathrm{Re}(s) = 0$.

Theorem 4. Let us assume that

(21) $$1 - \frac{cs-\lambda[1-\psi(s)]}{q} = \Phi^{+}(s,q)\Phi^{-}(s,q)$$

for $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(q) > 0$ where $\Phi^{+}(s,q)$ is a regular function of s in the domain $\mathrm{Re}(s) > 0$, continuous and free from zeros in $\mathrm{Re}(s) \geq 0$ and satisfies $\lim_{|s|\to\infty} [\log\Phi^{+}(s,q)]/s = 0$ $(\mathrm{Re}(s) \geq 0)$, furthermore $\Phi^{-}(s,q)$ is a regular function of s in the domain $\mathrm{Re}(s) < 0$, continuous and free from zeros in $\mathrm{Re}(s) \leq 0$ and satisfies $\lim_{|s|\to\infty} [\log\Phi^{-}(s,q)]/s = 0$ $(\mathrm{Re}(s) \leq 0)$. If $\{\xi(u) , 0 \leq u < \infty\}$ is a separable compound Poisson process defined by (2), then we have

(22) $$q\int_{0}^{\infty} e^{-qt}\, \underset{\sim}{E}\{e^{-s\eta(t)-(v-s)\xi(t)}\}dt = \frac{1}{\Phi^{+}(v,q)\Phi^{-}(v-s,q)}$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq \mathrm{Re}(v) \geq 0$. In particular, we have

(23) $$q\int_{0}^{\infty} e^{-qt}\, \underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \frac{1}{\Phi^{+}(s,q)\Phi^{-}(0,q)}$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq 0$.

Proof. Since the Laplace-Stieltjes transform of $F(x)$, defined by (3), is given by $\phi(s) = \lambda/(\lambda + s)$ for $\mathrm{Re}(s) > -\lambda$, we can write that in (54.51)

$$(24) \qquad 1-\phi(q-cs)\psi(s) = \frac{q}{\lambda+q-cs}\left[1-\frac{cs-\lambda[1-\psi(s)]}{q}\right]$$

for $\mathrm{Re}(s) = 0$. Since $\|\phi(q-cs)\psi(s)\| \leqq |\phi(\mathrm{Re}(q))| < 1$ for $\mathrm{Re}(q) > 0$, we can conclude by the results of Section 6 that the factorization (21) always exists, and $\Phi^{+}(s,q)$ and $\Phi^{-}(s,q)$ are determined up to a factor independent of s .

In the proof of (22) we shall distinguish two cases. If $c \geqq 0$, then by (24) and by (54.27) we obtain that

$$(25) \quad \begin{cases} \underset{\sim}{T}\{\log[1-\phi(q+cs-cv)\psi(v-s)]\} = \\ = \log \dfrac{q}{\lambda+q+cs-cv} + \log\Phi^{+}(v,q) + \log\Phi^{-}(v-s,q) \end{cases}$$

for $\mathrm{Re}(s) \geqq \mathrm{Re}(v) \geqq 0$ and $\mathrm{Re}(q) > 0$. Now if we put $\phi(s) = \lambda/(\lambda+s)$ in (54.28), then by Theorem 54.1 we get (22) for $c \geqq 0$.

If $c \leqq 0$, then (22) can be obtained by Theorem 54.2 . However, it is simpler to reduce the case $c \leqq 0$ to the case $c \geqq 0$. If we apply the result (22) to the process $\{-\xi(u)\ ,\ 0 \leqq u < \infty\}$ where $c \leqq 0$, then in **(21)** c should be replaced by $-c$ and $\psi(s)$ by $\psi(-s)$. Thus we obtain that

$$(26) \qquad 1-\frac{-cs-\lambda[1-\psi(-s)]}{q} = \Phi^{+}(-s,q)\Phi^{-}(-s,q)$$

for $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(q) > 0$ where now $\Phi^{-}(-s,q)$ is defined in the domain $\mathrm{Re}(s) \geqq 0$ and $\Phi^{+}(-s,q)$ in the domain $\mathrm{Re}(s) \leqq 0$. Thus by using the same correspondence which we used in the proof of Theorem 1 we obtain that

$$(27) \qquad q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s[\eta(t)-\xi(t)]+(v-s)\xi(t)}\}dt = \frac{1}{\Phi^-(-v,q)\Phi^+(s-v,q)}$$

for $\operatorname{Re}(q) > 0$ and $\operatorname{Re}(s) \geqq \operatorname{Re}(v) \geqq 0$ whenever $c \leqq 0$. If we replace v by $s-v$ in (27) , then we obtain (22) for $c \leqq 0$. This completes the proof of (22). If $v = s$ in (22), then we get (23).

We note that the following functions

$$(28) \qquad \Phi^+(s,q) = \exp\{-\int_0^\infty \frac{e^{-qu}}{u}[\int_{+0}^\infty e^{-sx}\,d\underset{\sim}{P}\{\xi(u) \leqq x\} - \underset{\sim}{P}\{\xi(u) > 0\}]du\}$$

for $\operatorname{Re}(s) \geqq 0$ and $\operatorname{Re}(q) > 0$ and

$$(29) \qquad \Phi^-(s,q) = \exp\{-\int_0^\infty \frac{e^{-qu}}{u}[\int_{-\infty}^{+0} e^{-sx}\,d\underset{\sim}{P}\{\xi(u) \leqq x\} - \underset{\sim}{P}\{\xi(u) \leqq 0\}]du\}$$

for $\operatorname{Re}(s) \leqq 0$ and $\operatorname{Re}(q) > 0$ satisfy the requirements in Theorem 4. In particular, we have

$$(30) \qquad \Phi^+(s,q)\Phi^-(s,q) = \exp\{-\int_0^\infty \frac{e^{-qu}}{u}[e^{csu-\lambda[1-\psi(s)]u}-1]du\} = \frac{q-cs+\lambda[1-\psi(s)]}{q}$$

for $\operatorname{Re}(s) = 0$ and $\operatorname{Re}(q) > 0$.

<u>Theorem 5.</u> <u>Let us assume that</u>

$$(31)' \qquad \int_{\varepsilon}^{\infty} \frac{\mathrm{P}\{\xi(u) > 0\}}{u} du < \infty$$

for some $\varepsilon > 0$, and that

$$(32) \qquad cs-\lambda[1-\psi(s)] = -\Phi^{+}(s)\Phi^{-}(s)$$

for $\mathrm{Re}(s) = 0$ where $\Phi^{+}(s)$ satisfies the requirements:

A_1 : $\Phi^{+}(s)$ is a regular function of s in the domain $\mathrm{Re}(s) > 0$,

A_2 : $\Phi^{+}(s)$ is continuous and free from zeros in $\mathrm{Re}(s) \geqq 0$,

A_3 : $\lim_{|s|\to\infty} [\log\Phi^{+}(s)]/s = 0$ whenever $\mathrm{Re}(s) \geqq 0$,

and $\Phi^{-}(s)$ satisfies the requirements:

B_1 : $\Phi^{-}(s)$ is a regular function of s in the domain $\mathrm{Re}(s) < 0$,

B_2 : $\Phi^{-}(s)$ is continuous in $\mathrm{Re}(s) \leqq 0$ and free from zeros in $\mathrm{Re}(s) < 0$,

B_3 : $\lim_{|s|\to\infty} [\log\Phi^{-}(s)]/s = 0$ whenever $\mathrm{Re}(s) < 0$.

If $\{\xi(u), 0 \leqq u < \infty\}$ is a separable compound Poisson process defined by (2), then the Laplace-Stieltjes transform of $W(x) = \mathrm{P}\{\sup_{0\leqq u<\infty} \xi(u) \leqq x\}$ is given by

$$(33) \qquad \Omega(s) = \frac{\Phi^{+}(0)}{\Phi^{+}(s)}$$

for $\mathrm{Re}(s) \geqq 0$.

Proof. First we shall prove that if (31) holds, then there exist two functions $\Phi^{+}(s)$ and $\Phi^{-}(s)$ which satisfy all the requirements and that

(33) is satisfied too.

Let us suppose that $\Phi^+(s,q)$ is defined by (28) and $\Phi^-(s,q)$ by (29). Then $\Phi^+(0,q) = 1$ and $\Phi^-(0,q) = 1$ for $\mathrm{Re}(q) > 0$.

If (31) holds, then by (28) we obtain that

$$(34) \qquad \Phi^+(s) = \lim_{q \to 0} \Phi^+(s,q) = \exp\{-\int_0^\infty \frac{1}{u} [\int_{+0}^\infty e^{-sx} dP\{\xi(u) \leq x\} - P\{\xi(u) > 0\}]du\}$$

exists for $\mathrm{Re}(s) \geq 0$. Since in this case $\Omega(s)$ is given by (14) for $\mathrm{Re}(s) \geq 0$ and since $\Phi^+(0) = 1$, it follows that (33) is satisfied. The function $\Phi^+(s)$ obviously satisfies the requirements A_1, A_2, A_3 .

If we take into consideration that

$$(35) \qquad q = e^{\log q} = e^{\int_0^\infty \frac{(e^{-u} - e^{-qu})}{u} du}$$

for $\mathrm{Re}(q) > 0$, then by (29) we obtain that

$$(36) \qquad \Phi^-(s) = \lim_{q \to +0} q\Phi^-(s,q) = \exp\{-\int_0^\infty \frac{1}{u} [\int_{-\infty}^{+0} e^{-sx} dP\{\xi(u) \leq x\} + P\{\xi(u) > 0\} - e^{-u}]du\}$$

exists for $\mathrm{Re}(s) \leq 0$, and $\Phi^-(s)$ satisfies the requirements B_1, B_2, B_3 . If we multiply (30) by q and let $q \to +0$, then we can see that (36) exists for $\mathrm{Re}(s) = 0$, and that (32) is satisfied for $\mathrm{Re}(s) = 0$.

In exactly the same way as in the proof of Theorem 43.15 we can prove that the requirements (32), A_1, A_2, A_3 and B_1, B_2, B_3 determine $\Phi^+(s)$ and $\Phi^-(s)$ up to a constant factor. Thus the theorem follows.

The limit distribution of $\eta(t)$ as $t \to \infty$ and the distribution of $\eta(t)$ for $t \geq 0$ for a general compound Poisson process was found in 1954 by H. Cramér [41],[42] .

Finally, we shall consider compound Poisson processes for which the distribution and the limit distribution of $\eta(t)$ can be determined explicitly.

First, let us suppose that

$$\chi(u) = \sum_{1 \leq n \leq \nu(u)} \chi_n \tag{37}$$

for $u \geq 0$ where $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ is a sequence of mutually independent and identically distributed positive random variables with distribution function $P\{\chi_n \leq x\} = H(x)$ and $\{\nu(u), 0 \leq u < \infty\}$ is a Poisson process of density λ which is independent of $\{\chi_n\}$.

We already considered the process $\{\chi(u), 0 \leq u < \infty\}$ in Section 48 (Definition 2) . By using Theorem 48.13 we can find the distribution of the supremum for the processes $\{\chi(u) - u, 0 \leq u < \infty\}$ and $\{u - \chi(u), 0 \leq u < \infty\}$.

We shall mention only briefly the following results which were found in 1962 by the author [202], [203], [205], [209] . For a more detailed account of these results see reference [210] .

Throughout the rest of this section we assume that $\{\chi(u), 0 \leq u < \infty\}$ is a separable compound Poisson process defined by (37). Then

$$(38) \qquad P\{\chi(u) \leq x\} = K(u,x) = \sum_{n=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^n}{n!} H_n(x)$$

where $H_n(x)$ is the n-th iterated convolution of $H(x)$ with itself and $H_0(x) = 1$ for $x \geq 0$ and $H_0(x) = 0$ for $x < 0$. We shall also use the notation

$$(39) \qquad \psi(s) = \int_0^{\infty} e^{-sx}\, dH(x)$$

for $\mathrm{Re}(s) \geq 0$.

In what follows we shall make frequent use of the following type of integral:

$$(40) \qquad \int_a^b g(u) d_u P\{\chi(u) \leq u+x\} = \sum_{n=0}^{\infty} \int_a^b g(u) e^{-\lambda u} \frac{(\lambda u)^n}{n!} d_u H_n(u+x)$$

where the integrals on the right-hand side exist. If the random variable $\chi(u)$ has a density function, then (40) reduces to

$$(41) \qquad \int_a^b g(u) \frac{\partial P\{\chi(u) \leq u+x\}}{\partial x} du ,$$

and if $\chi(u)$ is a discrete random variable, then (40) reduces to

$$(42) \qquad \sum_{a \leq u \leq b} g(u) P\{\chi(u) = u+x\}$$

where the sum is extended for all those $u \in [a,b]$ for which $P\{\chi(u) = u+x\} > 0$.

Let

(43) $$\alpha = \int_0^\infty x dH(x) .$$

Theorem 6. Let

(44) $$W(t,x) = P\{ \sup_{0 \leq u \leq t} [\chi(u) - u] \leq x\}$$

for $t \geq 0$. Then we have

(45) $$W(t,0) = \int_0^t (1-\frac{x}{t}) dP\{\chi(t) \leq x\}$$

for $t > 0$, and

(46) $$W(t,x) = P\{\chi(t) \leq t+x\} - \int_{+0}^t W(t-v, 0) d_v P\{\chi(v) \leq v+x\}$$

for all x and $t > 0$. If $x < 0$, then $W(t,x) = 0$.

Proof. First, by formula (48.100) it follows immediately that

(47) $$W(t,0) = E\{[1- \frac{\chi(t)}{t}]^+\}$$

for $t > 0$ and this proves (45). We shall prove that the subtrahend in (46) is the probability that $\chi(t) \leq t+x$ and $\chi(u) > u+x$ for some $u \in (0,t]$ and thus (46) follows. Let $v = \sup\{u : \chi(u) > u + x$ and $0 < u \leq t\}$. If there exists such a v, and $\chi(t) \leq t+x$, then $\chi(v) = v+x$ and $\chi(u) \leq u+x$ for $v \leq u \leq t$, or equivalently $\chi(u) - \chi(v) \leq u-v$ for $v \leq u \leq t$. The latter event has probability $W(t-v, 0)$ and thus

(46) follows by the theorem of total probability. Obviously $W(t,x) = 0$ if $x < 0$ and $t \geq 0$.

Theorem 7. Let

$$W(x) = \underset{\sim}{P}\{\sup_{0 \leq u < \infty} [\chi(u) - u] \leq x\} . \tag{48}$$

If $\lambda\alpha < 1$, then we have

$$W(0) = 1-\lambda\alpha \tag{49}$$

and

$$W(x) = 1-(1-\lambda\alpha) \int_{+0}^{\infty} d_u \underset{\sim}{P}\{\chi(u) \leq u+x\} \tag{50}$$

for all x.

If $\lambda\alpha \geq 1$, then $W(x) = 0$ for all x.

Proof. Since $\underset{\sim}{E}\{\chi(t)\} = \lambda\alpha t$ for $t \geq 0$ and $\chi(t)/t \Rightarrow \lambda\alpha$ as $t \to \infty$, it follows from (47) that

$$W(0) = \lim_{t \to \infty} W(t,0) = [1-\lambda\alpha]^{+} . \tag{51}$$

Let first $\lambda\alpha < 1$. Then by (47) we have

$$W(t,0) \geq \underset{\sim}{E}\{1- \frac{\chi(t)}{t}\} = 1 - \lambda\alpha . \tag{52}$$

Thus by (46) we obtain that

$$\int_{+0}^{t} d_u \underset{\sim}{P}\{\chi(u) \leq u+x\} \leq \frac{1}{1-\lambda\alpha} \tag{53}$$

for all $t \geq 0$. Furthermore, if $\lambda\alpha < 1$, then $\lim_{t\to\infty} P\{\chi(t) \leq t+x\} = 1$ for all x. Thus if $t \to \infty$ in (46), then we obtain (50).

If $\lambda\alpha > 1$, then $\lim_{t\to\infty} P\{\chi(t) \leq t+x\} = 0$ for all x, and the inequality

$$0 \leq P\{\sup_{0\leq u\leq t} [\chi(u) - u] \leq x\} \leq P\{\chi(t) \leq t+x\} \tag{54}$$

implies that $W(x) = 0$.

If $\lambda\alpha = 1$, then by (51) $W(0) = 0$. If $x < 0$, then obviously $W(x) = 0$. If $x > 0$, then we can find a y such that $0 < x < y$ and $P\{\chi(y) < y - x\} > 0$. Then the obvious inequality

$$P\{\chi(y) < y - x\}\, W(x) \leq W(0) = 0 \tag{55}$$

implies that $W(x) = 0$ for $x > 0$. This completes the proof of the theorem.

<u>Theorem 8</u>. <u>If</u> $\lambda\alpha < 1$, <u>then</u>

$$\Omega(s) = \int_0^\infty e^{-sx}\, dW(x) = \frac{1 - \lambda\alpha}{1-\lambda \frac{1-\psi(s)}{s}} \tag{56}$$

<u>for</u> $\mathrm{Re}(s) \geq 0$ <u>where the right-hand side of</u> (56) <u>is</u> 1 <u>if</u> $s = 0$.

<u>Proof</u>. If $0 < y$ and $0 \leq y + x$, then we can write down that

$$W(x) = \int_0^{y+x} W(y+x-z)\, d_z P\{\chi(y) \leq z\} - W(0) \int_0^y d_z P\{\chi(z) \leq z+x\} . \tag{57}$$

By forming the Laplace-Stieltjes transform of (57) we obtain that

$$\Omega(s) = e^{y[s-\lambda(1-\psi(s))]}\,\Omega(s) - W(0)\int_0^y e^{z[s-\lambda+\lambda\psi(s)]}dz \tag{58}$$

for all $y > 0$. Since

$$\int_0^y e^{z[s-\lambda+\lambda\psi(s)]}dz = \frac{e^{z[s-\lambda+\lambda\psi(s)]}-1}{s-\lambda+\lambda\psi(s)} \tag{59}$$

for $\mathrm{Re}(s) > 0$, it follows that

$$\Omega(s) = \frac{W(0)s}{s-\lambda+\lambda\psi(s)} \tag{60}$$

for $\mathrm{Re}(s) > 0$. This implies (56) for $\mathrm{Re}(s) \geq 0$ too.

We can also obtain (56) by Theorem 5. In this case $c = 1$ and in (32) we can choose $\Phi^+(s) = 1 - \lambda[1-\psi(s)]/s$ for $\mathrm{Re}(s) \geq 0$ and $\Phi^-(s) = -s$ for $\mathrm{Re}(s) \leq 0$.

Theorem 9. We have

$$\mathbf{P}\{\sup_{0\leq u\leq t}[u-\chi(u)] \leq x\} = 1 - \int_0^t \frac{x}{v}\, d_v \mathbf{P}\{\chi(v) \leq v - x\} \tag{61}$$

for $0 < x \leq t$.

Proof. We shall find the probability of the complementary event of $\{\sup_{0\leq u\leq t}[u-\chi(u)] \leq x\}$, that is, the probability that $u - \chi(u) > x$ for some $u \in (0,t]$. This latter event can occur in such a way that

$\inf\{u : u - \chi(u) > x\} = v$ where $0 \leq v \leq t$. Then $\chi(v) = v - x$ and $u - \chi(u) \leq x$ for $0 \leq u \leq v$, or equivalently, $\chi(v) - \chi(u) \leq v - u$ for $0 \leq u \leq v$. By Theorem 48.13 we have

$$(62) \qquad \underset{\sim}{P}\{\chi(v) - \chi(u) \leq v - u \text{ for } 0 \leq u \leq v \mid \chi(v) = v - x\} = \frac{x}{v}$$

for $0 < x \leq v$ where the conditional probability is defined up to an equivalence. By the theorem of total probability we get the subtrahend in (61) and this proves (61).

<u>Theorem 10.</u> <u>For</u> $x \geq 0$ <u>we have</u>

$$(63) \qquad \underset{\sim}{P}\{\sup_{0 \leq u < \infty} [u - \chi(u)] \leq x\} = 1 - e^{-\omega x}$$

<u>where</u> ω <u>is the largest real root of the equation</u>

$$(64) \qquad \lambda[1 - \psi(\omega)] = \omega .$$

<u>If</u> $\lambda\alpha \leq 1$, <u>then</u> $\omega = 0$ <u>and if</u> $\lambda\alpha > 1$, <u>then</u> $\omega > 0$.

<u>Proof.</u> By using Rouché's theorem we can prove that if $\lambda\alpha \leq 1$, then

$$(65) \qquad \lambda[1 - \psi(s)] = s$$

has a single root $s = 0$ in the domain $\mathrm{Re}(s) \geq 0$, whereas if $\lambda\alpha > 1$, then (65) has two roots $s = 0$ and $s = \omega$ in $\mathrm{Re}(s) \geq 0$ where ω is a positive real number. Thus Theorem 10 can be obtained by Theorem 5.

We note that if $\lambda\alpha > 1$, then by Lagrange's expansion we obtain that

$$\omega = \lambda[1- \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \int_0^{\infty} e^{-\lambda x} x^{n-1}\, dH_n(x)] \,. \tag{66}$$

As a further example for compound Poisson processes, let us suppose that

$$\xi(u) = \sum_{1 \leqq r \leqq \nu(u)} (\nu_r - 1) \tag{67}$$

for $u \geqq 0$ where $\nu_1, \nu_2, \ldots, \nu_r, \ldots$ is a sequence of mutually independent and identically distributed discrete random variables taking on nonnegative integers only and $\{\nu(u)\ ,\ 0 \leqq u < \infty\}$ is a Poisson process of density λ which is independent of $\{\nu_r\}$.

We shall also consider the process

$$\xi^*(u) = -\xi(u) = \sum_{1 \leqq r \leqq \nu(u)} (1-\nu_r) \tag{68}$$

for $u \geqq 0$.

Let us introduce the following notation

$$N_r = \nu_1 + \nu_2 + \ldots + \nu_r \tag{69}$$

for $r = 1,2,\ldots,$ $N_0 = 0$,

$$h(z) = \underset{\sim}{E}\{z^{\nu_r}\} \tag{70}$$

for $|z| \leqq 1$ and

$$\underset{\sim}{E}\{\nu_r\} = \gamma \tag{71}$$

(possibly $\gamma = \infty$) .

As a particular case of Lemma 20.2 we have the following result:

$$(72) \qquad \underset{\sim}{P}\{N_r < r \text{ for } r = 1,2,\ldots,n \mid N_n = k\} = \begin{cases} 1 - \frac{k}{n} & \text{if } k = 0,1,\ldots,n , \\ 0 & \text{otherwise} , \end{cases}$$

where the conditional probability is defined up to an equivalence.

By using (72) we can easily find the distribution of the supremum for the processes $\{\xi(u) , 0 \leq u < \infty\}$ and $\{\xi^*(u) , 0 \leq u < \infty\}$ defined by (67) and (68) respectively. The particular case where

$$(73) \qquad \underset{\sim}{P}\{\nu_r = 2\} = p \quad \text{and} \quad \underset{\sim}{P}\{\nu_r = 0\} = q$$

$(p > 0 , q > 0 , p + q = 1)$ has been considered by the author in reference [208] and the general case in reference [210]. In what follows we shall summarize these results.

<u>Theorem 11</u>. <u>If</u> $\{\xi(u) , 0 \leq u < \infty\}$ <u>is defined by</u> (67), <u>then we have</u>

$$(74) \qquad \underset{\sim}{P}\{\sup_{0\leq u\leq t}\xi(u) < k\} = \underset{\sim}{P}\{\xi(t) < k\} - \int_0^t \frac{\underset{\sim}{E}\{[\xi^*(t-u)]^+\}}{t-u} \underset{\sim}{P}\{\xi(u) = k\} du$$

<u>for</u> $k = 1,2,\ldots$ <u>where</u> $\xi^*(u) = -\xi(u)$ <u>for</u> $u \geq 0$.

<u>Proof</u>. The conditional probability $\underset{\sim}{P}\{ \sup_{0\leq u\leq t} \xi(u) < k \mid \nu(t) = n\}$ can be obtained by Theorem 20.1, and (74) follows by the theorem of total probabilities.

We note that in the particular case when the distribution of ν_r is given by (73), by (37.11) we have

(75) $$\underset{\sim}{P}\{\sup_{0\leq u\leq t} \xi(u) < k\} = \underset{\sim}{P}\{\xi(t) < k\} - \left(\frac{p}{q}\right)^k \underset{\sim}{P}\{\xi(t) < -k\}$$

for $k = 1,2,\ldots,$ and by (37.8) we have

(76) $$\underset{\sim}{P}\{-b < \xi(u) < a \text{ for } 0 \leq u \leq t\} =$$

$$= \sum_{j=-\infty}^{\infty} \left(\frac{p}{q}\right)^{-j(a+b)} \underset{\sim}{P}\{2j(a+b) - b < \xi(t) < 2j(a+b) + a\} -$$

$$- \sum_{j=-\infty}^{\infty} \left(\frac{p}{q}\right)^{j(a+b)+a} \underset{\sim}{P}\{-2(j+1)(a+b) + b < \xi(t) < -2j(a+b) - a\}$$

if a and b are positive integers.

<u>Theorem 12</u>. <u>Let us suppose that the process</u> $\{\xi(u), 0 \leq u < \infty\}$ <u>is defined by</u> (67) <u>and let</u>

(77) $$Q_k = \underset{\sim}{P}\{\sup_{0\leq u<\infty} \xi(u) < k\}$$

<u>for</u> $k = 1,2,\ldots$.

<u>If</u> $\gamma < 1$, <u>then</u>

(78) $$\sum_{k=1}^{\infty} Q_k z^k = \frac{(1-\gamma)z}{h(z)-z}$$

<u>for</u> $|z| < 1$. <u>If</u> $\gamma \geq 1$, <u>and</u> $\underset{\sim}{P}\{\nu_r = 1\} < 1$, <u>then</u> $Q_k = 0$ for $k = 1,2,\ldots$.

<u>Proof</u>. In this case we have

(79) $$Q_k = \underset{\sim}{P}\{\sup_{1\leq n<\infty} (N_n - n) < k\}$$

for $k = 1,2,\ldots$ and (78) can be obtained by Theorem 20.5 .

Theorem 13. *If the process* $\{\xi^*(u), 0 \leq u < \infty\}$ *is defined by* (68), *then we have*

$$(80)\qquad \mathbf{P}\{\sup_{0\leq u\leq t} \xi^*(u) < k\} = 1 - k\int_0^t \mathbf{P}\{\xi^*(u) = k\}\frac{du}{u}$$

for $k = 1,2,\ldots$.

Proof. The conditional probability $\mathbf{P}\{\sup_{0\leq u\leq t} \xi^*(u) < k \mid \nu(t) = n\}$ can be obtained by Theorem 20.2, and (80) follows by the theorem of total probabilities.

Theorem 14. *If the process* $\{\xi^*(u), 0 \leq u < \infty\}$ *is defined by* (68), *then we have*

$$(81)\qquad \mathbf{P}\{\sup_{0\leq u<\infty} \xi^*(u) < k\} = 1 - \delta^k$$

for $k = 1,2,\ldots$ *where* $z = \delta$ *is the smallest nonnegative real root of the equation*

$$(82)\qquad h(z) = z .$$

If $\gamma \leq 1$ *and* $\mathbf{P}\{\nu_r = 1\} < 1$, *then* $\delta = 1$, *and if* $\gamma > 1$ *or* $\mathbf{P}\{\nu_r = 1\} = 1$, *then* $\delta < 1$.

Proof. In this case we have

$$(83)\qquad \mathbf{P}\{\sup_{0\leq u<\infty} \xi^*(u) < k\} = \mathbf{P}\{\sup_{1\leq n<\infty} (n-N_n) < k\}$$

for $k = 1,2,\ldots$ and (81) can be obtained by Theorem 20.6 .

56. Processes with Independent Increments. In this section we assume that $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a homogeneous real stochastic process with independent increments and that $\mathbf{P}\{\xi(0) = 0\} = 1$. We have already defined such processes in Section 51 and we saw that

(1) $$\mathbf{E}\{e^{-s\xi(u)}\} = e^{u\Psi(s)}$$

exists for $\operatorname{Re}(s) = 0$ and the most general form of $\Psi(s)$ is given by

(2) $$\Psi(s) = -as + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{-0} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dM(x) + \int_{+0}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dN(x)$$

where a is a real constant, σ^2 is a nonnegative constant, $M(x)$ $(-\infty < x < 0)$ and $N(x)$ $(0 < x < \infty)$ are nondecreasing functions of x satisfying the conditions $\lim_{x \to -\infty} M(x) = 0$, $\lim_{x \to \infty} N(x) = 0$ and

(3) $$\int_{-\varepsilon}^{-0} x^2 dM(x) + \int_{+0}^{\varepsilon} x^2 dN(x) < \infty$$

for some $\varepsilon > 0$.

If we suppose that the process $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is separable, then

(4) $$\eta(t) = \sup_{0 \leq u \leq t} \xi(u)$$

is a random variable for every $t \geq 0$ and our aim is to give mathematical methods for finding the distribution of $\eta(t)$. This problem was solved in 1957 by G. Baxter and M. D. Donsker [8] . In this section we follow a different approach based on the results of Section 55.

We shall approximate the process $\{\xi(u), 0 \leq u < \infty\}$ by a sequence of compound Poisson processes $\{\xi_n(u), 0 \leq u < \infty\}$ in such a way that the finite dimensional distribution functions of the process $\{\xi_n(u), 0 \leq u < \infty\}$ converge to the finite dimensional distribution functions of the process $\{\xi(u), 0 \leq u < \infty\}$ as $n \to \infty$. If

(5) $$E\{e^{-s\xi_n(u)}\} = e^{u\Psi_n(s)}$$

for $Re(s) = 0$ where

(6) $$\Psi_n(s) = c_n - \lambda_n[1 - \psi_n(s)]$$

and c_n is a real constant, λ_n is a positive constant and $\psi_n(s)$ is the Laplace-Stieltjes transform of a real random variable, then $\{\xi_n(u), 0 \leq u < \infty\}$ converges to $\{\xi(u), 0 \leq u < \infty\}$ in distribution if and only if

(7) $$\lim_{n \to \infty} \Psi_n(s) = \Psi(s)$$

for $Re(s) = 0$. We can easily see that for any $\Psi(s)$ we can find a sequence $\{\Psi_n(s)\}$ such that (7) is satisfied.

If we suppose that $\{\xi(u), 0 \leq u < \infty\}$ and $\{\xi_n(u), 0 \leq u < \infty\}$ are separable processes and if (7) is satisfied, then by Theorem 52.3 we can conclude that

$$(8)\qquad \lim_{n\to\infty} \underset{\sim}{P}\{\sup_{0\le u\le t} \xi_n(u) \le x\} = \underset{\sim}{P}\{\sup_{0\le u\le t} \xi(u) \le x\}$$

in every continuity point of the distribution function on the right-hand side. The left-hand side in (8) can be obtained by Theorem 55.1 and thus the right-hand is also determined.

Theorem 1. If $\{\xi(u), 0 \le u < \infty\}$ is a separable, homogeneous, real stochastic process with independent increments for which $\underset{\sim}{P}\{\xi(0) = 0\} = 1$ and if $\eta(t)$ is defined by (4) for $t \ge 0$, then we have

$$(9)\qquad q\int_0^\infty e^{-qt}\underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \exp\{\int_0^\infty \frac{e^{-qu}}{u}\left[\int_{-0}^\infty e^{-sx}\, d\underset{\sim}{P}\{\xi(u) \le x\} - 1\right]du\}$$

for $Re(q) > 0$ and $Re(s) \ge 0$.

Proof. Let

$$(10)\qquad \eta_k(t) = \sup_{0\le u\le t} \xi_k(u)$$

for $t \ge 0$. By Theorem 55.1 we can conclude that (55.6) holds for the process $\{\xi_k(u), 0 \le u < \infty\}$. If $k \to \infty$, then by the continuity theorem for Laplace-Stieltjes transforms we obtain (9).

We note that (55.5) holds unchangeably for the process $\{\xi(u), 0 \le u < \infty\}$ too.

Let

$$(11)\qquad W(x) = \underset{\sim}{P}\{\sup_{0\le u<\infty} \xi(u) \le x\}$$

and

(12) $$\Omega(s) = \int_{-0}^{\infty} e^{-sx} \, dW(x)$$

for $\operatorname{Re}(s) \geqq 0$.

<u>Theorem 2.</u> <u>If</u> $\{\xi(u), 0 \leqq u < \infty\}$ <u>is a separable, homogeneous, real stochastic process with independent increments for which</u> $\underset{\sim}{P}\{\xi(0) = 0\} = 1$ <u>and if</u>

(13) $$\int_{\varepsilon}^{\infty} \frac{\underset{\sim}{P}\{\xi(u) > 0\}}{u} \, du < \infty$$

<u>for some positive</u> ε , <u>then</u> $W(x)$ <u>is a proper distribution function and</u>

(14) $$\Omega(s) = \exp\{\int_{0}^{\infty} \frac{1}{u} [\int_{-0}^{\infty} e^{-sx} \, d\underset{\sim}{P}\{\xi(u) \leqq x\} - 1] du\}$$

<u>for</u> $\operatorname{Re}(s) \geqq 0$. <u>If</u> (13) <u>is not satisfied, then</u> $W(x) = 0$ <u>for every</u> x <u>and</u> $\Omega(s) \equiv 0$ <u>for</u> $\operatorname{Re}(s) \geqq 0$.

<u>Proof</u>. By using the same method as in the proof of Theorem 1 we can prove Theorem 2 by Theorem 55.2 and Theorem 55.3 .

We note that if $\underset{\sim}{E}\{\xi(t)\}$ exists and

(15) $$\underset{\sim}{E}\{\xi(t)\} = \rho t$$

for $t \geqq 0$, then $W(\infty) = 1$ if and only if $\rho < 0$. If $\rho \geqq 0$, then $W(x) = 0$ for every x .

We can also determine the distribution and the limiting distribution of $\eta(t)$ by using the method of factorization.

Theorem 3. Let us assume that

$$1 - \frac{\Psi(s)}{q} = \Phi^{+}(s,q)\Phi^{-}(s,q) \tag{16}$$

for $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(q) > 0$ where the functions $\Phi^{+}(s,q)$ and $\Phi^{-}(s,q)$ satisfy the same requirements as in Theorem 55.4 . If $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a separable, homogeneous, real stochastic process with independent increments for which (1) holds, then we have

$$q \int_0^\infty e^{-qt} \mathbf{E}\{e^{-s\eta(t)}\}dt = \frac{1}{\Phi^{+}(s,q)\Phi^{-}(0,q)} \tag{17}$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq 0$.

Proof. If we define $\Phi^{+}(s,q)$ by (55.28) and $\Phi^{-}(s,q)$ by (55.29), then these functions satisfy all the requirements. In exactly the same way as in the proof of Theorem 43.15 we can prove that the functions $\Phi^{+}(s,q)$ and $\Phi^{-}(s,q)$ are determined by the requirements up to a factor independent of s . If we apply (55.23) for each of the processes $\{\xi_k(u)\ ,\ 0 \leq u < \infty\}$, then by the limiting procedure $k \to \infty$ we obtain (17) which was to be proved.

We note that (55.22) holds unchangeably for the process $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ too.

Theorem 4. Let us assume that $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a separable, homogeneous, real stochastic process with independent increments for which

$P\{\xi(0) = 0\} = 1$ and

(18) $$\int_{\varepsilon}^{\infty} \frac{P\{\xi(u) > 0\}}{u} \, du < \infty .$$

If

(19) $$\Psi(s) = -\Phi^{+}(s)\Phi^{-}(s)$$

for $\mathrm{Re}(s) = 0$ where $\Phi^{+}(s)$ and $\Phi^{-}(s)$ satisfy the requirements A_1, A_2, A_3 and B_1, B_2, B_3 respectively in Theorem 55.5, then

(20) $$\Omega(s) = \frac{\Phi^{+}(0)}{\Phi^{+}(s)}$$

for $\mathrm{Re}(s) \geq 0$.

Proof. The proof of this theorem follows along the same lines as the proof of Theorem 55.5 .

Examples. Let us suppose that $\{\xi(u) , 0 \leq u < \infty\}$ is a separable stable process of type $S(\alpha,\beta,c,0)$ where either $0 < \alpha < 1$, $1 < \alpha < 2$, $-1 \leq \beta \leq 1$ and $c > 0$ or $\alpha = 1$, $\beta = 0$, $c > 0$. In this case either

(21) $$\Psi(s) = -c|s|^{\alpha}\left(1 + \beta \frac{s}{|s|} \tan \frac{\alpha\pi}{2}\right)$$

where $0 < \alpha < 2$, $\alpha \neq 1$, $-1 \leq \beta \leq 1$ and $c > 0$ or

(22) $$\Psi(s) = -c|s|$$

where $c > 0$. Our aim is to find the distribution of

(23) $$\eta(t) = \sup_{0 \leqq u \leqq t} \xi(u)$$

for $t \geqq 0$. (See also Theorem 45.11.)

Now the random variable $\xi(u)$ has a stable distribution of type $S(\alpha,\beta,cu,0)$ and thus by the solution of Problem 46.8 we have

(24) $$\int_{-0}^{\infty} e^{-sx}\, d\underset{\sim}{P}\{\xi(u) \leqq x\} = \underset{\sim}{T}\{e^{u\Psi(s)}\} = \\ = 1 - \frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi} \int_0^{\infty} \frac{1 - e^{-cux^\alpha s^\alpha/\cos\frac{\gamma\pi}{2}}}{1 - 2x\sin\frac{\gamma\pi}{2\alpha} + x^2}\, dx$$

for $\mathrm{Re}(s) \geqq 0$ where

(25) $$\gamma = \frac{2}{\pi}\,\mathrm{arc\,tan}\,(\beta \tan\frac{\alpha\pi}{2})$$

and $-1 < \gamma < 1$.

Thus by Theorem 1 we obtain that

(26) $$q\int_0^{\infty} e^{-qt}\, \underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \exp\{-\frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi}\int_0^{\infty} \frac{\log[1 + \frac{cx^\alpha s^\alpha}{q\cos\frac{\gamma\pi}{2}}]}{1-2x\sin\frac{\gamma\pi}{2\alpha} + x^2}\, dx\}$$

for $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geqq 0$. Hence $\underset{\sim}{P}\{\eta(t) \leqq x\}$ can be obtained by inversion.

We observe that $\eta(t)$ has the same distribution as $t^{1/\alpha}\eta(1)$ and this makes possible some simplification in finding $\underset{\sim}{P}\{\eta(t) \leqq x\}$.

Let

(27) $$H(x) = \underset{\sim}{P}\{\eta(1) \leqq x\}$$

and

(28) $$Q(y) = \exp\{-\frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi}\int_0^\infty \frac{\log[1+\frac{cx^\alpha}{y^\alpha\cos\frac{\gamma\pi}{2}}]}{1-2x\sin\frac{\gamma\pi}{2\alpha}+x^2}dx\}$$

for $y > 0$. (We note that $Q(y) = G((\cos\frac{\gamma\pi}{2})^{1/\alpha}\, y/c^{1/\alpha})$ where $G(x)$ is defined by (45.233).)

Since $\underset{\sim}{P}\{\eta(t) \leqq x\} = H(xt^{-1/\alpha})$ for $t > 0$, if we put $q = 1$ and $s = 1/y$ in (26), then for $y > 0$ we get

(29) $$\int_0^\infty I\,(\frac{y}{x})dH(x) = Q(y)$$

where

(30) $$I(x) = \int_0^\infty e^{-t-t^{1/\alpha}/x}\,dt$$

for $x \geqq 0$.

The function $I(x)$ can be considered as a distribution function of a positive random variable. Thus by (29) we can interpret $Q(y)$ as the distribution function of the product of two independent positive random variables having distribution functions $I(x)$ and $H(x)$ respectively.

The unknown $H(x)$ can be obtained from (29) by using Mellin-Stieltjes transform. Since

(31) $$\int_0^\infty x^s d\,I\,(x) = \Gamma(1-s)\Gamma(1+\frac{s}{\alpha})$$

if $-\alpha < s < 1$, we obtain from (29) that

(32) $$\int_0^\infty x^s H(x) = \frac{1}{\Gamma(1-s)\Gamma(1+\frac{s}{\alpha})} \int_0^\infty y^s d\,Q\,(y)$$

if $-\alpha < s < 1$ and $-1 < s < \alpha$. By inversion we can determine $H(x)$ and thus

(33) $$\underset{\sim}{P}\{\eta(t) \leq x\} = H(xt^{-1/\alpha})$$

for $t > 0$ and $x \geq 0$.

In the particular case where $\alpha = 1$, $\beta = 0$ and $c = 1$ we have

(34) $$Q(y) = \exp\{-\frac{1}{\pi}\int_0^\infty \frac{\log(1+\frac{x}{y})}{1+x^2}\,dx\}$$

for $y > 0$ and (32) reduces to

(35) $$\int_0^\infty x^s dH(x) = \frac{\sin\pi s}{\pi s}\int_0^\infty y^s d\,Q\,(y)$$

for $-1 < s < 1$. Hence

(36) $$x\,\frac{dH(x)}{dx} = \frac{Q(xe^{\pi i}) - Q(xe^{-\pi i})}{2\pi i}$$

for $x > 0$ where the definition of $Q(s)$ is extended by analytical continuation to the complex plane cut along the negative real axis from the origin to infinity. By evaluating (36) we get

(37) $$\frac{dH(x)}{dx} = \frac{1}{\pi x^{1/2}(1+x^2)^{3/4}} \exp\{-\frac{1}{\pi}\int_0^x \frac{\log y}{1+y^2}\, dy\}$$

for $x > 0$. This result is due to D. A. Darling [46].

In the particular case where $1 < \alpha < 2$, $\beta = -1$, and $c > 0$, we have

(38) $$P\{\eta(t) \leq x\} = 1 - \frac{P\{\xi(t) > x\}}{P\{\xi(t) > 0\}}$$

for $x \geq 0$ and $t > 0$. This result is due to A. V. Skorokhod [185 p. 157]. If we take into consideration that in this case $P\{\xi(t) > 0\} = P\{\xi(1) > 0\}$ for all $t > 0$, then we have the obvious relation

(39) $$P\{\xi(t) > x\} = P\{\xi(1) > 0\} P\{\sup_{0 \leq u \leq t} \xi(u) > x\}$$

for $t > 0$ and $x > 0$ and this implies (38). To prove (39) we note that the event $\{\xi(t) > x\}$ can occur in such a way that $\inf\{u : \xi(u) > x\} = v$ where $0 \leq v \leq t$, and $\xi(t) - \xi(v) > 0$. The last event has probability $P\{\xi(t) - \xi(v) > 0\} = P\{\xi(1) > 0\}$ regardless of v. We note that in this case $P\{\xi(1) > 0\} = 1/\alpha$.

The problem of finding the distribution of the supremum for stable processes has also been studied by G. Baxter and M. D. Donsker [8], C. C. Heyde [87], and the author [212].

Stochastic processes with independent increments having either no negative jumps or no positive jumps. Let $\{\xi(u), 0 \leq u < \infty\}$ be a homogeneous, real stochastic process with independent increments for which the sample functions have no negative jumps and vanish at $u = 0$ with probability 1 . Then

(40) $$\underset{\sim}{E}\{e^{-s\xi(u)}\} = e^{u\Psi(s)}$$

exists for $\mathrm{Re}(s) \geq 0$ and the most general form of $\Psi(s)$ is given by

(41) $$\Psi(s) = as + \frac{1}{2}\sigma^2 s^2 + \int_{+0}^{\infty} (e^{-sx} - 1 + \frac{sx}{1+x^2}) dN(x)$$

where a is a real constant, σ^2 is a nonnegative constant, and $N(x)$ $(0 < x < \infty)$ is a nondecreasing function of x satisfying the conditions $\lim_{x \to \infty} N(x) = 0$ and

(42) $$\int_{+0}^{\varepsilon} x^2 dN(x) < \infty$$

for some (any) $\varepsilon > 0$.

If $\xi^*(u) = -\xi(u)$ for $0 \leq u < \infty$, where the process $\{\xi(u) , 0 \leq u < \infty\}$ is defined above , then $\{\xi^*(u) , 0 \leq u < \infty\}$ is a homogeneous, real stochastic process with independent increments for which the sample functions have no positive jumps and vanish at $u = 0$ with probability 1 . Conversely, every such process $\{\xi^*(u) , 0 \leq u < \infty\}$ can be represented in the way mentioned above.

In what follows we shall consider simultaneously the processes $\{\xi(u) , 0 \leq u < \infty\}$ and $\{\xi^*(u) , 0 \leq u < \infty\}$ where $\xi^*(u) = -\xi(u)$ for $0 \leq u < \infty$. We shall demonstrate that the distribution of the supremum for the processes

$\{\xi(u), 0 \leq u < \infty\}$ and $\{\xi^*(u), 0 \leq u < \infty\}$ can be determined explicitly. See V. M. Zolotarev [227] and the author [210 pp. 83-89].

In the following we shall need the following type of integral

$$(43) \qquad \int_0^t g(u)\mathbf{P}\{x < \xi(u) < x + du\} .$$

To define (43) let us subdivide the interval $[0,t]$ by partition points $0 = u_0 < u_1 < \ldots < u_n = t$. Let $\Delta u_i = u_i - u_{i-1}$ and $u_i^* \in [u_{i-1}, u_i]$ for $i = 1,2,\ldots, n$. If for any partition of the interval and for any choice of u_i^* the sums

$$(44) \qquad \sum_{i=1}^{n} g(u_i^*)\mathbf{P}\{x < \xi(u_i^*) \leq x + \Delta u_i\}$$

have a common limit as $\max_{1\leq i\leq n} \Delta u_i \to 0$, then we say that the integral (43) exists and is equal to the common limit of the sums (44).

If $\xi(u)$ is a discrete random variable, then (43) reduces to the sum

$$(45) \qquad \sum_{0\leq u\leq t} g(u)\mathbf{P}\{\xi(u) = x\}$$

where the summation is extended for all those $u \in [0,t]$ for which $\mathbf{P}\{\xi(u) = x\} > 0$.

If $\xi(u)$ has a density function, then (43) reduces to

$$(46) \qquad \int_0^t g(u) \frac{\partial \mathbf{P}\{\xi(u) \leq x\}}{\partial x} du .$$

We define

(47) $$\int_0^\infty g(u)\underset{\sim}{P}\{x < \xi(u) < x + du\} = \lim_{t\to\infty}\int_0^t g(u)\underset{\sim}{P}\{x < \xi(u) < x + du\}$$

provided that the limit exists.

To find the distribution of the supremum for the processes $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ and $\{\xi^*(u)\ ,\ 0 \leqq u < \infty\}$ we shall approximate the process $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ by a sequence of compound Poisson processes $\{\xi_n(u)\ ,\ 0 \leqq u < \infty\}$ is such a way that the finite dimensional distribution functions of the process $\{\xi_n(u)\ ,\ 0 \leqq u < \infty\}$ converge to the finite dimensional distribution functions of the process $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ as $n \to \infty$.

If we suppose that

(48) $$\xi_n(u) = c_n \sum_{1\leqq i\leqq \nu_n(u)} (\xi_{ni} - 1)$$

for $u \geqq 0$ where c_n is a positive constant, $\xi_{n1}, \xi_{n2}, \ldots, \xi_{ni}, \ldots$ is a sequence of mutually independent and identically distributed discrete random variables taking on nonnegative integers only and $\{\nu_n(u)\ ,\ 0 \leqq u < \infty\}$ is a Poisson process of density λ_n which is independent of the sequence $\{\xi_{ni}\ ;\ i = 1,2,\ldots\}$ and if we choose the parameters c_n, λ_n and $h_n(z) = \underset{\sim}{E}\{z^{\xi_{ni}}\}$ in such a way that

(49) $$\lim_{n\to\infty} \lambda_n[e^{c_n s}\ h(e^{-c_n s}) - 1] = \Psi(s)$$

for $Re(s) \geqq 0$ where $\Psi(s)$ is given by (41), then the process $\{\xi_n(u)\ ,\ 0 \leqq u < \infty\}$ satisfies the desired properties. We can easily see that for any $\Psi(s)$ given by (41) we can find suitable c_n, λ_n and $h_n(z)$ such that (49) is satisfied.

If we suppose that $\{\xi(u), 0 \leq u < \infty\}$ and $\{\xi_n(u), 0 \leq u < \infty\}$ are separable processes and if (49) is satisfied, then by Theorem 52.3 we can conclude that

$$\lim_{n\to\infty} \mathbf{P}\{\sup_{0\leq u\leq t} \xi_n(u) \leq x\} = \mathbf{P}\{\sup_{0\leq u\leq t} \xi(u) \leq x\} \tag{50}$$

in every continuity point of the distribution function on the right-hand side. If we write $\xi^*(u) = -\xi(u)$ and $\xi_n^*(u) = -\xi_n(u)$ for $0 \leq u < \infty$, then we have also

$$\lim_{n\to\infty} \mathbf{P}\{\sup_{0\leq u\leq t} \xi_n^*(u) \leq x\} = \mathbf{P}\{\sup_{0\leq u\leq t} \xi^*(u) \leq x\} \tag{51}$$

in every continuity point of the distribution function on the right-hand side.

The probabilities on the left-hand side of (50) and (51) can be obtained by Theorem 55.11 and by Theorem 55.13 respectively. Thus the probabilities on the right-hand side of (50) and (51) are also determined.

The limiting case $t = \infty$ can be obtained by Theorem 55.12 and by Theorem 55.14.

If for the process $\{\xi(u), 0 \leq u < \infty\}$ we have (40) where $\Psi(s)$ is given by (41), then

$$\mathbf{E}\{\xi(u)\} = -\rho u \tag{52}$$

for $u \geq 0$ where

$$\rho = a - \int_{+0}^{\infty} \frac{x^3}{1+x^2}\, dN(x) . \tag{53}$$

It is possible that $\rho = -\infty$; however, $\rho = \infty$ is impossible. **In what follows we shall exclude the trivial cases $P\{\xi(u) \geq 0\} = 1$ for $u \geq 0$ and $P\{\xi(u) \leq 0$ =1 for $u \geq 0$**

Theorem 5. If $\{\xi(u) , 0 \leq u < \infty\}$ is a separable, homogeneous, real stochastic process with independent increments for which (40) holds with $\psi(s)$ given by (51), then we have

$$P\{\sup_{0 \leq u \leq t} \xi(u) \leq x\} = P\{\xi(t) \leq x\} - \int_0^t \frac{E\{[\xi^*(t-u)]^+\}}{t-u} P\{x < \xi(u) < x + du\} \tag{54}$$

for $x > 0$ where $\xi^*(u) = -\xi(u)$ for $u \geq 0$.

Proof. If we apply Theorem 55.11 to the process $\{\xi_n(u)/c_n , 0 \leq u < \infty\}$ and if $k = [x/c_n]$, then we obtain (54) by letting $n \to \infty$ in (55.74) .

Theorem 6. Let

$$W(x) = P\{\sup_{0 \leq u < \infty} \xi(u) \leq x\} \tag{55}$$

where the process $\{\xi(u) , 0 \leq u < \infty\}$ is the same as in Theorem 5. If $\rho > 0$, then we have

$$W(x) = 1 - \rho \int_0^{\infty} P\{x < \xi(u) < x + du\} \tag{56}$$

for $x > 0$ and

$$\Omega(s) = \int_0^{\infty} e^{-sx}\, W(x)dx = \frac{\rho}{\Psi(s)} \tag{57}$$

for $\mathrm{Re}(s) > 0$. *If* $\rho \leq 0$ *and* $\underset{\sim}{P}\{\xi(u) = 0\} < 1$ *for* $u > 0$, *then* $W(x) = 0$ *for all* x .

Proof. Formula (56) can be deduced from (54) and formula (57) from (55.78) .

Theorem 7. *If* $\xi^*(u) = -\xi(u)$ *for* $u \geq 0$ *where the process* $\{\xi(u) , 0 \leq u < \infty\}$ *is the same as in Theorem 5, then we have*

$$\underset{\sim}{P}\{\sup_{0\leq u\leq t} \xi^*(u) \leq x\} = 1 - \int_0^t \frac{x}{u}\underset{\sim}{P}\{x < \xi^*(u) < x + du\} \tag{58}$$

for $x > 0$.

Proof. If we apply Theorem 55.13 to the process $\{\xi_n(u)/c_n , 0 \leq u < \infty\}$ and if $k = [x/c_n]$, then we obtain (58) by letting $n \to \infty$ in (55.80) .

Theorem 8. *If the process* $\{\xi^*(u) , 0 \leq u < \infty\}$ *is the same as in Theorem 7, then we have*

$$\underset{\sim}{P}\{\sup_{0\leq u<\infty} \xi^*(u) \leq x\} = 1 - \int_0^\infty \frac{x}{u}\underset{\sim}{P}\{x < \xi^*(u) < x + du\} \tag{59}$$

for $x > 0$, *or*

$$\underset{\sim}{P}\{\sup_{0\leq u<\infty} \xi^*(u) \leq x\} = 1 - e^{-\omega x} \tag{60}$$

for $x > 0$, *where* ω *is the largest nonnegative real root of the equation*

$$\Psi(s) = 0 . \tag{61}$$

If $\rho \geqq 0$ and $P\{\xi^*(u) = 0\} < 1$, for $u > 0$ then $\omega = 0$, and if $\rho < 0$ or $P\{\xi^*(u) = 0\} = 1$ for $u \geqq 0$, then $\omega > 0$.

Proof. If we let $t \to \infty$ in (58), then we get (59). Formula (60) can be obtained by Theorem 55.14.

We note that if

$$\int_{+0}^{\varepsilon} x \, dN(x) < \infty \tag{62}$$

for some $\varepsilon > 0$, then (41) can be reduced to the following form

$$\Psi(s) = \bar{a}s + \frac{1}{2}\sigma^2 s^2 + \int_{+0}^{\infty} (e^{-sx} - 1)dN(x) \tag{63}$$

where $\bar{a}$ is a real constant. If

$$\int_{\varepsilon}^{\infty} x dN(x) < \infty \tag{64}$$

for some $\varepsilon > 0$, then (41) can be reduced to the following form

$$\Psi(s) = \bar{a}s + \frac{1}{2}\sigma^2 s^2 + \int_{+0}^{\infty} (e^{-sx} - 1 + sx)dN(x) \tag{65}$$

where $\bar{a}$ is again a real constant, but it is, in general, not the same constant as in (63).

In the particular case when

$$\Psi(s) = s + \int_{+0}^{\infty} (e^{-sx} - 1)dN(x) \tag{66}$$

for $\text{Re}(s) \geqq 0$ where $N(x)$ is a nondecreasing function of x in the interval $(0, \infty)$ for which $\lim_{x \to \infty} N(x) = 0$ and

(67) $$\int_{+0}^{\varepsilon} x dN(x) < \infty$$

for some $\varepsilon > 0$, Theorems 5, 6, 7, 8 can also be deduced from Theorems 7, 8, 9, 10 in Section 55. In the particular case of (66) we can also prove Theorems 5, 6, 7, 8 directly by using Theorem 51.8 .

Examples. First let us suppose that $\{\xi(u), 0 \leq u < \infty\}$ is a general Brownian motion process which we defined in Section 50. Let

(68) $$\zeta(u) = au + \sigma\xi(u)$$

for $u \geq 0$ where a is a real constant and σ is a positive constant. Then

(69) $$\underset{\sim}{P}\{\zeta(u) \leq x\} = \Phi\left(\frac{x-au}{\sigma u^{1/2}}\right)$$

for all x where

(70) $$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

is the normal distribution function.

In this case we have $\underset{\sim}{E}\{\zeta(u)\} = au$ and $\underset{\sim}{\mathrm{Var}}\{\zeta(u)\} = \sigma^2 u$ for $u \geq 0$ and

(71) $$\underset{\sim}{E}\{e^{-s\zeta(u)}\} = e^{u\Psi(s)}$$

for $u \geq 0$ where

(72) $$\Psi(s) = -as + \frac{1}{2}\sigma^2 s^2 .$$

Now we can apply Theorems 5, 6, 7, 8 in finding the distribution of the supremum for the process $\{\zeta(u) , 0 \leq u < \infty\}$, however, we shall show that it is simpler to use formula (55.75).

We observe that if we define the process $\{\xi_n(u) , 0 \leq u < \infty\}$ by (48) where now $c_n = \sigma/\sqrt{n}$, $\lambda_n = n$ and

$$(73) \quad \underset{\sim}{P}\{\xi_{ni} = 2\} = p_n = \frac{1}{2} + \frac{a}{2\sigma\sqrt{n}} , \underset{\sim}{P}\{\xi_{ni} = 0\} = q_n = \frac{1}{2} - \frac{a}{2\sigma\sqrt{n}}$$

for $n > a^2/\sigma^2$, then the finite dimensional distribution functions of the process $\{\xi_n(u) , 0 \leq u < \infty\}$ converge to the finite dimensional distribution functions of the process $\{\zeta(u) , 0 \leq u < \infty\}$. This follows from the fact that

$$(74) \quad \lim_{n \to \infty} n(q_n e^{\frac{\sigma s}{\sqrt{n}}} + p_n e^{-\frac{\sigma s}{\sqrt{n}}} - 1) = -as + \frac{1}{2}\sigma^2 s^2$$

for all s .

For the process $\{\xi_n(u) , 0 \leq u < \infty\}$ we have by (55.75) that

$$(75) \quad \underset{\sim}{P}\{\sup_{0 \leq u \leq t} \xi_n(u) < \frac{k\sigma}{\sqrt{n}}\} = \underset{\sim}{P}\{\xi_n(t) < \frac{k\sigma}{\sqrt{n}}\} - \left(\frac{p_n}{q_n}\right)^k \underset{\sim}{P}\{\xi_n(t) < -\frac{k\sigma}{\sqrt{n}}\}$$

for $k = 1,2,\ldots$. If we put $k = [x\sqrt{n}/\sigma]$ in (75) and let $n \to \infty$, then we obtain that

$$(76) \quad \begin{aligned} \underset{\sim}{P}\{\sup_{0 \leq u \leq t} \zeta(u) \leq x\} &= \underset{\sim}{P}\{\zeta(t) \leq x\} - e^{2ax/\sigma^2}\underset{\sim}{P}\{\zeta(t) \leq -x\} \\ &= \Phi\left(\frac{x-at}{\sigma t^{1/2}}\right) - e^{2ax/\sigma^2}\Phi\left(\frac{-x-at}{\sigma t^{1/2}}\right) \end{aligned}$$

for $x > 0$.

If $a < 0$ and if we let $t \to \infty$ in (76), then we obtain that

(77) $$\underset{\sim}{P}\{\sup_{0 \leqq u < \infty} \zeta(u) \leqq x\} = 1 - e^{2ax/\sigma^2}$$

for $x > 0$.

We note that in a similar way we can determine the probability

(78) $$\underset{\sim}{P}\{-y < \zeta(u) < x \quad \text{for} \quad 0 \leqq u \leqq t\}$$

if x and y are positive real numbers by using formula (55.76).

If in (77) we put $\sigma = 1$ and $a = -y$, and if we use the representation (68), then we obtain that for a separable Brownian motion process $\{\xi(u) ,$ $0 \leqq u < \infty\}$ we have

(79) $$\underset{\sim}{P}\{\xi(u) > x + uy \quad \text{for some} \quad u \in [0, \infty)\} = e^{-2xy}$$

for $x > 0$ and $y > 0$. For another proof of (79) we refer to J. L. Doob [58] .

As a second example let us suppose that $\{\nu(u) , 0 \leqq u < \infty\}$ is a separable Poisson process of density λ . Then

(80) $$\underset{\sim}{P}\{\nu(u) = k\} = e^{-\lambda u} \frac{(\lambda u)^k}{k!}$$

for $k = 0,1,2,\ldots$ and $u \geqq 0$.

In this case the distribution functions of $\sup_{0 \leq u \leq t} [\nu(u)-u]$ and $\sup_{0 \leq u \leq t} [u-\nu(u)]$ can be determined by Theorem 55.6 and Theorem 55.9 respectively.

Here we shall mention another method for finding the probabilities

$$(81) \qquad W(t,r,k) = \underset{\sim}{P}\{\nu(u) \leq u + r \quad \text{for} \quad 0 \leq u \leq t \mid \nu(t) = k\}$$

for $0 \leq k \leq t + r$ and $t > 0$, and

$$(82) \qquad W(t,r) = \underset{\sim}{P}\{\nu(u) \leq u + r \quad \text{for} \quad 0 \leq u \leq t\}$$

for $r = 0,1,2,\ldots$ and $t > 0$.

By Theorem 48.6 we have

$$(83) \qquad W(t,0,k) = 1 - \frac{k}{t}$$

for $0 \leq k \leq t$, and by (48.57) we have

$$(84) \qquad W(t,0) = \underset{\sim}{P}\{\nu(t) \leq t\} - \lambda \underset{\sim}{P}\{\nu(t) \leq t - 1\}$$

for $t > 0$.

Starting from (83) we can obtain $W(t,r,k)$ for $r = 1,2,\ldots$ by the following recurrence formula

$$(85) \qquad W(t,r+1,k) = \frac{(t+1)^k}{t^k} W(t+1,r,k) - \sum_{j=1}^{r} \binom{k}{j} \frac{1}{t^j} W(t,r+1-j,k-j) .$$

This follows from the equation

$$(86) \qquad W(t+1,r,k) = \sum_{j=0}^{r} \binom{k}{j} \frac{t^{k-j}}{(t+1)^k} W(t,r+1-j,k-j) .$$

In proving (86) we take into consideration that the event $\{\nu(u) \leq u+r$ for $0 \leq u \leq t+1\}$ can occur in such a way that $\nu(1) = j$ where $j = 0,1,2,\ldots$. Since

$$\mathbf{P}\{\nu(1) = j \mid \nu(t+1) = k\} = \binom{k}{j} \frac{t^{k-j}}{(t+1)^k} \tag{87}$$

for $j = 0,1,\ldots, k$, the equation (86) follows easily.

By (85) we obtain that

$$W(t,1,k) = \frac{(t+1-k)(t+1)^{k-1}}{t^k} \tag{88}$$

for $0 \leq k \leq t+1$, and

$$W(t,2,k) = \frac{(t+2-k)[(t+2)^{k-1} - (t+1)^{k-2}]}{t^k} \tag{89}$$

for $0 \leq k \leq t+2$. (See Problem 58. 2.)

In a similar way, starting from (84), we can obtain $W(t,r)$ for $r = 1,2,\ldots$ by the following recurrence formula

$$W(t,r+1) = e^{\lambda} W(t+1,r) - \sum_{j=1}^{r} \frac{\lambda^j}{j!} W(t,r+1-j) . \tag{90}$$

This follows from the equation

$$W(t+1,r) = \sum_{j=0}^{r} e^{-\lambda} \frac{\lambda^j}{j!} W(t,r+1-j) , \tag{91}$$

which can be proved by taking into consideration that the event $\{\nu(u) \leq u+r$ for $0 \leq u \leq t+1\}$ can occur in such a way that $\nu(1) = j$ where $j = 0,1,2,\ldots$.

By (90) we obtain that

(92) $$W(t,1) = e^{\lambda}\,[P\{\nu(t+1) \leq t+1\} - \lambda P\{\nu(t+1) \leq t\}]$$

for $t > 0$.

Finally, we mention an interesting result for Poisson process-es which is a particular case of a more general result found in 1959 by S. Karlin and J. McGregor [101] .

Theorem 9. *Let* $\{\nu_r(u)\ ,\ 0 \leq u < \infty\}$ $(r = 1,2,\ldots, m)$ *be independent separable Poisson processes of density* λ . *Let* $0 \leq a_1 < a_2 <\ldots < a_m$ *and* $c_1 < c_2<\ldots < c_m$ *be integers. Then we have*

(93) $$P\{\nu_1(u)+c_1 < \nu_2(u)+c_2<\ldots<\nu_m(u)+c_m \text{ for } 0 \leq u \leq t\,|\,\nu_r(t) = a_r \text{ for } r = 1,2,\ldots, m\} = \operatorname{Det}\left|\frac{a_j!}{(a_j+ c_j- c_i)!}\right|_{i,j=1,2,\ldots,m}$$

where the right-hand side of (93) *is an* $m \times m$ *determinant. In* (93) $1/x! = 0$ *if* $x = -1, -2,\ldots$.

Proof. Let $\xi_r(u) = \nu_r(u) + c_r$ for $0 \leq u < \infty$ and let $b_r = a_r+ c_r$. We shall prove that if $0 \leq a_1 < a_2 <\ldots < a_m$ and $b_1 < b_2 <\ldots < b_m$, then

(94) $$P\{\xi_1(u) < \xi_2(u)<\ldots<\xi_m(u) \text{ for } 0 \leq u \leq t \text{ and } \xi_r(t) = b_r \text{ for } r = 1,2,\ldots,m\} = \operatorname{Det}\left|P\{\xi_r(t) = b_s\}\right|_{r,s=1,2,\ldots,m} .$$

Let

(95) $$D_m = \operatorname{Det}\left|\underset{\sim}{P}\{\xi_r(t) = b_s\}\right|_{r,s=1,2,\ldots,m} .$$

Since

(96) $$D_m = \operatorname{Det}\left|\frac{e^{-\lambda t}(\lambda t)^{a_s+c_s-c_r}}{(a_s+c_s-c_r)!}\right| = \frac{e^{-m\lambda t}(\lambda t)^{a_1}\ldots(\lambda t)^{a_m}}{a_1!\ \ldots\ a_m!}\operatorname{Det}\left|\frac{a_s!}{(a_s+c_s-c_r)!}\right|$$

where $1/x! = 0$ for $x = -1, -2,\ldots,$ therefore (94) implies (93).

Now we shall prove (94). Denote by C the event that at least two paths $\{\xi_r(u)\ ,\ 0 \leq u \leq t\}$ $(r = 1,2,\ldots,m)$ coincide for some u $(0 \leq u \leq t)$. Let us write that

(97) $$D_m = \sum_{(i_1,\ldots,i_m)} (-1)^I \underset{\sim}{P}\{\xi_1(t) = b_{i_1},\ldots,\ \xi_m(t) = b_{i_m}\}$$

where the summation is extended over all permutations $(i_1, i_2,\ldots, i_m)$ of $(1,2,\ldots, m)$ and I is the number of inversions in the permutation $(i_1, i_2,\ldots, i_m)$. Let us express each term in (97) in the following way:

(98) $$\underset{\sim}{P}\{\xi_1(t) = b_{i_1},\ldots,\ \xi_m(t) = b_{i_m}\} =$$
$$= \underset{\sim}{P}\{\xi_1(t) = b_{i_1},\ldots,\ \xi_m(t) = b_{i_m},\ C\} + \underset{\sim}{P}\{\xi_1(t) = b_{i_1},\ldots,\ \xi_m(t) = b_{i_m},\ \overline{C}\} .$$

We have

$$\sum_{(i_1,\ldots,i_m)} (-1)^I \underset{\sim}{P}\{\xi_1(t) = b_{i_1},\ldots,\ \xi_m(t) = b_{i_m},\ C\} = \tag{99}$$

$$= -\sum_{(i_1,\ldots,i_m)} (-1)^I \underset{\sim}{P}\{\xi_1(t) = b_{i_1},\ldots,\ \xi_m(t) = b_{i_m},\ C\} .$$

For if C occurs, then at least two paths $\{\xi_r(u)\ ,\ 0 \leqq u \leqq t\}$ $(r = 1,2,\ldots,m)$ coincide for some u $(0 \leqq u \leqq t)$. Let us suppose that the r-th and the s-th paths coincide for the first time in the interval $[0, t]$. After this coincidence let us interchange the remaining parts of the two paths. By this interchange, on the one hand, the sum (99) remains unchangead, and on the other hand it is multiplied by -1. For

$$\underset{\sim}{P}\{\ldots,\ \xi_r(t) = b_{i_r},\ldots,\ \xi_s(t) = b_{i_s},\ldots,\ C\} = \tag{100}$$

$$= \underset{\sim}{P}\{\ldots,\ \xi_r(t) = b_{i_s},\ldots,\ \xi_s(t) = b_{i_r},\ldots,\ C\}$$

and the number of inversions in the two permutations $(\ldots,i_r,\ldots,i_s,\ldots)$ and $(\ldots,i_s,\ldots,i_r,\ldots)$ differ by an odd number. This implies that the sum (99) is necessarily 0.

Thus by (97)

$$D_m = \sum_{(i_1,\ldots,i_m)} (-1)^I \underset{\sim}{P}\{\xi_1(t) = b_{i_1},\ldots,\ \xi_m(t) = b_{i_m},\ \overline{C}\}. \tag{101}$$

Obviously, every term in (101) is 0 except one which corresponds to $i_1 = 1$, $i_2 = 2,\ldots,\ i_m = m$. Thus we have

$$D_m = \underset{\sim}{P}\{\xi_1(t) = b_1,\ldots,\xi_m(t) = b_m,\ \overline{C}\} \tag{102}$$

which completes the proof of the theorem.

57. <u>First Passage Time Problems</u>. Let $\{\xi(u), 0 \leq u < \infty\}$ be a separable real stochastic process for which $P\{\xi(0) = 0\} = 1$. For $x \geq 0$ let us define

$$\theta(x) = \inf\{u : \xi(u) \geq x \text{ and } 0 \leq u < \infty\} \tag{1}$$

and $\theta(x) = \infty$ if $\xi(u) < x$ for all $u \in [0, \infty)$. We can interpret $\theta(x)$ as the first passage time of the process $\{\xi(u), 0 \leq u < \infty\}$ through x. For every $x \geq 0$ the first passage time $\theta(x)$ is a nonnegative random variable (possibly ∞).

It is of some importance to determine the distribution of $\theta(x)$ for a wide class of stochastic processes $\{\xi(u), 0 \leq u < \infty\}$. Since obviously

$$P\{\theta(x) \leq t\} = P\{\sup_{0 \leq u \leq t} \xi(u) \geq x\} \tag{2}$$

for all $x \geq 0$ and $t \geq 0$, the problem of finding the distribution of $\theta(x)$ can be reduced to the problem of finding the distribution of $\sup_{0 \leq u \leq t} \xi(u)$.

In the previous section we determined the distribution of $\sup_{0 \leq u \leq t} \xi(u)$ for a separable, homogeneous, real process $\{\xi(u), 0 \leq u < \infty\}$ with independent increments and thus by (2) we can also find the distribution of $\theta(x)$ for $x \geq 0$.

In what follows we shall mention a few processes $\{\xi(u), 0 \leq u < \infty\}$ for which simple explicit results can be obtained.

The first result concerning first passage time problems was obtained in 1708 by <u>A. De Moivre</u> [52]. A. De Moivre's result can be formulated in the following way: Let us suppose that a particle performs a random walk on the x-axis. It starts at $x = 0$ and at times $n = 1,2,3,\ldots$ it moves a unit distance to the right with probability p or a unit distance to the left with probability q where $p > 0$, $q > 0$ and $p+q = 1$. Let us suppose that the successive displacements are mutually independent random variables. Denote by $\xi(n)$ $(n = 0,1,2,\ldots)$ the position of the particle immediately after time n, and denote by $\theta(k)$ the first passage time through $x = k$ $(k = 1,2,\ldots)$. By the result of A. De Moivre [52] we have

$$\mathbf{P}\{\theta(k) = n\} = \frac{k}{n}\mathbf{P}\{\xi(n) = k\} \tag{3}$$

for $n = k, k+2,\ldots$ and $k = 1,2,\ldots$.

By a result of É. Barbier [4] which was found in 1887 we can conclude that the result (3) is valid under more general circumstances. If we suppose that in the random walk process mentioned above the successive displacements $\xi_1, \xi_2,\ldots, \xi_n,\ldots$ are mutually independent and identically distributed random variables for which $\mathbf{P}\{\xi_n = 1\} = p$ and $\mathbf{P}\{\xi_n = -\mu\} = q$ where μ is a positive integer, $p > 0$, $q > 0$ and $p + q = 1$, then (3) holds unchangeably for $n = k, k+2\mu,\ldots$ and $k = 1,2,\ldots$.

In 1960 the author [199], [200] proved a generalization of the classical ballot theorem and this implies that (3) is valid if we suppose that $\xi_1, \xi_2,\ldots, \xi_n,\ldots$ are mutually independent and identically distributed discrete random variables taking on integers ≤ 1 only.

This last results implies immediately the following more general result: Let us suppose that

$$\xi(u) = \sum_{1 \leq i \leq \nu(u)} \xi_i \tag{4}$$

for $u \geq 0$ where $\{\nu(u), 0 \leq u < \infty\}$ is a Poisson process of density λ and $\xi_1, \xi_2, \ldots, \xi_i, \ldots$ are mutually independent and identically distributed discrete random variables taking on integers ≤ 1 only and that $\{\xi_i\}$ and $\{\nu(u), 0 \leq u < \infty\}$ are independent. If $\theta(k)$ denotes the first passage time through k where $k = 1, 2, \ldots$, then we have

$$\mathrm{P}\{\theta(k) \leq t\} = \int_0^t \frac{k}{u} \mathrm{P}\{\xi(u) = k\} du \tag{5}$$

for $t \geq 0$. By (5) we have also

$$\frac{\partial \mathrm{P}\{\theta(k) \leq t\}}{\partial t} = \frac{k}{t} \mathrm{P}\{\xi(t) = k\} \tag{6}$$

for $t > 0$ and $k = 1, 2, \ldots$.

Now let us suppose that $\{\chi(u), 0 \leq u < \infty\}$ is a separable compound Poisson process whose sample functions are nondecreasing functions of u with probability 1. Let $\xi(u) = u - \chi(u)$ for $u \geq 0$ and denote by $\theta(x)$ the first passage time of the process $\{\xi(u), 0 \leq u < \infty\}$ through x where $x \geq 0$. By a result of the author [209] obtained in 1961 we have

$$\mathrm{P}\{\theta(x) \leq t\} = \int_x^t \frac{x}{u} d_u \mathrm{P}\{\chi(u) \leq u - x\} \tag{7}$$

for $0 < x \leq t$. If we suppose only that $\{\chi(u) , 0 \leq u < \infty\}$ is a separable, homogeneous process with independent increments almost all of whose sample functions are nondecreasing step functions vanishing at $u = 0$, then (7) remains valid unchangeably. This was pointed out by the author [209] in 1962. See also [203], [204], [205] . This last result was stated in 1957 by D. G. Kendall [106], however, he did not prove it.

It is interesting to mention that if $\{\xi(u) , 0 \leq u < \infty\}$ is a separable Brownian motion process for which

$$\mathbf{P}\{\xi(t) \leq x\} = \Phi\left(\frac{x}{\sqrt{t}}\right) \tag{8}$$

for $t > 0$ where $\Phi(x)$ is the normal distribution function and if $\theta(x)$ denotes the first passage time through x where $x > 0$, then we have

$$\frac{\partial \mathbf{P}\{\theta(x) \leq t\}}{\partial t} = \frac{x}{t} \frac{\partial \mathbf{P}\{\xi(t) \leq x\}}{\partial x} = \frac{x}{\sqrt{2\pi t^3}} e^{-x^2/2t} \tag{9}$$

for $x > 0$ and $t > 0$. This result can be deduced from a result found in 1900 by L. Bachelier [3].

From (5) we can deduce a more general result. Let $\{\xi(u) , 0 \leq u < \infty\}$ be a separable, homogeneous stochastic process with independent increments for which the sample functions have no negative jumps and vanish at $u = 0$ with probability 1 . The trivial cases $\mathbf{P}\{\xi(u) \geq 0\} = 1$ for $u \geq 0$ and $\mathbf{P}\{\xi(u) \leq 0\} = 1$ for $u \geq 0$ will be excluded. Let $\xi^*(u) = -\xi(u)$ for $u \geq 0$ and denote by $\theta^*(x)$ the first passage time of the process $\{\xi^*(u) , 0 \leq u < \infty\}$ through x where $x > 0$. Then we have

(10) $$\underset{\sim}{P}\{\theta^*(x) \leq t\} = \int_0^t \frac{x}{u} \underset{\sim}{P}\{x < \xi^*(u) < x + du\}$$

for $x > 0$ and $t > 0$. This result was proved in 1964 by V. M. Zolotarev [228]. His proof is based on a result of A. V. Skorokhod [185 pp. 129–134]. See also the author [210 pp. 83-89].

In this case $\{\theta^*(x), 0 \leq x < \infty\}$ is also a homogeneous stochastic process with independent increments. The sample functions of the process $\{\theta^*(x), 0 \leq x < \infty\}$ are nondecreasing functions of x and vanishing at $x = 0$ with probability 1.

Let

(11) $$\underset{\sim}{E}\{e^{-s\xi(u)}\} = e^{u\Psi(s)}$$

for $Re(s) \geq 0$ and $u \geq 0$ where $\Psi(s)$ is given by (56.41). Then we can write that

(12) $$\underset{\sim}{E}\{e^{-s\theta^*(x)}\} = e^{-x\omega^*(s)}$$

for $Re(s) \geq 0$ and $x \geq 0$ where

(13) $$\omega^*(s) = a^*s - \int_{+0}^{\infty} (e^{-sx} - 1)dN^*(x)$$

for $Re(s) \geq 0$ and a^* is a nonnegative real number, and $N^*(x)$ is a nondecreasing function of x in the interval $(0, \infty)$ for which $\lim_{x \to \infty} N^*(x) = 0$ and

(14) $$\int_{+0}^{\varepsilon} x dN^*(x) < \infty$$

for some $\varepsilon > 0$.

We can prove that if $Re(s) \geq 0$, then $z = \omega^*(s)$ is a root of the equation

$$(15) \qquad \Psi(z) = s$$

in the domain $Re(z) \geq 0$. Actually, we can prove that if $Re(s) > 0$, then $z = \omega^*(s)$ is the only root of the equation in the domain $Re(z) > 0$. See A. V. Skorokhod [185 pp. 129-134] .

In 1961 V. M. Zolotarev [228] demonstrated that for some $\Psi(s)$ defined by (56.41) for $Re(s) \geq 0$ there exists a $\omega^*(s)$ defined for $Re(s) \geq 0$ which can be represented in the form (13) and which satisfies $\Psi(\omega^*(s)) = s$ for $Re(s) \geq 0$. He also showed that for the corresponding processes $\{\xi^*(u) , 0 \leq u < \infty\}$ and $\{\theta^*(x) , 0 \leq x < \infty\}$ (10) holds; however, he did not demonstrate that $\{\theta^*(x) , 0 \leq x < \infty\}$ is the first passage time process of $\{\xi^*(u) , 0 \leq u < \infty\}$.

Example. In accordance with the notation of Section 42 let us denote by $f(x ; \alpha,\beta,c,0)$ the density function of a stable distribution function of type $S(\alpha,\beta,c,0)$ where $c > 0$.

Let us suppose that $\{\xi(u) , 0 \leq u < \infty\}$ is a stable process of type $S(\alpha,1,c,0)$ where $1 < \alpha \leq 2$ and $c > 0$. Then

$$(16) \qquad \frac{\partial \mathbf{P}\{\xi(t) \leq x\}}{\partial x} = \frac{1}{t^{1/\alpha}} f\left(\frac{x}{t^{1/\alpha}} ; \alpha,1,c,0\right)$$

for $t > 0$ and all x .

If $\{\theta^*(x), 0 \leqq x < \infty\}$ is the first passage time process which corresponds to $\xi^*(u) = -\xi(u)$ for $0 \leqq u < \infty$, then by (10) we have

(17) $$\frac{\partial \underset{\approx}{P}\{\theta^*(x) \leqq t\}}{\partial t} = \frac{x}{t^{1+1/\alpha}} f\left(\frac{x}{t^{1/\alpha}}; \alpha, -1, c, 0\right)$$

for $t > 0$ and $x > 0$.

On the other hand by (42.173) we have

(18) $$\Psi(s) = -\frac{cs^{\alpha}}{\cos(\alpha\pi/2)}$$

for $\mathrm{Re}(s) \geqq 0$ in (11). Thus by (15) we have

(19) $$\omega(s) = \left(-\frac{c}{\cos(\alpha\pi/2)}\right)^{1/\alpha} s^{1/\alpha}$$

for $\mathrm{Re}(s) \geqq 0$ in (12). Hence by (42.171) we can conclude that $\{\theta^*(x), 0 \leqq x < \infty\}$ is a stable process of type

(20) $$S\left(\frac{1}{\alpha}, 1, \left(\frac{-\cos(\alpha\pi/2)}{c}\right)^{1/\alpha} \cos\frac{\pi}{2\alpha}, 0\right).$$

Accordingly we have

(21) $$\frac{\partial \underset{\approx}{P}\{\theta^*(x) \leqq t\}}{\partial t} = \frac{1}{x^{\alpha}} f\left(\frac{t}{x^{\alpha}}; \frac{1}{\alpha}, 1, \left(\frac{-\cos(\alpha\pi/2)}{c}\right)^{1/\alpha} \cos\frac{\pi}{2\alpha}, 0\right)$$

for $t > 0$ and $x > 0$.

If we compare (17) and (21), then we obtain that the identity

(22) $$f(\frac{x}{t^{1/\alpha}};\alpha,-1,c,0) = \frac{t^{1+\frac{1}{\alpha}}}{x^{\alpha+1}} f(\frac{t}{x^{\alpha}}, \frac{1}{\alpha}, 1; (\frac{-\cos(\alpha\pi/2)}{c})^{1/\alpha} \cos\frac{\pi}{2\alpha}, 0)$$

holds for $t > 0$ and $x > 0$.

If put $t = 1$ and $c = -\cos(\alpha\pi/2)$ in (22), then we get

(23) $$f(x;\alpha,-1, -\cos\frac{\alpha\pi}{2}, 0) = \frac{1}{x^{\alpha+1}} f(\frac{1}{x^{\alpha}}; \frac{1}{\alpha}, 1, \cos\frac{\pi}{2\alpha}, 0)$$

for $x > 0$ and $1 < \alpha \leq 2$.

Conversely, (23) implies (22). This follows from the relation

(24) $$f(x;\alpha,\beta,cu,0) = \frac{1}{u^{1/\alpha}} f(\frac{x}{u^{1/\alpha}};\alpha,\beta,c,0)$$

which holds for all $c > 0$ and $u > 0$ whenever $\alpha \neq 1$.

The identity (23) is indeed true. If $\alpha = 2$, then this follows from (42.108) and (42.116). If $1 < \alpha < 2$ and if we use the notation (42.128), then we can express (23) in the following equivalent form

(25) $$h(x;\alpha,2-\alpha) = \frac{1}{x^{\alpha+1}} h(\frac{1}{x^{\alpha}}; \frac{1}{\alpha}, \frac{1}{\alpha})$$

for $x > 0$ and $1 < \alpha < 2$, and this is true by Theorem 42.7.

By using Theorem 56.5 we can determine explicitly the distribution of $\theta(x)$ for a separable, homogeneous stochastic process $\{\xi(u), 0 \leq u < \infty\}$ with independent increments for which the sample functions have no negative jumps and vanish at $u = 0$ with probability 1.

The following problem is connected with the first passage time problem. Let $\{\xi(u), 0 \leq u < \infty\}$ be a separable, homogeneous stochastic process with independent increments for which $\mathbf{P}\{\xi(0) = 0\} = 1$. Then by Theorem 51.4 the limits $\xi(u+0)$ and $\xi(u-0)$ exist for all $u \in (0, \infty)$ with probability 1.

For $a \geq 0$ let us define

(26) $$\theta(a) = \inf\{u : \xi(u) \geq a \text{ and } 0 \leq u < \infty\}$$

and $\theta(a) = \infty$ if $\xi(u) < a$ for all $u \in [0, \infty)$.

If $\theta(a)$ is finite, then let us define

(27) $$\gamma'(a) = \xi(\theta(a) + 0) - a,$$

(28) $$\gamma''(a) = a - \xi(\theta(a) - 0),$$

and

(29) $$\gamma(a) = \gamma'(a) + \gamma''(a)$$

for $a \geq 0$. The problem arises to find the distributions of the random variables $\gamma'(a)$, $\gamma''(a)$ and $\gamma(a)$ if they exist.

In 1955 E. B. Dynkin [61] determined the distributions of $\gamma'(a)$, $\gamma''(a)$ and $\gamma(a)$ in the case where $\{\xi(u), 0 \leq u < \infty\}$ is a separable stable process of type $S(\alpha,1,c,0)$ where $0 < \alpha < 1$ and $c > 0$. Then we have

(30) $$\mathbf{E}\{e^{-s\xi(u)}\} = e^{-ucs^{\alpha}}$$

for $\mathrm{Re}(s) \geq 0$ and $u \geq 0$.

If $a > 0$, then $\underset{\sim}{P}\{\theta(a) < \infty\} = 1$ and we have

$$(31) \qquad \underset{\sim}{P}\{ \frac{\gamma'(a)}{a} \leq x\} = H_\alpha(x)$$

and

$$(32) \qquad \underset{\sim}{P}\{ \frac{\gamma'(a)}{a} > x , \frac{\gamma''(a)}{a} > y\} = 1 - H_\alpha(\frac{x+y}{1-y})$$

for $x > 0$ and $0 < y < 1$ where

$$(33) \qquad H_\alpha(x) = \begin{cases} 1 & \text{for } x \geq 1 , \\ \frac{\sin\alpha\pi}{\pi} \int_0^x \frac{du}{u^\alpha(1+u)} & \text{for } 0 < x < 1 , \\ 0 & \text{for } x < 0 , \end{cases}$$

and furthermore,

$$(34) \qquad \underset{\sim}{P}\{ \frac{\gamma(a)}{a} \leq x\} = K_\alpha(x)$$

for $x \geq 0$ where

$$(35) \qquad K_\alpha(x) = \frac{\sin\alpha\pi}{\pi} \int_0^x \frac{q(u)}{u^{\alpha+1}} du$$

and

$$(36) \qquad q(u) = \begin{cases} 1 - (1-u)^\alpha & \text{for } 0 \leq u \leq 1 , \\ 1 & \text{for } u > 1 . \end{cases}$$

The above results can easily be deduced from (49.176), (49.183) and

(49.184). It is sufficient to prove (32) because (31) and (34) follow from (32). In (49.183) let us define the random variables χ_t and η_t for the sequence $\theta_k = \xi(k) - \xi(k-1)$ $(k = 1,2,\ldots)$ and for $t > 0$. Then we have

$$(37) \qquad \underset{\sim}{P}\{\frac{\gamma'(a)}{a} > x, \frac{\gamma''(a)}{a} > y\} = \lim_{h\to 0} \underset{\sim}{P}\{\chi_{a/h} > \frac{ax}{h}, \eta_{a/h} > \frac{ay}{h}\} = 1-H_\alpha(\frac{x+y}{1-y})$$

for $x > 0$ and $0 < y < 1$ and this proves (32). To prove the first equality in (37) let us apply (49.183) first to the sequence $\tau_k = \xi(kh^\alpha)$ where $k = 0,1,2,\ldots$ and $h > 0$, and then to the sequence $\tau_k = h\xi(k)$ where $k = 0,1,2,\ldots$ and $h > 0$ and let $h \to 0$. Since $\xi(kh^\alpha)$ and $h\xi(k)$ have the same distribution, it follows that the first equality is true in (37).

If we refer to Note 2 in Section 49 and if we use the solution of Problem 53.6, then we can easily extend the previous results to a separable stable process of type $S(\alpha,\beta,c,0)$ where either $0 < \alpha < 1$ or $1 < \alpha < 2$, and $-1 < \beta \leq 1$ and $c > 0$. If $a > 0$, then $\underset{\sim}{P}\{\theta(a) < \infty\} = 1$, and we have

$$(38) \qquad \underset{\sim}{P}\{\frac{\gamma'(a)}{a} \leq x\} = H_{\alpha q}(x)$$

and

$$(39) \qquad \underset{\sim}{P}\{\frac{\gamma'(a)}{a} > x, \frac{\gamma''(a)}{a} > y\} = 1 - H_{\alpha q}(\frac{x+y}{1-y})$$

for $x > 0$ and $0 < y < 1$, furthermore

$$(40) \qquad \underset{\sim}{P}\{\frac{\gamma(a)}{a} \leq x\} = K_{\alpha q}(x)$$

for $x > 0$ where $H_\alpha(x)$ and $K_\alpha(x)$ are defined for $0 < \alpha < 1$ by (33) and (35) respectively and where

$$(41) \qquad q = \frac{1}{2} + \frac{\gamma}{2\alpha}$$

with

$$(42) \qquad \gamma = \frac{2}{\pi} \text{arc tan}(\beta \tan \frac{\alpha\pi}{2})$$

and $-\frac{\pi}{2} < \text{arc tan } x < \frac{\pi}{2}$. These results are due to Ya. G. Sinai [182].

We note that in this case by (42.192) we have $\underset{\sim}{P}\{\xi(u) > 0\} = q$ for all $u > 0$ and that $0 < \alpha q < 1$ always holds.

Finally, we mention the papers of D. V. Gusak [74], E. A. Pecherskii and B. A. Rogozin [142], and E. S. Shtatland [178] in which the joint distribution of $\theta(a)$ and $\gamma'(a)$ and the distribution of $\tau = \inf\{u : \xi(u) = \sup_{0\le u\le t} \xi(u)$ are determined for some homogeneous processes with independent increments.

58. Problems

58.1. In a ballot candidates $A_1, A_2, \ldots, A_m$ score $a_1, a_2, \ldots, a_m$ votes respectively. Denote by $\alpha_1^{(r)}, \alpha_2^{(r)}, \ldots, \alpha_m^{(r)}$ the number of votes registered for $A_1, A_2, \ldots, A_m$ respectively among the first r votes recorded. Let us suppose that all the possible voting records are equally probable. Let $c_1 < c_2 < \ldots < c_m$ be integers. Prove that

$$P = \mathbf{P}\{\alpha_1^{(r)} + c_1 < \alpha_2^{(r)} + c_2 < \ldots < \alpha_m^{(r)} + c_m \text{ for } r = 1, 2, \ldots, a_1 + \ldots + a_m\} =$$

$$= \operatorname{Det} \left| \frac{a_j!}{(a_j + c_j - c_i)!} \right|_{i,j=1,2,\ldots,m}$$

where $1/x! = 0$ for $x = -1, -2, \ldots$. (See D. E. Barton and C. L. Mallows [6].)

58.2. Let $\{\nu(u), 0 \leq u < \infty\}$ be a separable Poisson process of density λ . Find

$$W(t,r,k) = \mathbf{P}\{\nu(u) \leq u + r \text{ for } 0 \leq u \leq t \mid \nu(t) = k\}$$

for $0 \leq k \leq t + r$, $t > 0$ and $r = 0,1$. (See N. V. Smirnov [187].)

58.3. Let $\{\chi(u), 0 \leq u < \infty\}$ be a compound Poisson process defined by (55.37). Give a direct method for finding $W(t, x) = \mathbf{P}\{\chi(u) \leq u + x$ for $0 \leq u \leq t\}$.

58.4 Let $\{\chi(u), 0 \leq u < \infty\}$ be a separable compound Poisson process defined by (55.37) where $H(x) = 1-e^{-\mu x}$ for $x \geq 0$. Find

$$W(t, x) = \underset{\sim}{P}\{ \sup_{0 \le u \le t} [\chi(u) - u] \le x\} .$$

(See G. Arfwedson [2].)

58.5. Let $\{\chi(u) , 0 \le u < \infty\}$ be a separable compound Poisson process defined by (55.37) where $H(x) = 1-e^{-\mu x}$ for $x \geq 0$. Find

$$V(t, x) = \underset{\sim}{P}\{ \sup_{0 \le u \le t} [u - \chi(u)] \le x\} .$$

(See G. Arfwedson [2].)

58.6. Let

$$\chi(u) = \sum_{0<\tau_n \le u} \chi_n$$

for $u \geq 0$ where $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots, \tau_0 = 0)$ and χ_n $(n = 1,2,\ldots)$ are independent sequences of mutually independent random variables. We suppose that $\tau_n - \tau_{n-1}$ $(n = 1,2,\ldots)$ are positive random variables for which $\underset{\sim}{E}\{e^{-s(\tau_n-\tau_{n-1})}\} = \phi(s)$ for $Re(s) \geq 0$ and χ_n $(n = 1,2,\ldots)$ are real random variables for which $\underset{\sim}{E}\{e^{-s\chi_n}\} = \psi(s)$ for $Re(s) = 0$. Give a method for finding the distribution function of

$$\eta(t) = \sup_{0 \le u \le t} [\chi(u) - u] .$$

58.7. Let us consider Problem 58.6 in the particular case when χ_n $(n = 1,2,\ldots)$ are nonnegative random variables. Give a method for finding the distribution function of

$$\eta^*(t) = \sup_{0 \le u \le t} [u - \chi(u)] .$$

58.8. Prove that if $q \neq 0$ and $\alpha \leqq \beta$, then

$$(s-q)\int_{\alpha}^{\beta} e^{-qt-s[\gamma-t]^{+}}dt = e^{-qt-s[\gamma-t]^{+}} - \frac{s}{q}\, e^{-qt-q[\gamma-t]^{+}}\Big|_{\alpha}^{\beta},$$

$$= \{e^{-q\beta-s[\gamma-\beta]^{+}} - e^{-q\alpha-s[\gamma-\alpha]^{+}}\} - \frac{s}{q}\{e^{-q\beta-q[\gamma-\beta]^{+}} - e^{-q\alpha-q[\gamma-\alpha]^{+}}\} .$$

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