CHAPTER VII

STOCHASTIC PROCESSES

47. Basic Theorems. We start this section with the definition of a stochastic process. Let $(\Omega, \mathcal{B}, \underset{\sim}{P})$ be a probability space where Ω is the sample space with sample points $\omega \in \Omega$, $\mathcal{B}$ is a σ-algebra of subsets of Ω, and $\underset{\sim}{P}$ is a probability measure defined on $\mathcal{B}$. We may assume without loss of generality that the probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$ is complete. A probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$ is said to be complete if $A \in \mathcal{B}$, $\underset{\sim}{P}\{A\} = 0$ and $B \subset A$ imply that $B \in \mathcal{B}$. Every probability space can always be completed. Let T be an infinite parameter (index) set. For each $t \in T$ let $\xi(t) = \xi(t, \omega)$ be a random variable defined on Ω, that is, for each $t \in T$, $\xi(t, \omega)$ is a measurable function of ω with respect to $\mathcal{B}$. We say that the family of random variables $\xi(t)$, $t \in T$, forms a stochastic process. That is, a stochastic process is any infinite family of random variables $\{\xi(t), t \in T\}$.

In this section we shall consider only real stochastic processes, but more generally we can consider also complex, vector or abstract stochastic processes.

In most applications t can be considered as time and then T is the time range involved. If T is an infinite sequence, e.g., $T = \{0, 1, 2, \ldots\}$ or $T = \{\ldots, -1, 0, 1, \ldots\}$, then $\{\xi(t), t \in T\}$ is called a discrete parameter stochastic process. If T is a finite or infinite interval, e.g.,

$T = [0,1]$, $T = [0,\infty)$ or $T = (-\infty, \infty)$, then $\{\xi(t), t \in T\}$ is called a continuous parameter stochastic process.

For any fixed $\omega \in \Omega$ the function $\xi(t) = \xi(t, \omega)$ defined for $t \in T$ is called a sample function of the process.

For any finite subset $(t_1, t_2, \ldots, t_n)$ of the parameter set T, the joint distribution function of the random variables $\xi(t_1), \xi(t_2), \ldots, \xi(t_n)$ is called a finite dimensional distribution function of the process. The finite dimensional distribution functions of the process,

$$(1) \quad F_{t_1,t_2,\ldots,t_n}(x_1, x_2,\ldots,x_n) = \underset{\sim}{P}\{\xi(t_1) \leq x_1, \xi(t_2) \leq x_2,\ldots, \xi(t_n) \leq x_n\},$$

defined for all finite sets $(t_1, t_2,\ldots, t_n) \subset T$ and for all real $x_1, x_2,\ldots, x_n$, are the basic distributions and we shall classify stochastic processes according to the properties of their finite dimensional distribution functions.

It is obvious that the distribution functions $F_{t_1,t_2,\ldots,t_n}(x_1, x_2,\ldots,x_n)$ defined by (1) must be consistent in the sense that if $(i_1, i_2,\ldots, i_n)$ is a permutation of $(1,2,\ldots,n)$, then

$$(2) \quad F_{t_{i_1},t_{i_2},\ldots,t_{i_n}}(x_{i_1}, x_{i_2},\ldots, x_{i_n}) = F_{t_1,t_2,\ldots,t_n}(x_1, x_2,\ldots,x_n)$$

and if $1 \leq m < n$, then

$$(3) \quad F_{t_1,t_2,\ldots,t_m}(x_1, x_2,\ldots, x_m) = \lim_{\substack{x_j \to \infty \\ (j=m+1,\ldots,n)}} F_{t_1,t_2,\ldots,t_n}(x_1,x_2,\ldots,x_n)$$

for all $t_{m+1},\ldots,t_n$.

A. N. Kolmogorov [55] proved that the consistency conditions (2) and (3) are the only conditions which the finite dimensional distributions of a stochastic process should satisfy. Kolmogorov's result can be formulated in the following way.

Theorem 1. *If the distribution functions*

(4)
$$F_{t_1,t_2,\ldots,t_n}(x_1,x_2,\ldots,x_n)$$

are defined for any finite subset $(t_1, t_2,\ldots, t_n)$ *of a parameter set* T , *and if they satisfy the conditions* (2) *and* (3), *then there exists a probability space* $(\Omega,\mathcal{B},P)$ *and a family of random variables* $\xi(t) = \xi(t, \omega)$ $(t \in T,$ $\omega \in \Omega)$ *such that*

(5)
$$P\{\xi(t_1) \leq x_1, \xi(t_2) \leq x_2,\ldots, \xi(t_n) \leq x_n\} = F_{t_1,t_2,\ldots,t_n}(x_1,x_2,\ldots,x_n)$$

for every finite subset $(t_1, t_2,\ldots, t_n)$ *of* T .

Proof. In what follows we shall introduce some convenient terminology and reformulate the theorem in the new terminology.

The set $R = (-\infty, \infty)$ of all finite real numbers ω is called a real line. A set A in R is called an elementary set if it can be represented as the union of a finite number of intervals in R . Denote by $\mathcal{B}$ the minimal σ-algebra which contains all the intervals in R . The elements of $\mathcal{B}$ are called Borel sets in R .

Let T be a parameter set and for each $t \in T$ let R_t be a real line with points ω_t . We define the product space

(6) $$R_T = \underset{t\in T}{X} R_t$$

as the space with points $\omega_T = (\omega_t, t \in T)$ where $\omega_t \in R_t$. A set

(7) $$A_T = \underset{t\in T}{X} A_t$$

with points $\omega_T = (\omega_t, t \in T)$ where $\omega_t \in A_t$ is called a product set in R_T.

Let $T_n = (t_1, t_2, \ldots, t_n)$ be a finite subset of the parameter set T. A set A_{T_n} in the product space R_{T_n} is called an elementary set if it can be represented as the union of a finite number of such n-dimensional intervals in R_{T_n} whose sides are parallel to the coordinate axes. Denote by $\mathcal{B}_{T_n}$ the minimal σ-algebra which contains all these n-dimensional intervals in R_{T_n}. The elements of $\mathcal{B}_{T_n}$ are called Borel sets in R_{T_n}.

Let A_{T_n} be a set in the product space R_{T_n}. The set

(8) $$A_{T_n} X R_{T-T_n}$$

is called a cylinder set in R_T with base A_{T_n}. If A_{T_n} is a Borel set in R_{T_n}, then (8) is called a Borel cylinder. If A_{T_n} is a product set in R_{T_n}, then (8) is called a product cylinder.

The minimal σ-algebra which contains all the Borel cylinders in R_T is called the product σ-algebra of $\mathcal{B}_t$ for $t \in T$ and is denoted by

(9) $$\mathcal{B}_T = \underset{t\in T}{X} \mathcal{B}_t .$$

If $\underset{\sim}{P}_T$ is a probability measure defined on $\mathcal{B}_T$, that is, if $(R_T, \mathcal{B}_T, \underset{\sim}{P}_T)$ is a probability space, and if $T_n = (t_1, t_2, \ldots, t_n)$ is any finite subset of T, then we can define a probability measure $\underset{\sim}{P}_{T_n}$ on $\mathcal{B}_{T_n}$ by assigning

(10) $$\underset{\sim}{P}_{T_n}\{A_{T_n}\} = \underset{\sim}{P}_T\{A_{T_n} \times R_{T-T_n}\}$$

to every Borel set A_{T_n} in R_{T_n}. The probability measure $\underset{\sim}{P}_{T_n}$ is called the marginal probability or the projection of $\underset{\sim}{P}_T$ on $\underset{\sim}{R}_{T_n}$.

In what follows we shall prove that there is a unique probability measure $\underset{\sim}{P}_T$ defined on $\mathcal{B}_T$ for which

(11) $$\underset{\sim}{P}_T\{A_T\} = F_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n)$$

whenever $A_T = A_{T_n} \times R_{T-T_n}$ with

(12) $$A_{T_n} = \{(\omega_{t_1}, \omega_{t_2}, \ldots, \omega_{t_n}) : \omega_{t_i} \leqq x_i \text{ for } i = 1,2,\ldots,n\}$$

and $T_n = (t_1, t_2, \ldots, t_n)$ is any finite subset of T. This implies that if we consider the probability space $(\Omega_T, \mathcal{B}_T, \underset{\sim}{P}_T)$ and if we define the random variables $\xi(t)$ for $t \in T$ by

(13) $$\xi(t) = \xi(t, \omega_T) = \omega_t$$

where $\omega_T = (\omega_t, t \in T)$, then we have

(14) $$\underset{\sim}{P}\{\xi(t_1) \leqq x_1, \xi(t_2) \leqq x_2, \ldots, \xi(t_n) \leqq x_n\} = F_{t_1,t_2,\ldots,t_n}(x_1,x_2,\ldots,x_n)$$

for all finite subsets $(t_1, t_2, \ldots, t_n)$ of T. By proving the above formulated statement we shall have proved Theorem 1.

We note that for every finite subset $T_n = (t_1, t_2, \ldots, t_n)$ of T the distribution function $F_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n)$ uniquely determines a probability measure $\underset{\sim}{P}_{T_n}$ on $\mathcal{B}_{T_n}$ in such a way that $\underset{\sim}{P}_{T_n}\{A_{T_n}\} = F_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n)$ whenever $A_{T_n} = \{(\omega_{t_1}, \omega_{t_2}, \ldots, \omega_{t_n}) : \omega_{t_i} \leq x_i \text{ for } 1 \leq i \leq n\}$. (See Theorem 2.2 in the Appendix.) Thus the distribution function $F_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n)$ induces a probability space $(R_{T_n}, \mathcal{B}_{T_n}, \underset{\sim}{P}_{T_n})$. If we define $\xi(t_i) = \xi(t_i, \omega_{T_n}) = \omega_{t_i}$ for $\omega_{T_n} = (\omega_{t_1}, \omega_{t_2}, \ldots, \omega_{t_n})$ and $i = 1,2,\ldots,n$, then the random variables $\xi(t_1), \xi(t_2), \ldots, \xi(t_n)$ have the joint distribution function $F_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n)$.

By the assumptions (2) and (3) the probabilities $\underset{\sim}{P}_{T_n}$ are consistent in the following sense. If $T_n = (t_1, t_2, \ldots, t_n)$ and $T_m = (t_1, t_2, \ldots, t_m)$, where $1 \leq m < n$, are finite subset of T, then the projection of $\underset{\sim}{P}_{T_n}$ on R_{T_m} coincides with $\underset{\sim}{P}_{T_m}$.

Theorem 1 states that consistent probabilities $\underset{\sim}{P}_{T_n}$ on all finite product σ-algebras $\mathcal{B}_{T_n}$ determine uniquely a probability $\underset{\sim}{P}_T$ on the σ-algebra $\mathcal{B}_T$ in such a way that every $\underset{\sim}{P}_{T_n}$ is a projection of $\underset{\sim}{P}_T$ on R_{T_n}.

Now we are going to prove this last statement.

Denote by C_T the space of all those product cylinders $A_{T_n} \times R_{T-T_n}$ for which $T_n = (t_1, t_2, \ldots, t_n)$ is any finite subset of T and A_{T_n} is the union of a finite number of product sets of the form $\mathop{X}\limits_{i=1}^{n} A_{t_i}$ where each A_{t_i} is an elementary set in R_{t_i} or equivalently each A_{t_i} is an interval in R_{t_i}.

To every set $A_{T_n} \times R_{T-T_n}$ in C_T let us assign the probability

$$\underset{\sim}{P}_T\{A_{T_n} \times R_{T-T_n}\} = \underset{\sim}{P}_{T_n}\{A_{T_n}\} \tag{15}$$

which is uniquely determined by $F_{t_1,t_2,\ldots,t_n}(x_1, x_2, \ldots, x_n)$.

It is easy to see that C_T is an algebra of subsets of R_T and $\underset{\sim}{P}_T$ is a non-negative and finitely additive set function on C_T. Obviously $\underset{\sim}{P}_T\{R_T\} = 1$.

Next we shall prove that $\underset{\sim}{P}_T$ is σ-additive on C_T. Since $\underset{\sim}{P}_T$ is finitely additive on C_T, it is sufficient to prove that $\underset{\sim}{P}_T$ is continuous at the empty set, and this implies σ-additivity. Having proved that $\underset{\sim}{P}_T$ is σ-additive on C_T by Carathéodory's extension theorem (see Theorem **1.2** in the Appendix) we can extend the definition of $\underset{\sim}{P}_T$ to $\mathcal{B}_T$, the minimal σ-algebra which contains C_T, in such a way that $\underset{\sim}{P}_T$ remains a nonnegative and σ-additive set function and the extension is unique.

Now let us prove that $\underset{\sim}{P}_T$ when defined on C_T is continuous at the empty set, that is, if $A_n \in C_T$ for $n = 1,2,\ldots$, $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ and $\lim_{n \to \infty} A_n = \prod_{n=1}^{\infty} A_n = \Theta$, then $\lim_{n \to \infty} \underset{\sim}{P}_T\{A_n\} = 0$.

Since every cylinder set depends only on a finite number of parameters, the set of all parameters involved in defining the sequence $\{A_n\}$ is countable. By interchanging, if necessary, the parameters and by including or removing some cylinder sets we may assume without loss of generality that there is a sequence of parameters $t_1, t_2, \ldots, t_n, \ldots$ such that in the sequence $\{A_n\}$ each set A_n is a cylinder set in R_T with base B_n where B_n is the union of a finite number of intervals in $R_{t_1} \times R_{t_2} \times \ldots \times R_{t_n}$.

We shall prove the continuity of $\underset{\sim}{P}_t$ at θ by contradiction. We shall show that if $\lim_{n \to \infty} \underset{\sim}{P}_T\{A_n\} = \varepsilon > 0$, then $\lim_{n \to \infty} A_n = \prod_{n=1}^{\infty} A_n$ is not empty.

Accordingly, let us assume that

$$\underset{\sim}{P}_T\{A_n\} = \underset{\sim}{P}_{T_n}\{B_n\} \geq \varepsilon > 0 \tag{16}$$

for $n = 1,2,\ldots$ where $T_n = (t_1, t_2, \ldots, t_n)$. We shall prove that $\prod_{n=1}^{\infty} A_n$ is not empty.

To simplify notation let us write $\underset{\sim}{P} = \underset{\sim}{P}_T$ and $\underset{\sim}{P}_n = \underset{\sim}{P}_{T_n}$ for $n = 1,2,\ldots$.

The set function $\underset{\sim}{P}_n$ is a probability on $\mathcal{B}_{T_n}$. Thus $\underset{\sim}{P}_n$ is σ-additive and therefore it is continuous on $\mathcal{B}_{T_n}$. Consequently B_n contains a bounded and closed Borel set B_n^* such that

$$\underset{\sim}{P}_n\{B_n - B_n^*\} < \frac{\varepsilon}{2^{n+1}} . \tag{17}$$

For B_n is the union of a finite number of intervals in R_{T_n} and each constituent interval in B_n contains a bounded and closed interval whose $\underset{\sim}{P}_n$ - measure is arbitrarily near to the $\underset{\sim}{P}_n$ - measure of the original interval. If A_n^* denotes the cylinder set in R_T with base B_n^*, then by (17) we have

$$\underset{\sim}{P}\{A_n - A_n^*\} = \underset{\sim}{P}_n\{B_n - B_n^*\} < \frac{\varepsilon}{2^{n+1}} . \tag{18}$$

Let $C_n = A_1^* A_2^* \ldots A_n^*$. Since $C_n \subset A_n^* \subset A_n$ and $A_n\overline{C}_n = A_n\overline{A}_1^* + A_n\overline{A}_2^* + \ldots + A_n\overline{A}_n^* \subset A_1\overline{A}_1^* + A_2\overline{A}_2^* + \ldots + A_n\overline{A}_n^*$, it follows that

$$(19)\qquad \underset{\sim}{P}\{A_n\} - \underset{\sim}{P}\{C_n\} = \underset{\sim}{P}\{A_n - C_n\} \leqq \sum_{k=1}^{n} \underset{\sim}{P}\{A_k - \bar{A}_k^*\} < \frac{\varepsilon}{2}\ .$$

By (16) we have $\underset{\sim}{P}\{A_n\} \geqq \varepsilon$ and therefore (19) implies that

$$(20)\qquad \underset{\sim}{P}\{C_n\} > \frac{\varepsilon}{2}$$

for $n = 1,2,\ldots$. Thus C_n is not empty and we can select in it a point $a(n) = (a_t(n), t \in T)$. Since $C_n \subset C_m$ if $n \geqq m$, therefore $a(n) \in C_m$ if $n \geqq m$. Hence

$$(21)\qquad (a_{t_1}(n), a_{t_2}(n), \ldots, a_{t_m}(n)) \in B_m^*$$

for $n \geqq m$. The set B_m^* is bounded for every $m = 1,2,\ldots$. Thus the sequence $\{a(n)\}$ contains a subsequence $\{a(n_k^{(1)})\}$ for which $a_{t_1}(n_k^{(1)}) \to a_{t_1}$ as $k \to \infty$. Furthermore, the sequence $\{a(n_k^{(1)})\}$ contains a subsequence $\{a(n_k^{(2)})\}$ for which $a_{t_2}(n_k^{(2)}) \to a_{t_2}$ as $k \to \infty$. Continuing in this way for each $i = 1,2,\ldots$ we can define a sequence $\{a(n_k^{(i)})\}$ such that $\{a(n_k^{(i)})\}$ is a subsequence of $\{a(n_k^{(i-1)})\}$ and $a_{t_i}(n_k^{(i)}) \to a_{t_i}$ as $k \to \infty$. Then the diagonal sequence $\{a(n_k^{(k)})\}$ has the property that

$$(22)\qquad \lim_{k \to \infty} a_{t_i}(n_k^{(k)}) = a_{t_i}$$

exists for $i = 1,2,\ldots$. Let $a = (a_t, t \in T)$ where a_t is defined by (22) for $t = t_1, t_2, \ldots, t_i, \ldots$ and $a_t = 0$, say, for $t \neq t_i$ $(i = 1,2,\ldots)$.

Since $(a_{t_1}(n_k^{(k)}), a_{t_2}(n_k^{(k)}), \dots, a_{t_m}(n_k^{(k)})) \varepsilon B_m^*$ for $k = 1,2,\dots,$ and since the set B_m^* is closed by (22) it follows that

$$(23) \qquad (a_{t_1}, a_{t_2}, \dots, a_{t_m}) \varepsilon B_m^* \subset B_m$$

and consequently $a \varepsilon A_m = B_m X R_{T-T_m}$ for $m = 1,2,\dots$. This proves that $\prod_{m=1}^{\infty} A_m$ is not empty which was to be proved.

Accordingly, we have proved that the probability measure $\underset{\sim}{P}_T$ defined by (15) on C_T is σ-additive. By Carathéodory's extension theorem the definition of $\underset{\sim}{P}_T$ can uniquely extended over $\mathcal{B}_T$ in such a way that $\underset{\sim}{P}_T$ remains nonnegative and σ-additive. Thus there exists a probability space $(R_T, \mathcal{B}_T, \underset{\sim}{P}_T)$ and every $\underset{\sim}{P}_{T_n}$ is a projection of $\underset{\sim}{P}_T$ on R_{T_n} .

If we define the random variables $\xi(t)$ for $t \varepsilon T$ by $\xi(t) = \xi(t, \omega_T) = \omega_t$ where $\omega_T = (\omega_t, t \varepsilon T)$, then (5) holds for every finite subset $(t_1, t_2, \dots, t_n)$ of T . This completes the proof of Theorem 1.

Theorem 1 was proved in 1933 by A. N. Kolmogorov [55]. In some particular cases, Theorem 1 can be deduced from some results found in 1917 by P. J. Daniell [21],[22] for integrals in a space of an infinite number of dimensions. For the theory of abstract integrals we refer to M. Fréchet [43], A. N. Kolmogorov [54] and B. Jessen [47].

In the above discussion we considered real stochastic processes. In general we can consider vector stochastic processes or stochastic processes taking values in a metric space and we can demonstrate that the appropriate version of Theorem 1 is valid for such processes too. That is if we assume that each R_t $(t \varepsilon T)$ is a finite dimensional Euclidean space or a metric space, if $\mathcal{B}_t$ denotes the class of Borel subsets of R_t, that is, if $\mathcal{B}_t$ is the minimal σ-algebra which contains all the open sets in R_t, and if $\underset{\sim}{P}_{T_n}$ $(T_n \subset T)$ are consistent probabilities defined on all finite product σ-algebras $\mathcal{B}_{T_n} = \underset{t \varepsilon T_n}{X} \mathcal{B}_t$, then there is a unique probability measure $\underset{\sim}{P}_T$ defined on the σ-algebra $\mathcal{B}_T = \underset{t \varepsilon T}{X} \mathcal{B}_t$ in such a way that every $\underset{\sim}{P}_{T_n}$ is a projection of $\underset{\sim}{P}_T$ on R_{T_n}. The proof of Theorem 1 can easily be extended to stochastic processes taking values in a finite dimensional Euclidean space or in a metric space. However, in general, the appropriate version of Theorem 1 is not valid anymore for abstract stochastic processes. That is, if we assume that each R_t $(t \varepsilon T)$ is an abstract set, if $\mathcal{B}_t$ is a σ-algebra of subsets of R_t, and if $\underset{\sim}{P}_{T_n}$ $(T_n \subset T)$ are consistent probabilities defined on all finite product σ-algebras $\mathcal{B}_{T_n} = \underset{t \varepsilon T_n}{X} \mathcal{B}_t$, then, in general, we cannot define a probability measure $\underset{\sim}{P}_T$ on the σ-algebra $\mathcal{B}_T = \underset{t \varepsilon T}{X} \mathcal{B}_t$ in such a way that every $\underset{\sim}{P}_{T_n}$ is a projection of $\underset{\sim}{P}_T$ on R_{T_n}. In 1938 J. L. Doob [26] and in 1944 E. S. Andersen [2] believed that they have proved the abstract version of Theorem 1, but in 1946 E. S. Andersen and B. Jessen [3] pointed out that these proofs were incorrect. In 1948 E. S. Andersen and B. Jessen [4] constructed an example which shows that in fact the abstract version of Theorem 1 is not valid in general.

It should be noted that in the particular case where the finite dimensional probability measures are consistent product measures the abstract version of Theorem 1 is valid. This result was formulated for the first time in 1934 by Z. Łomnicki and S. Ulam [59], but their proof contains an error which was pointed out in 1946 by E. S. Andersen and B. Jessen [3]. For the proof of the extension theorem for product measures in abstract product sets we refer to J. v. Neumann [65 pp. 122-148], B. Jessen [48], S. Andersen and B. Jessen [3] and S. Kakutani [50].

In the proof of Theorem 1 we actually constructed a probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$ and a family of real random variables $\xi(t)$, $t \in T$, such that the finite dimensional distribution functions of the process $\{\xi(t) , t \in T\}$ are the prescribed distribution functions (4). However, this is not the only possible construction. We can construct infinitely many probability spaces $(\Omega, \mathcal{B}, \underset{\sim}{P})$ and on each probability space we can define infinitely many families of random variables $\{\xi(t) , t \in T\}$ having the given finite dimensional distribution functions (4). In fact if $(\Omega, \mathcal{B}, \underset{\sim}{P})$ is a probability space and $\{\xi(t) , t \in T\}$ and $\{\xi^*(t) , t \in T\}$ are two families of random variables for which

$$\underset{\sim}{P}\{\xi(t) = \xi^*(t)\} = 1 \tag{24}$$

for all $t \in T$, then both $\{\xi(t), t \in T\}$ and $\{\xi^*(t) , t \in T\}$ have the same finite dimensional distributions. In this case we say that $\{\xi(t) , t \in T\}$ and $\{\xi^*(t) , t \in T\}$ are equivalent stochastic processes. Accordingly, we can replace every stochastic process $\{\xi(t) , t \in T\}$ by an equivalent stochastic process $\{\xi^*(t) , t \in T\}$ without changing its finite dimensional distribution functions.

If we want to construct a stochastic process $\{\xi(t) , t \in T\}$ with given finite dimensional distribution functions, then we can choose among infinitely many possible versions. Some versions may have desirable properties and in this case it is reasonable to choose such a version. To see the differences among the possible versions of a stochastic process let us consider the following simple example. Let Ω be the interval $[0,1]$, $\mathcal{B}$, the class of Lebesgue-measurable subsets of $[0,1]$, and $\underset{\sim}{P}$, the Lebesgue measure. Then $(\Omega,\mathcal{B},\underset{\sim}{P})$ is a complete probability space. Let $\{\xi(t)\}$ be a family of random variables defined for $t \in T = [0,1]$ for which

$$\underset{\sim}{P}\{\xi(t) = 0\} = 1 \tag{25}$$

for all $t \in T$. The finite dimensional distribution functions of $\{\xi(t)\,,\ 0 \leq t \leq 1\}$ are uniquely determined by (25) and they are consistent. Thus by Theorem 1 it follows that indeed there exists a process $\{\xi(t)\,,\ 0 \leq t \leq 1$ for which (25) holds.

By (25) it follows that

$$\underset{\sim}{P}\{\xi(t) \equiv 0 \quad \text{for} \quad t \in S\} = 1 \tag{26}$$

for any finite or countably infinite subset S of $[0,1]$. For many purposes it would be desirable to conclude from (26) that

$$\underset{\sim}{P}\{\xi(t) \equiv 0 \quad \text{for all } t \in [0,1]\} = 1 . \tag{27}$$

However, (27) does not follow from (26) in general, unless we choose some suitable version of the process $\{\xi(t)\,,\ 0 \leq t \leq 1\}$. For example if M is a subset of $[0,1]$ and if we define $\xi(t) = \xi(t\,,\omega)$ for $t \in [0,1]$ and $\omega \in [0,1]$ in the following way

$$\xi(t,\omega) = \begin{cases} 0 & \text{if} \quad t \in M\,,\ \omega \in [0,1]\,, \\ 0 & \text{if} \quad t \notin M\,,\ \omega \neq t\,, \\ 1 & \text{if} \quad t \notin M\,,\ \omega = t\,, \end{cases} \tag{28}$$

then (25) and (26) are satisfied, and $\{\omega : \xi(t,\omega) = 0 \quad \text{for} \quad 0 \leq t \leq 1\} = \{\omega : \omega \in M\}$. Now if $M \in \mathcal{B}$, then $\{\xi(t) \equiv 0 \quad \text{for} \quad 0 \leq t \leq 1\} \in \mathcal{B}$ and

$\underset{\sim}{P}\{\xi(t) \equiv 0 \text{ for } 0 \leq t \leq 1\} = \mu(M)$ where $\mu(M)$ is the Lebesgue measure of M, whereas, if $M \notin \mathcal{B}$, then $\{\xi(t) \equiv 0 \text{ for } 0 \leq t \leq 1\} \notin \mathcal{B}$ and we cannot speak about the probability of $\{\xi(t) \equiv 0 \text{ for } 0 \leq t \leq 1\}$, that is, the finite dimensional distributions of the process do not determine the probability of $\{\xi(t) \equiv 0 \text{ for } 0 \leq t \leq 1\}$. If we choose $M = [0,1]$ or M is any Borel subset of $[0,1]$ with Lebesgue measure 1, then (27) holds. This is of course the desirable case but we cannot exclude the other cases without imposing some restriction on the stochastic process to be chosen. The simplest and the most useful criterion in choosing the stochastic process $\{\xi(t), t \in T\}$ is the criterion of separability which was introduced in 1937 by *J. L. Doob* [25]. See also *J. L. Doob* [28],[29], *W. Ambrose* [1], *J. L. Doob* and *W. Ambrose* [33], *J. L. Doob* [30],[31],[32], *P. A. Meyer* [61], [62 pp. 55-64], and *I. I. Gikhman* and *A. V. Skorokhod* [44 pp. 150-156].

Definition 1. *Let* $\{\xi(t), t \in T\}$ *be a real stochastic process with arbitrary linear parameter set* T. *Let the random variables* $\xi(t)$, $t \in T$, *be defined on a probability space* $(\Omega, \mathcal{B}, \underset{\sim}{P})$ *and let* $\xi(t)$ *have value* $\xi(t,\omega)$ *at* $\omega \in \Omega$. *The process* $\{\xi(t), t \in T\}$ *is said to be separable if there is a countable subset* S *of* T *and a set* $\Lambda \in \mathcal{B}$ *with* $\underset{\sim}{P}\{\Lambda\} = 0$ *such that if* A *is any closed set of the real line and if* I *is any open interval of the real line, then*

$$(29) \quad \{\omega : \xi(t,\omega) \in A \text{ for } t \in IS\} - \{\omega : \xi(t,\omega) \in A \text{ for } t \in IT\} \subset \Lambda .$$

The set S is called a separability set of the process, and Λ, an exceptional set. Since every open set can be represented as a countable union of open intervals, it is obvious that the above definition remains valid unchangeably if we assume that I is any open set.

The advantage of a separable process of is evident. Let us consider the process $\{\xi(t), t \in T\}$ in the above definition. If A is a closed set and I is an open interval, then in general the set $\{\omega : \xi(t,\omega) \in A \text{ for } t \in IT\}$ does not belong to $\mathcal{B}$. However, if the probability space $(\Omega,\mathcal{B},\underset{\sim}{P})$ is complete and if the process is separable, then $\{\omega : \xi(t,\omega) \in A \text{ for } t \in IT\}$ belongs to $\mathcal{B}$ and

$$(30) \qquad \underset{\sim}{P}\{\xi(t) \in A \text{ for } t \in IT\} = \underset{\sim}{P}\{\xi(t) \in A \text{ for } t \in IS\} .$$

For example, if there is a separable stochastic process $\{\xi(t), 0 \leq t \leq 1\}$ defined on a complete probability space and if (25) holds for $t \in [0,1]$, then (27) is true. As we have already seen (27) is not true without some hypothesis for the process $\{\xi(t), 0 \leq t \leq 1\}$.

If $\{\xi(t), t \in T\}$ is a separable stochastic process defined on a probability space $(\Omega,\mathcal{B},\underset{\sim}{P})$ which is complete, then we can define the probabilities of such events as that the sample functions are bounded, are continuous, are integrable and so on.

We note that if $\{\xi(t,\omega), t \in T\}$ is a separable stochastic process defined on a complete probability space, if S is a separability set and if $\omega \notin \Lambda$ where Λ is an exceptional set, then

$$(31) \qquad \inf_{t\in IT} \xi(t,\omega) = \inf_{t\in IS} \xi(t,\omega) \quad \text{and} \quad \sup_{t\in IT} \xi(t,\omega) = \sup_{t\in IS} \xi(t,\omega)$$

for every open interval I. Conversely, if there is a set $\Lambda \in \mathcal{B}$ with $\underset{\sim}{P}\{\Lambda\} = 0$ such that if $\omega \notin \Lambda$ it follows that (31) is true for every open interval I, then the process $\{\xi(t,\omega), t \in T\}$ is obviously separable.

If the process $\{\xi(t), t \in T\}$ is separable and if I is any open interval, then $\inf_{t\in IT} \xi(t)$, $\sup_{t\in IT} \xi(t)$, $\liminf_{t\to u} \xi(t)$ and $\limsup_{t\to u} \xi(t)$ are all (finite or infinite valued) random variables.

We have demonstrated that a separable process has many desirable properties. The problem arises what restrictions should we impose on a process in order to be separable. We shall prove that every stochastic process $\{\xi(t), t \in T\}$ has a separable version $\{\xi^*(t), t \in T\}$ which has the same finite dimensional distribution functions as the original process. This is the best possible result which we can expect. The proof of this result is based on the following two auxiliary theorems.

<u>Lemma 1</u>. <u>Let</u> $\{\xi(t,\omega), t \in T\}$ <u>be a real stochastic process with an arbitrary parameter set</u> T. <u>To each linear Borel set</u> A <u>there corresponds a countable sequence</u> $\{t_k\}$ <u>such that</u>

(32) $$\mathbf{P}\{\xi(t_k,\omega) \in A \text{ for } k \geqq 1 \text{ and } \xi(t,\omega) \notin A\} = 0$$

<u>for all</u> $t \in T$.

<u>Proof</u>. Let t_1 be any point of T. If $t_1, t_2, \ldots, t_n$ have already been chosen, then let us define

(33) $$a_n = \sup_{t\in T} \mathbf{P}\{\xi(t_k,\omega) \in A \text{ for } k \leqq n \text{ and } \xi(t,\omega) \notin A\}.$$

Then $1 \geqq a_1 \geqq a_2 \geqq \ldots \geqq a_n \geqq \ldots \geqq 0$. If $a_n = 0$, then $(t_1, t_2, \ldots, t_n)$ satisfies (32). If $a_n > 0$, then let us choose t_{n+1} as any value for which

(34) $$\mathbf{P}\{\xi(t_k,\omega) \in A \text{ for } k \leq n \text{ and } \xi(t_{n+1},\omega) \notin A\} > \frac{a_n}{2} .$$

If $a_n > 0$ for all $n = 1,2,\ldots$, then we have

(35) $$\mathbf{P}\{\xi(t_k,\omega) \in A \text{ for } k \geq 1 \text{ and } \xi(t,\omega) \notin A\} \leq \lim_{n \to \infty} a_n$$

for all $t \in T$.

Since the sets $\{\omega : \xi(t_k,\omega) \in A$ for $k \leq n$ and $\xi(t_{n+1},\omega) \notin A\}$ for $n = 1,2,\ldots$ are disjoint, we have

(36) $$\sum_{n=1}^{\infty} \mathbf{P}\{\xi(t_k,\omega) \notin A \text{ for } k \leq n \text{ and } \xi(t_{n+1},\omega) \notin A\} \leq 1 ,$$

whence $\lim_{n \to \infty} \mathbf{P}\{\xi(t_k,\omega) \notin A$ for $k \leq n$ and $\xi(t_{n+1},\omega) \notin A\} = 0$. By (34) we obtain that $\lim_{n \to \infty} a_n = 0$. Finally (35) implies (32) which completes the proof.

The following auxiliary theorem follows easily from the previous one.

Lemma 2. Let $\{\xi(t) , t \in T\}$ *be a real stochastic process with an arbitrary parameter set* T. *Let* A_0 *be a countable class of linear Borel sets, and let* A *be the class of sets which are the intersections of sets belonging to* A_0. *Then there exists a countable set of points* $t_1, t_2, \ldots, t_k, \ldots$ *such that to each* $t \subset T$ *there corresponds an* ω-*set* Λ_t *with* $\mathbf{P}\{\Lambda_t\} = 0$ *and*

(37) $$\{\xi(t_k,\omega) \in A \text{ for } k \geq 1 \text{ and } \xi(t,\omega) \notin A\} \subset \Lambda_t$$

for each $A \in A$.

Proof. For each $A \in A_0$ there is a countable parameter set such that (32) holds. Obviously (32) holds for each $A \in A_0$ if $\{t_k\}$ is chosen as the union of all these parameter sets. Let

$$(38) \qquad \Lambda_t = \bigcup_{A \varepsilon \mathcal{A}_0} \{\omega : \xi(t_k,\omega) \,\varepsilon\, A \quad \text{for} \quad k \geqq 1 \quad \text{and} \quad \xi(t,\omega) \not\varepsilon\, A\} .$$

with the above definition of $\{t_k\}$.

If $A \,\varepsilon\, \mathcal{A}$ and $A \subset A_0 \,\varepsilon\, \mathcal{A}_0$, then

$$(39) \qquad \{\xi(t_k,\omega) \,\varepsilon\, A \quad \text{for} \quad k \geqq 1 \quad \text{and} \quad \xi(t,\omega) \not\varepsilon\, A_0\} \subset$$

$$\subset \{\xi(t_k,\omega) \,\varepsilon\, A_0 \quad \text{for} \quad k \geqq 1 \quad \text{and} \quad \xi(t,\omega) \not\varepsilon\, A_0\} \subset \Lambda_t$$

and hence (37) follows because each $A \,\varepsilon\, \mathcal{A}$ is the intersection of a sequence of sets in $\mathcal{A}_0$. This completes the proof of the lemma.

Theorem 2. Let $\{\xi(t) , t \,\varepsilon\, T\}$ be a real stochastic process with linear parameter set T defined on a probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$. There exists a separable stochastic process $\{\xi^*(t) , t \,\varepsilon\, T\}$ defined on the same probability space such that

$$(40) \qquad \underset{\sim}{P}\{\xi^*(t) = \xi(t)\} = 1$$

for all $t \,\varepsilon\, T$. The random variables $\xi^*(t)$ $(t \,\varepsilon\, T)$ may take on the values $+\infty$ and $-\infty$.

Proof. We note that (40) implies that the finite dimensional distribution functions of the process $\{\xi^*(t) , t \,\varepsilon\, T\}$ are the same as the corresponding finite dimensional distribution functions of the process $\{\xi(t) , t \,\varepsilon\, T\}$, that is, if we replace a stochastic process by its separable version, then all the finite dimensional distribution functions remain unchanged.

We note also that for each $t \varepsilon T$ the set $\{\omega : \xi^*(t,\omega) \neq \xi(t,\omega)\}$ has probability 0, but this set may vary with t. If the union $\bigcup_{t \varepsilon T} \{\omega : \xi^*(t,\omega) \neq \xi(t,\omega)\}$ has probability 0, then the process $\{\xi(t), t \varepsilon T\}$ itself is separable.

To prove the theorem let A_0 be the class of linear sets which are finite unions of open or closed intervals with rational or infinite endpoints, and let A be the class of sets which are intersections of sequences of sets in A_0. Then A includes the closed sets.

For any open interval I with rational or infinite endpoints let us consider the stochastic process $\{\xi(t), t \varepsilon IT\}$ and apply Lemma 2 with A_0 and A as just defined. By Lemma 2 there is a countable set $S(I) \subset IT$ and an ω-set $\Lambda_t(I)$ such that $P\{\Lambda_t(I)\} = 0$ for $t \varepsilon IT$ and

(41) $$\{\xi(s,\omega) \varepsilon A \text{ for } s \varepsilon S(I) \text{ and } \xi(t,\omega) \not\varepsilon A\} \subset \Lambda_t(I)$$

for $A \varepsilon A$ and $t \varepsilon IT$. Define

(42) $$S = \bigcup_I S(I) \quad \text{and} \quad \Lambda_t = \bigcup_I \Lambda_t(I)$$

where the union is taken for all open intervals I with rational or infinite endpoints.

For fixed ω let $A(I,\omega)$ be the closure of the set of values $\xi(s,\omega)$ as s varies in IS. The set $A(I,\omega)$ may include the values $+\infty$ and $-\infty$. It is closed, nonempty, and

(43) $$\xi(t,\omega) \varepsilon A(I,\omega) \text{ if } t \varepsilon IT \text{ and } \omega \not\varepsilon \Lambda_t .$$

If we define

$$(44) \qquad A(t,\omega) = \bigcap_{I \ni t} A(I,\omega)$$

where the intersection is taken for all those specified intervals which contain t , then $A(t,\omega)$ is closed, nonempty, and

$$(45) \qquad \xi(t,\omega) \in A(t,\omega) \quad \text{if} \quad t \in T \quad \text{and} \quad \omega \notin \Lambda_t \ .$$

Now let us define $\xi^*(t,\omega)$ for $t \in T$ and $\omega \in \Omega$ as follows:

$$(46) \qquad \xi^*(t,\omega) = \xi(t,\omega) \quad \text{if} \quad t \in S \quad \text{or} \quad t \notin S \quad \text{and} \quad \omega \notin \Lambda_t \ ,$$

and $\xi^*(t,\omega)$ is any value in $A(t,\omega)$ if $t \notin S$ and $\omega \in \Lambda_t$.

The process $\{\xi^*(t,\omega) \ , \ t \in T\}$ obviously satisfies the condition (40). It remains to prove that $\{\xi^*(t) \ , \ t \in T\}$ is separable.

Let A be a closed set and let I be an open interval with rational or infinite endpoints. Suppose that ω has the property that

$$(47) \qquad \xi^*(s,\omega) \in A \quad \text{if} \quad s \in IS \ .$$

Then $A(I,\omega) \subset A$ necessarily holds. It follows from the definition of $\xi^*(t,\omega)$ that if $t \in IT$, then

$$(48) \qquad \xi^*(t,\omega) = \xi(t,\omega) \in A(I,\omega) \quad \text{for} \quad t \in S \quad \text{and} \quad \text{for} \quad t \notin S \ , \ \omega \notin \Lambda_t$$

and

$$(49) \qquad \xi^*(t,\omega) \in A(t,\omega) \subset A(I,\omega) \subset A \quad \text{for} \quad t \notin S \ , \ \omega \in \Lambda_t \ .$$

Thus

(50) $$\{\xi^*(s,\omega) \in A \quad \text{for} \quad s \in IS\} = \{\xi^*(t,\omega) \in A \quad \text{for} \quad t \in IT\}$$

if A is a closed set and if I is an open interval with rational or infinite endpoints. Since any open interval can be expressed as the union of a countable number of open intervals with rational or infinite endpoints, it follows from (50) that (50) is true for any open interval I. This completes the proof of the theorem.

We observe that we cannot exclude infinite values for $\xi^*(t,\omega)$, since the set $A(t,\omega)$ above may contain no finite values.

Theorem 2 and the above proof are due to J. L. Doob [30 pp. 57-60].

In many cases it is necessary to specify the separability set S of a stochastic process. The following theorem shows that for a large class of stochastic processes we can easily find separability sets.

Theorem 3. *Let* $\{\xi(t), t \in T\}$ *be a separable, real stochastic process with linear parameter set* T. *If for every* $\varepsilon > 0$ *we have*

(51) $$P\{|\xi(t) - \xi(u)| > \varepsilon\} \to 0 \quad \text{as} \quad |t-u| \to 0 ,$$

then any countable and everywhere dense subset S *of* T *is a separability set of the process.*

Proof. Let $\{\xi(t), t \in T\}$ be defined on a probability space $(\Omega, \mathcal{B}, P)$. Let S be a separability set of the process and let Λ be an exceptional set. Then (29) holds for any closed set A and $\Lambda \in \mathcal{B}$ and $P\{\Lambda\} = 0$.

Let S^* be any countable and everywhere dense subset of T. We shall prove that there is a set Λ^* such that $\Lambda^* \in \mathcal{B}$, $P\{\Lambda^*\} = 0$ and

(52) $\{\omega : \xi(t,\omega) \varepsilon A \text{ for } t \varepsilon IS^*\} - \{\omega : \xi(t,\omega) \varepsilon A \text{ for } t \varepsilon IT\} \subset \Lambda \cup \Lambda^*$

for any closed set A. This implies that S^* is a separability set of the process.

For any open interval I with rational or infinite endpoints and for fixed ω denote by $B(I,\omega)$ the set of values of $\xi(s,\omega)$ as s varies in IS^*. Then we have

(53) $$\underset{\sim}{P}\{\xi(t,\omega) \not\varepsilon B(I,\omega)\} = 0$$

for all $t \varepsilon IT$. To prove (53) for each $t \varepsilon IT$ let us choose a sequence $\{t_k\}$ such that $t_k \varepsilon IS^*$ and $t_k \to t$ as $k \to \infty$. Then we have

(54) $$\underset{\sim}{P}\{\xi(t,\omega) \not\varepsilon B(I,\omega)\} \leqq \lim_{m \to \infty} \underset{\sim}{P}\{\liminf_{k \to \infty} |\xi(t_k) - \xi(t)| > \frac{1}{m}\}$$
$$\leqq \lim_{m \to \infty} \liminf_{k \to \infty} \underset{\sim}{P}\{|\xi(t_k) - \xi(t)| > \frac{1}{m}\} = 0 .$$

This implies (53).

Let

(55) $$\Lambda^* = \bigcup_{I} \bigcup_{t \varepsilon IS} \{\omega : \xi(t,\omega) \not\varepsilon B(I,\omega)\}$$

where the union is taken for all open intervals with rational or infinite endpoints. We have $\Lambda^* \varepsilon \mathcal{B}$ and by (53) $\underset{\sim}{P}\{\Lambda^*\} = 0$.

Now if $\omega \not\varepsilon \Lambda \cup \Lambda^*$ and $\xi(t,\omega) \varepsilon A$ for all $t \varepsilon IS^*$ where A is a closed set, then for every $t \varepsilon IS$ we have

(56) $$\xi(t,\omega) \varepsilon B(I,\omega) \subset A .$$

Finally by (29) we can conclude that (56) implies that

(57) $$\xi(t,\omega) \subset A$$

for all $t \in IT$ whenever $\omega \notin \Lambda \cup \Lambda^*$. This proves (52).

Note. The notion of separability and Theorem 2 and Theorem 3 can also be extended to abstract valued processes. We shall mention here some results for the case when $\{\xi(t), t \in T\}$ is a stochastic process with state space X and parameter set T where X and T are metric spaces. That is let $(\Omega, \mathcal{B}, P)$ be a probability space and for each $t \in T$ let $\xi(t) = \xi(t,\omega)$ be a measurable function of $\omega \in \Omega$ taking values in X.

Definition 2. The process $\{\xi(t), t \in T\}$ is said to be separable if there is a countable subset S of T and a set $\Lambda \in \mathcal{B}$ with $P\{\Lambda\} = 0$ such that if A is any closed set in X and I is any open set in T, then

(58) $$\{\omega : \xi(t,\omega) \in A \text{ for } t \in IS\} - \{\omega : \xi(t,\omega) \in A \text{ for } t \in IT\} \subset \Lambda .$$

In exactly the same way as we proved Theorem 2 and Theorem 3 we can prove the following more general theorems. (See I. I. Gikhman and A. V. Skorokhod [44 pp. 150-156].)

Theorem 4. If X is a compact metric space and T is a separable metric space, then there exists a separable stochastic process $\{\xi^*(t), t \in T\}$ defined on the same probability space as $\{\xi(t), t \in T\}$ and having the same state space X as $\{\xi(t), t \in T\}$ such that

(59) $$\underset{\sim}{P}\{\xi^*(t) = \xi(t)\} = 1$$

for all $t \in T$.

Theorem 5. If X is a separable and locally compact metric space and T is a separable metric space, then there exists a separable stochastic process $\{\xi^*(t), t \in T\}$ defined on the same probability space as $\{\xi(t), t \in T\}$ and having state space $X^* \supset X$ where X^* is some compact extension of X such that

(60) $$\underset{\sim}{P}\{\xi^*(t) = \xi(t)\} = 1$$

for all $t \in T$.

Theorem 6. Let $\{\xi(t), t \in T\}$ be a separable stochastic process with state space X and parameter set T where X is a metric space with metric $\rho(x,y)$ and T is a separable metric space with metric $r(t,u)$. If for every $\varepsilon > 0$ we have

(61) $$\underset{\sim}{P}\{\rho(\xi(t), \xi(u))\} \to 0 \text{ as } r(t,u) \to 0,$$

then any countable and everywhere dense subset S of T is a separability set of the process.

48. Poisson and Compound Poisson Processes. Before introducing the notion of Poisson and compound Poisson processes it is necessary to deal with the Poisson distribution. We say that a random variable ξ has a Poisson distribution with parameter a where a is a positive number if

(1) $$P\{\xi = k\} = e^{-a} \frac{a^k}{k!}$$

for $k = 0,1,2,\ldots$.

The Poisson distribution appears for the first time in connection with the matching problem. In 1713 N. Bernoulli and P. R. Montmort (see [143 pp. 301-302]) found that

(2) $$P_k(n) = \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}$$

is the probability that exactly k matches occur if we draw all the n cards from a box which contains n cards numbered $1,2,\ldots,n$ and if all the $n!$ possible results are equally probable. Both L. Euler [112] and A. De Moivre [109] observed that the sum in (2) tends to $1/e$ as $n \to \infty$ and $k = 0,1,2,\ldots$. Thus in the middle of the eighteenth century L. Euler and A. De Moivre encountered an instance of the Poisson distribution preceding S. D. Poisson by nearly a century.

In 1837 S. D. Poisson [149],[150 pp. 171-172] demonstrated that if we consider n Bernoulli trials with probability p for success and if we suppose that $n \to \infty$ and $p \to 0$ in such a way that $np \to a$ where a is a positive number, then the limiting distribution of the number of successes is a Poisson distribution with parameter a , that is,

(3) $$\lim_{\substack{n \to \infty \\ np \to a}} \binom{n}{k} p^k (1-p)^{n-k} = e^{-a} \frac{a^k}{k!}$$

for $k = 0,1,2,\ldots$.

In 1898 L. v. Bortkiewicz [102] provided a thorough study of the Poisson distribution and he observed that in several cases when instantaneous random events occur in time, then with good approximation the number of events occurring in any one interval has a Poisson distribution. L. v. Bortkiewicz considered examples such as the occurrence of accidental deaths by horse kick in the Prussian Army over a 20 years period, and he found that the observations were in agreement with the Poisson distribution.

At the beginning of the twentieth century several researchers considered random phenomena which obey the Poisson law.

In 1903 F. Lundberg [134] assumed in his research that insurance claims happen according to the Poisson law. In 1909 A. K. Erlang [111] applied the Poisson law for the incoming calls in a telephone exchange. In investigating the nature of radioactive disintegration in 1910 E. Rutherford and H. Geiger [122], [167] observed the number of α-particles reaching a counter in consecutive intervals and their data showed good agreement with the Poisson law. In 1918 W. Schottky [172] assumed in his investigations that electron emission from metals occurs according to the Poisson law.

The first explanations of the appearance of the Poisson distribution in the random phenomena mentioned above were based on the Poisson approximation of the Bernoulli distribution. In 1910 H. Bateman [97],[98] demonstrated that if a random phenomenon satisfies some plausible conditions, then the number of events occurring in any interval necessarily has a Poisson distribution. This was the first result in which the Poisson distribution appeared as an exact distribution and not an approximating distribution. In 1921 M. Fujiwara [121] considered more general assumptions than H. Bateman and deduced the Poisson law as a particular case of a more general law. In 1953 K. Florek, E. Marczewski and C. Ryll-Nardzewski [117] weakened further the assumptions which lead to the Poisson law.

Now we are going to deduce the Poisson law under general assumptions. If we observe instantaneous random events occurring in the time interval $(0, \infty)$, then it is convenient to introduce the random variable $\nu(t)$ denoting the number of events occurring in the time interval $(0, t]$. The family of random variables $\{\nu(t), 0 \leq t < \infty\}$ is said to form a point process. We say that the random phenomenon obeys the Poisson law if for every $u \geq 0$ and $t > 0$, the random variable $\nu(u+t) - \nu(u)$, that is, the number of events occurring in the time interval $(u, u+t]$, has a Poisson distribution. Our aim is to find conditions under which the point process $\{\nu(t), 0 \leq t < \infty\}$ obeys the Poisson law.

Let us suppose that in the time interval $(0, \infty)$ instantaneous events occur at random and denote by $\nu(t)$ the number of events occurring in the time interval $(0, t]$. We shall study point processes which satisfy some or all the following conditions:

(a) Independence. For any $0 \leq t_0 < t_1 < \dots < t_n$ and for nonnegative integers $k_1, k_2, \dots, k_n$, where $n = 2,3,\dots$, the events $\{\nu(t_j) - \nu(t_{j-1}) = k_j\}$ for $j = 1,2,\dots, n$ are mutually independent.

(b) Homogenity. The probability of the event $\{\nu(u+t) - \nu(u) = k\}$ where $u \geq 0$, $t \geq 0$, $k = 0,1,2,\ldots$ does not depend on u.

(c) Orderliness. In any interval $(0, t]$ events occur singly with probability one.

The following result is the main result for point processes defined above an it leads to the definition of the basic Poisson process.

Theorem 1. If $\nu(t)$ denotes the number of events occurring in the time interval $(0, t]$ in a random point process and if $\{\nu(t), 0 \leq t < \infty\}$ satisfies the conditions (a), (b) and (c), then there exists a nonnegative constant λ such that

$$(4) \qquad \underset{\sim}{P}\{\nu(u+t) - \nu(u) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

for $u \geq 0$, $t \geq 0$ and $k = 0,1,2,\ldots$.

Proof. If we want to describe mathematically a desired random point process defined in the time interval $[0, \infty)$, then we should construct a probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$ and we should define a family of random variables $\nu(t) = \nu(t,\omega)$ $(0 \leq t < \infty, \omega \in \Omega)$ such that conditions (a), (b), (c) are satisfied.

It is natural to assume that Ω contains all those real functions $\omega(t)$ defined for $t \geq 0$ which take on only nonnegative integers, are nondecreasing, continuous on the right, and satisfy $\omega(0) = 0$. Let us assume that $\mathcal{B}$ is the smallest σ-algebra which contains the events $\{\omega : \omega(t) = k\}$ for all $t \geq 0$ and $k = 0,1,2,\ldots$. For every $t \geq 0$ define the random variable $\nu(t) = \nu(t,\omega) = \omega(t)$ if $\omega = \omega(t)$. We shall show that there exists a probability measure $\underset{\sim}{P}$ such that (a), (b) and (c) are satisfied and $\underset{\sim}{P}$ depends only on a nonnegative real parameter λ .

Let

$$\underset{\sim}{P}\{\nu(t) = k\} = P_k(t) \tag{5}$$

for $t \geq 0$ and $k = 0,1,\ldots$. We shall prove that necessarily

$$P_k(t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \tag{6}$$

for $t \geq 0$ and $k = 0,1,2,\ldots$ where $\lambda \geq 0$.

By using some simple properties of the Poisson distribution we can prove that by (5) and (6) the probability $\underset{\sim}{P}\{A\}$ is uniquely determined for $A \in \mathcal{A}$ where $\mathcal{A}$ is the smallest algebra which contains the events $\{\omega : \omega(t) = k\}$ for all $t \geq 0$ and $k = 0,1,2,\ldots$. By Carathéodory's extension theorem (Theorem 1.2 in the Appendix) the definition of $\underset{\sim}{P}\{A\}$ can uniquely be extended to $\mathcal{B}$. That is, there exists indeed a probability space $(\Omega,\mathcal{B},\underset{\sim}{P})$ and a family of random variables $\{\nu(t) , 0 \leq t < \infty\}$ for which the conditions (a), (b), and (c) are satisfied. It remains to prove that (6) holds with some $\lambda \geq 0$.

Since the event $\{\nu(t+u) = k\}$ occurs if and only if $\{\nu(t) = k-j\}$ and

$\{\nu(t+u) - \nu(t) = j\}$ for at least one $j = 0,1,\ldots,k$, by the conditions (a) and (b) we obtain that

(7) $$P_k(t+u) = \sum_{j=0}^{k} P_{k-j}(t)P_j(u)$$

for $t \geqq 0$, $u \geqq 0$ and $k = 0,1,2,\ldots$.

If $k = 0$, then (7) reduces to

(8) $$P_0(t+u) = P_0(t)P_0(u)$$

for $t \geqq 0$ and $u \geqq 0$. We shall prove that (8) implies that

(9) $$P_0(t) = [P_0(1)]^t$$

for all $t \geqq 0$.

From (9) it follows that either $P_0(t) = 1$ for all $t \geqq 0$, or $P_0(t) = 0$ for all $t > 0$, or

(10) $$P_0(t) = e^{-\lambda t}$$

for $t \geqq 0$ where λ is a finite positive number. For there are three possibilities $P_0(1) = 1$ or $P_0(1) = 0$ or $0 < P_0(1) < 1$. If $P_0(1) = 1$, then by (9) $P_0(t) = 1$ for all $t \geqq 0$. If $P_0(1) = 0$, then by (9) $P_0(t) = 0$ for all $t > 0$. If $0 < P_0(1) < 1$, then there exists a finite positive λ such that $P_0(1) = e^{-\lambda}$ and then (10) follows from (9).

Since $0 \leqq P_0(t) \leqq 1$ for all $t \geqq 0$, it follows from (8) that $P_0(t+u) \leqq P_0(u)$ for $t \geqq 0$ and $u \geqq 0$. If r and s are positive integers, then by the repeated application of (8) we obtain that $P_0(\frac{r}{s}) = [P_0(\frac{1}{s})]^r$ and if $r = s$, then $P_0(1) = [P_0(\frac{1}{s})]^s$. Thus it follows that

$$P_0(\tfrac{r}{s}) = [P_0(1)]^{\frac{r}{s}} \tag{11}$$

for any positive rational number r/s. If $t > 0$, then for every sufficiently large s there is an $r \geq 2$ such that $r-1 \leq ts < r$. Then $P_0(\frac{r}{s}) \leq P_0(t) \leq P_0(\frac{r-1}{s})$. By (11) $\lim_{s\to\infty} P_0(\frac{r}{s}) = \lim_{s\to\infty} P_0(\frac{r-1}{s}) = [P_0(1)]^t$ and this proves (9) for $t > 0$. Since necessarily $P_0(0) = 1$, therefore (9) is true for all $t \geq 0$.

If $P_0(t) = 1$ for all $t \geq 0$, then $P_k(t) = 0$ for all $k = 1,2,\ldots,$ and $t \geq 0$. This corresponds to the degenerate case when with probability one no events occur in any interval $(0, t]$. This proves (6) for $\lambda = 0$.

If $P_0(t) = 0$ for all $t > 0$, then by (7) it follows that $P_k(t) = 0$ for all $k = 1,2,\ldots$ and $t > 0$. This case is meaningless and should be excluded. This case can be considered as (6) with $\lambda = \infty$.

Now we shall prove that if $P_0(t) = e^{-\lambda t}$ for $t \geq 0$ where λ is a finite positive number then (6) holds for all $t \geq 0$ and $k = 0,1,2,\ldots$.

If $P_0(t) = e^{-\lambda t}$ for all $t \geq 0$ where λ is a finite positive number, then by (7) we obtain that

$$P_1(t+u) = P_1(t)e^{-\lambda u} + P_1(u)e^{-\lambda t} \tag{12}$$

for $t \geq 0$ and $u \geq 0$. Let $f(t) = e^{\lambda t} P_1(t)$ for $t \geq 0$. Then by (12) we have

$$f(t+u) = f(t) + f(u) \tag{13}$$

for $t \geqq 0$ and $u \geqq 0$. Obviously $0 \leqq f(t) \leqq e^{\lambda}$ for $0 \leqq t \leqq 1$. The only solution of (13) which is bounded in the interval $[0, 1]$ is

$$f(t) = \lambda_1 t \tag{14}$$

where λ_1 is a real constant. For if we define $g(t) = f(t) - tf(1)$ for $t \geqq 0$, then by (13)

$$g(t+u) = g(t) + g(u) \tag{15}$$

for all $t \geqq 0$ and $u \geqq 0$. On the other hand by definition $g(1) = 0$, and this implies that $g(t+1) = g(t)$ for all $t \geqq 0$. Since $g(t)$ is bounded in the interval $[0, 1]$, therefore $g(t)$ is bounded in the interval $[0, \infty)$. If $g(t) \neq 0$ for some $t > 0$, then $g(nt) = ng(t)$ is arbitrarily large for sufficiently large n values. This, however, contradicts to the boundedness of $g(t)$ in $[0, \infty)$. Therefore $g(t) = 0$ for all $t > 0$, that is, $f(t) = tf(1)$ for all $t > 0$. Obviously $f(0) = 0$. This proves (14). By definition $\lambda_1 \geqq 0$. Thus we proved that

$$P_1(t) = e^{-\lambda t} \lambda_1 t \tag{16}$$

for $t \geqq 0$ where $\lambda_1 \geqq 0$. Since

$$P_0(t) + P_1(t) = e^{-\lambda t}(1+\lambda_1 t) \leqq 1 \tag{17}$$

for all $t \geqq 0$, it follows that necessarily $\lambda_1 \leqq \lambda$.

Now we shall prove that condition (c) implies that $\lambda_1 = \lambda$. According to condition (c) in any finite interval $(0, t]$ the discontinuities of the sample functions of the process are jumps of magnitude 1 with probability 1 . This condition can be stated in the following way: If

$$(18) \qquad A_m = \{\nu(\tfrac{jt}{2^m}) - \nu(\tfrac{(j-1)t}{2^m}) \leq 1 \quad \text{for} \quad 1 \leq j \leq 2^m\}$$

for $m = 1, 2, \ldots,$ then

$$(19) \qquad \underset{\sim}{P}\{\sum_{m=1}^{\infty} A_m\} = \lim_{m \to \infty} \underset{\sim}{P}\{A_m\} = 1 \; ,$$

or

$$(20) \qquad \lim_{m \to \infty} [P_0(\tfrac{t}{2^m}) + P_1(\tfrac{t}{2^m})]^{2^m} = 1$$

for all $t \geq 0$. By (10) and (16) it follows from (20) that $e^{-\lambda t + \lambda_1 t} = 1$ for all $t \geq 0$, that is, $\lambda_1 = \lambda$. This proves (6) for $k = 1$.

Having proved that (6) is true for $k = 0$ and $k = 1$, by mathematical induction we can prove that (6) is true for all $k \geq 0$.

If $k \geq 2$, then

$$(21) \qquad 0 \leq \sum_{j=2}^{k} P_{k-j}(t) P_j(u) \leq \sum_{j=2}^{k} P_j(u) \leq 1 - P_0(u) - P_1(u)$$

for all $t \geq 0$ and $u \geq 0$. Since $1 - P_0(u) - P_1(u) = o(u)$ where $\lim_{u \to 0} o(u)/u = 0$, it follows from (7) that

$$(22) \qquad \frac{P_k(t+u) - P_k(t)}{u} = -P_k(t)\,\frac{1-P_0(u)}{u} + P_{k-1}(t)\,\frac{P_1(u)}{u} + \frac{o(u)}{u}$$

for $t \geq 0$, $u \geq 0$ and $k \geq 1$. If $u \to 0$ in (22) , then we have

$$\frac{dP_k(t)}{dt} = -\lambda P_k(t) + \lambda P_{k-1}(t) \tag{23}$$

for $t \geq 0$ and $k \geq 1$. If we multiply (23) by $e^{\lambda t}$, then we get

$$\frac{de^{\lambda t} P_k(t)}{dt} = \lambda e^{\lambda t} P_{k-1}(t) \tag{24}$$

for $t \geq 0$ and $k \geq 1$. Since $P_0(0) = 1$, therefore $P_k(0) = 0$ for $k \geq 1$ and by integrating (24) we obtain that

$$P_k(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda u} P_{k-1}(u)du \tag{25}$$

for $k \geq 1$ and $t \geq 0$. Starting from $P_0(t) = e^{-\lambda t}$ for $t \geq 0$ we can obtain $P_k(t)$ for every $k = 1,2,\ldots$ and $t \geq 0$ by (25). By mathematical induction it follows immediately that (6) is true if λ is a finite positive number. This completes the proof of the theorem.

Now we can define the notion of a homogeneous Poisson process.

Definition 1. *We say that a family of real random variables* $\{\nu(t), 0 \leq t < \infty\}$ *forms a homogeneous Poisson process with parameter* λ *where* λ *is a finite positive number, if for any* $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$ $(n = 2,3,\ldots)$ *the random variables* $\nu(t_1) - \nu(t_0)$, $\nu(t_2) - \nu(t_1), \ldots, \nu(t_n) - \nu(t_{n-1})$ *are mutually independent*, $P\{\nu(0) = 0\} = 1$, *and*

$$P\{\nu(u+t) - \nu(u) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \tag{26}$$

for all $t \geq 0$, $u \geq 0$ *and* $k = 0,1,2,\ldots$.

By Theorem 1 we can conclude that such a process exists, and if we exclude the trivial case when $P\{\nu(t) = 0\} = 1$ for all $t \geq 0$, then the conditions (a), (b), and (c) determine the distribution (26) up to the parameter λ.

The parameter λ has a simple probability interpretation. To see this let us calculate the expectation of $\nu(t)$. We have

$$(27) \qquad \underset{\sim}{E}\{\nu(t)\} = \sum_{k=0}^{\infty} k\, e^{-\lambda t} \frac{(\lambda t)^k}{k!} = \lambda t .$$

Accordingly $\underset{\sim}{E}\{\nu(t+1) - \nu(t)\} = \lambda$, that is, the expected number of events occurring in any interval $(t, t+1]$ of length 1 is just λ. For this reason we shall call λ the density of the process. The knowledge of this single parameter completely determines the finite dimensional distributions of a homogeneous Poisson process. In what follows we shall add various remarks to the notion of a homogeneous Poisson process.

First, we observe that condition (c) can be replaced by the following equivalent condition

$$(28) \qquad \lim_{t \to 0} \frac{\underset{\sim}{P}\{\nu(t) > 1\}}{t} = 0 .$$

For (20) holds if and only if

$$(29) \qquad \lim_{t \to 0} \frac{1-P_0(t) - P_1(t)}{t} = 0 .$$

Obviously, condition (c) could be replaced by any other condition which guarantees that in (16) $\lambda_1 = \lambda$. For example, if we exclude the trivial case when $\underset{\sim}{P}\{\nu(t) = 0\} = 1$ for all $t \geq 0$, then condition (c) can be replaced by

$$(30) \qquad \lim_{t \to 0} \frac{P_1(t)}{1-P_0(t)} = 1 .$$

If $\{\nu(t), 0 \leq t < \infty\}$ is a homogeneous Poisson process of density λ, then

$$(31)\qquad \underset{\sim}{P}\{\nu(t)=0\}=1-\lambda t+o(t)\ ,\ \underset{\sim}{P}\{\nu(t)=1\}=\lambda t+o(t)\ \text{ and }\ \underset{\sim}{P}\{\nu(t)>1\}=o(t)$$

where $\lim_{t \to 0} o(t)/t = 0$. Conversely, if instead of condition (c) we assume that $\underset{\sim}{P}\{\nu(t) = 1\} = \lambda t+o(t)$ where λ is a positive constant, and $\underset{\sim}{P}\{\nu(t) > 1\} = o(t)$, then these conditions together with (a) and (b) imply that $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ is a homogeneous Poisson process of density λ .

We can easily determine the moments of the distribution

$$(32)\qquad \underset{\sim}{P}\{\nu(t) = k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

where $k = 0,1,2,\ldots$ and $t \geq 0$. The r-th binomial moment of $\nu(t)$ is equal to

$$(33)\qquad \underset{\sim}{E}\left\{\binom{\nu(t)}{r}\right\} = \sum_{k=r}^{\infty}\binom{k}{r}e^{-\lambda t}\frac{(\lambda t)^k}{k!} = \frac{(\lambda t)^r}{r!}$$

for $r = 0,1,2,\ldots$ and the r-th moment of $\nu(t)$ is equal to

$$(34)\qquad \underset{\sim}{E}\{[\nu(t)]^r\} = \sum_{j=1}^{r}\mathfrak{S}_r^j\,(\lambda t)^j$$

for $r = 1,2,\ldots$ where $\mathfrak{S}_r^j$ $(j = 1,2,\ldots,r)$ are Stirling numbers of the second kind defined by

$$(35)\qquad \mathfrak{S}_r^j = \frac{1}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}i^r\ .$$

We note that the process $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ which we constructed in the proof of Theorem 1 is obviously a separable process. Conversely, if we suppose that $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ is a separable, homogeneous Poisson process, then with probability 1 its sample functions are nondecreasing step functions which increase only by jumps of magnitude 1 and which vanish at the origin.

Let $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ be a separable Poisson process of density λ. Denote by $\rho(S)$ the sum of all positive jumps $\nu(t+0) - \nu(t-0)$ for $t \in S$, that is, $\rho(S)$ is the number of events occurring in the set S. In 1953 E. Marczewski[136] proved that if S is a Borel subset of $[0, \infty)$, then $\rho(S)$ is a random variable for which

$$(36) \qquad \underset{\sim}{P}\{\rho(S) = k\} = e^{-\lambda\mu(S)} \frac{[\lambda\mu(S)]^k}{k!}$$

if $k = 0,1,2,\ldots$ and $\mu(S)$ is the Lebesgue measure of S. Furthermore, if $S_1, S_2,\ldots, S_n$ $(n = 2,3,\ldots)$ are disjoint Borel subsets of $[0, \infty)$, then $\rho(S_1), \rho(S_2),\ldots, \rho(S_n)$ are mutually independent random variables.

Let $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ be a point process for which $\underset{\sim}{P}\{\nu(0) = 0\} = 1$ and

$$(37) \qquad \underset{\sim}{P}\{\nu(u+t) - \nu(u) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

for $u \geq 0$, $t \geq 0$ and $k = 0,1,2,\ldots,$ and λ is a positive constant. By our definition, $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ is a Poisson process if and only if condition (a) is satisfied for every $n = 2,3,\ldots$. Actually when we deduced (37) we used condition (a) only in the particular case when $n = 2$. We needed condition (a) for every $n = 2,3,\ldots$ only in proving that (37) uniquely determines the probability measure $\underset{\sim}{P}\{A\}$ for all $A \in \mathcal{B}$.

The following problem arises naturally: Does there exist a point process $\{\nu(t), 0 \leq t < \infty\}$ for which (37) holds and condition (a) is not satisfied. The answer is affirmative. L. Shepp (see J. R. Goldman [124 pp. 778–779]) and P. A. P. Moran [145] constructed point processes $\{\nu(t), 0 \leq t < \infty\}$ for which (37) holds but condition (a) is not satisfied.

Let us suppose more generally that $\{\nu(t), 0 \leq t < \infty\}$ is a point process and if $\rho(S)$ is defined as above, then (36) holds for a class F of Borel subsets of $[0, \infty)$. How large should F be in order that (36) imply condition (a). If F is the class of intervals in $[0, \infty)$, then as we already mentioned condition (a) is not satisfied necessarily. A. Rényi [164] proved that if F is the class of the unions of a finite number of disjoint finite intervals in $[0, \infty)$, then (36) implies condition (a). See also P. M. Lee [130].

Next we shall prove a few basic theorems for homogeneous Poisson processes. These theorems have many useful applications in the theory of stochastic processes.

Some results of S. O. Rice [396 pp. 299-301] make it plausible the validity of the following theorem. See also J. L. Doob [30 pp. 400-401], C. Ryll-Nardzewski [169] and the author [178].

Theorem 2. *Let* $\{\nu(t), 0 \leq t < \infty\}$ *be a homogeneous Poisson process of density* λ. *Under the condition that in the interval* $(0, t]$ *exactly* n $(n = 1,2,\ldots)$ *events occur, the joint distribution of the occurrence times of these* n *events agrees with the joint distribution of the coordinates arranged in increasing order of magnitude of* n *random points distributed independently and uniformly in the interval* $(0, t]$.

Proof. The proof of this theorem is based on the following simple remarks.

Suppose that n random points are distributed in the interval $(0, t]$. Denote by $\tau_1, \tau_2, \ldots, \tau_n$ their coordinates arranged in increasing order of magnitude. Divide the interval $(0, t]$ into r subintervals by partition points $0 = t_0 < t_1 < \ldots < t_r = t$ and let $(n_1, n_2, \ldots, n_r)$ be a partition of n into nonnegative integers, that is, $n_1 + n_2 + \ldots + n_r = n$. Denote by $P_{n_1, n_2, \ldots, n_r}(t_1, t_2, \ldots, t_r)$ the probability that the interval $(t_{i-1}, t_i]$ contains exactly n_i points for $i = 1,2,\ldots,r$.

If we know the joint distribution function of the random variables $\tau_1, \tau_2, \ldots, \tau_n$, then the probabilities $P_{n_1, n_2, \ldots, n_r}(t_1, t_2, \ldots, t_r)$ are uniquely determined, and conversely if we know the probabilities $P_{n_1, n_2, \ldots, n_r}(t_1, t_2, \ldots, t_r)$ for all partitions of $(0, t]$ and n, then the joint distribution function of $\tau_1, \tau_2, \ldots, \tau_n$ is uniquely determined by these probabilities.

If we choose n points independently of each other in the interval $(0, t]$ and if the random points have a uniform distribution over $(0, t]$, then

$$(38)\quad P_{n_1,n_2,\ldots,n_r}(t_1,t_2,\ldots,t_r) = \frac{n!}{n_1!n_2!\ldots n_r!}\left(\frac{t_1 - t_0}{t}\right)^{n_1}\left(\frac{t_2 - t_1}{t}\right)^{n_2}\cdots\left(\frac{t_r - t_{r-1}}{t}\right)^{n_r}$$

for $0 = t_0 < t_1 < \ldots < t_r$ and $n_1 + n_2 + \ldots + n_r = n$.

Conversely, if (38) holds for all partitions of $(0, t]$ and n, then the joint distribution function of $\tau_1, \tau_2, \ldots, \tau_n$ agrees with the joint distribution function of the coordinates arranged in increasing order of n random points distributed independently and uniformly in the interval $(0, t]$.

Now to prove the theorem let $0 = t_0 < t_1 < \dots < t_r = t$ and $n_1 + n_2 + \dots + n_r = n$ where $r = 1,2,\dots$. Then we have

$$(39)\quad \begin{aligned} &P\{\nu(t_i) - \nu(t_{i-1}) = n_i \quad \text{for } i = 1,2,\dots,r \mid \nu(t) = n\} = \\ &= \frac{\prod_{i=1}^{r} P\{\nu(t_i) - \nu(t_{i-1}) = n_i\}}{P\{\nu(t) = n\}} = \frac{\prod_{i=1}^{r} e^{-\lambda(t_i - t_{i-1})} \frac{[\lambda(t_i - t_{i-1})]^{n_i}}{n_i!}}{e^{-\lambda t} \frac{(\lambda t)^n}{n!}} = \\ &= \frac{n!}{n_1! n_2! \dots n_r!} \left(\frac{t_1 - t_0}{t}\right)^{n_1} \left(\frac{t_2 - t_1}{t}\right)^{n_2} \dots \left(\frac{t_r - t_{r-1}}{t}\right)^{n_r}. \end{aligned}$$

Accordingly, (38) holds for the distribution of the n points in the Poisson process in $(0, t]$ and therefore the theorem is true.

Theorem 3. *Let $\{\nu(t), 0 \le t < \infty\}$ be a homogeneous Poisson process of density λ. Denote by $\tau_1, \tau_2, \dots, \tau_n, \dots$ the occurrence times of the successive events occurring in the time interval $[0, \infty)$. Let $\theta_k = \tau_k - \tau_{k-1}$ for $k = 1,2,\dots$ where $\tau_0 = 0$. The random variables $\theta_1, \theta_2, \dots, \theta_k, \dots$ are mutually independent and identically distributed with distribution function*

$$(40)\qquad P\{\theta_k \le x\} = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Proof. We shall prove that

$$(41)\qquad P\{\theta_1 > x_1, \theta_2 > x_2, \dots, \theta_k > x_k\} = e^{-\lambda(x_1 + x_2 + \dots + x_k)}$$

for $k = 1,2,\dots$ and $x_1 > 0$, $x_2 > 0, \dots, x_k > 0$.

In what follows we shall make use of the inequalities

$$(42) \qquad 1 - \frac{\lambda}{n} < \sum_{j=0}^{\infty} e^{-\frac{\lambda j}{n}} e^{-\frac{\lambda}{n}} \frac{\lambda}{n} < 1$$

which are valid for $n = 1,2,\ldots,$ and which follow from

$$(43) \qquad \sum_{j=1}^{\infty} e^{-\frac{\lambda j}{n}} \frac{\lambda}{n} < \lambda \int_0^{\infty} e^{-\lambda x} dx < \sum_{j=0}^{\infty} e^{-\frac{\lambda j}{n}} \frac{\lambda}{n} .$$

If n is sufficiently large then we have the inequalities

$$(44) \qquad e^{-\lambda(x_1+\ldots+x_k)}\left(1-\frac{\lambda}{n}\right)^{k-1} < \underset{\sim}{P}\{\theta_1 > x_1,\ldots,\theta_k > x_k\} < e^{-\lambda(x_1+\ldots+x_k)} e^{\frac{2\lambda(k-1)}{n}} .$$

To prove the first inequality let us place consecutive intervals of lengths x_1, j_1/n, $1/n, x_2$, j_2/n, $1/n,\ldots,x_{k-1}$, j_{k-1}/n, $1/n$, x_k on the interval $[0, \infty)$ starting at the origin. If for some $j_1 = 0,1,2,\ldots,j_2 = 0,1,2,\ldots,j_{k-1} = 0,1,2,\ldots$ one event occurs in each of the $k-1$ intervals of length $1/n$ and no event occurs in the remaining intervals, then this event implies that $\{\theta_1 > x_1, \theta_2 > x_2,\ldots, \theta_k > x_k\}$. If we calculate the appropriate probabilities and use (42), then we obtain that the first inequality in (44) is valid for $n \geq \lambda$.

To prove the second inequality in (44) let us place consecutive intervals of lengths x_1, j_1/n, $1/n$, x_2-2/n, j_2/n, $1/n,\ldots$, $x_{k-1}-2/n$, j_{k-1}/n, $1/n$, x_k-2/n, where $n > 2/x_i$ for $i = 1,2,\ldots,n$, on the interval $[0, \infty)$ starting at the origin. If $\{\theta_1 > x_1, \theta_2 > x_2,\ldots, \theta_k > x_k\}$, then this event implies that for some $j_1 = 0,1,2,\ldots$, $j_2 = 0,1,2,\ldots$, $j_{k-1} = 0,1,2,\ldots$ no event occurs in any of the intervals of lengths x_1, $x_2-2/n,\ldots$, x_k-2/n . By calculating the probability of this event, we obtain that the second inequality in (44) is valid for $n \geqq 2/x_i$ $(i = 1,2,\ldots,n)$.

If we let $n \to \infty$ in (44), then we obtain (41). From (41) it follows that $\theta_1, \theta_2,\ldots, \theta_k$ are mutually independent random variables for $k = 2,3,\ldots$ and each variable has the distribution function (40). If every $x_i \to 0$ $(i = 1,2,\ldots, k)$ in (41) except x_j , and $x_j = x > 0$, then we obtain that

$$(45) \qquad \underset{\sim}{P}\{\theta_j > x\} = e^{-\lambda x}$$

for $j = 1,2,\ldots, k$ and $x > 0$. Thus by (41) and (45) we obtain that

$$(46) \qquad \underset{\sim}{P}\{\theta_1 > x_1, \theta_2 > x_2,\ldots,\theta_k > x_k\} = \underset{\sim}{P}\{\theta_1 > x_1\}\underset{\sim}{P}\{\theta_2 > x_2\}\ldots\underset{\sim}{P}\{\theta_k > x_k\}$$

for $k = 1,2,\ldots$ and $x_1 > 0, x_2 > 0,\ldots, x_k > 0$. By (45) and (46) we can conclude that the theorem is true.

Theorem 3 makes it possible to define a homogeneous Poisson process in a constructive way. Let us suppose that $\theta_1, \theta_2,\ldots, \theta_k,\ldots$ is a sequence of mutually independent and identically distributed random variables with distribution function

$$(47) \qquad F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqq 0 , \\ 0 & \text{for } x < 0 , \end{cases}$$

where λ is a positive constant.

Define $\tau_0 = 0$ and $\tau_k = \theta_1 + \theta_2 + \ldots + \theta_k$ for $k = 1,2,\ldots$. For every $t \geqq 0$ let $\nu(t)$ be a random variable which takes on only nonnegative integers and satisfies the relation

$$(48) \qquad \{\nu(t) \geq k\} = \{\tau_k \leq t\}$$

for all $t \geq 0$ and $k = 0,1,2,\ldots$.

By this definition the family of random variables $\{\nu(t) \,,\, 0 \leq t < \infty\}$ forms a Poisson process of density λ . This fact can easily be proved by using the following characteristic property of the exponential distribution function. If θ is a random variable for which $\mathbf{P}\{\theta \leq x\} = F(x)$ is given by (47), then for any $u \geq 0$ and $x \geq 0$ we have

$$(49) \qquad \mathbf{P}\{\theta \leq u+x \mid \theta > u\} = \frac{\mathbf{P}\{u < \theta \leq u+x\}}{\mathbf{P}\{\theta > u\}} = \frac{F(u+x) - F(u)}{1 - F(u)} = F(x) \,,$$

that is, the conditional probability (49) does not depend on u .

The possibility of the above constructive definition of the Poisson process was essentially observed in 1911 by H. Bateman [97].

The next two theorems deal with the superposition and decomposition of Poisson processes.

Theorem 4. *Let $\{\nu_i(t) \,,\, 0 \leq t < \infty\}$ $(i = 1,2,\ldots,r)$ be mutually independent Poisson processes with densities λ_i $(i = 1,2,\ldots,r)$. Let $\nu(t) = \nu_1(t) + \nu_2(t) + \ldots + \nu_r(t)$ for $t \geq 0$. Then $\{\nu(t) \,,\, 0 \leq t < \infty\}$ is a Poisson process of density $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_r$.*

Proof. Obviously the point process $\{\nu(t) \,,\, 0 \leq t < \infty\}$ satisfies conditions (a) and (b). Since

(50)
$$\mathbf{P}\{\nu(t) = k\} = \sum_{k_1+k_2+\dots+k_r=k} \mathbf{P}\{\nu_1(t) = k_1\}\mathbf{P}\{\nu_2(t) = k_2\}\dots\mathbf{P}\{\nu_r(t) = k_r\}$$
$$= e^{-\lambda t} \sum_{k_1+k_2+\dots+k_r=k} \frac{(\lambda_1 t)^{k_1}(\lambda_2 t)^{k_2}\dots(\lambda_r t)^{k_r}}{k_1!\, k_2!\dots k_r!} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

for $k = 0,1,2,\dots$ and $t \geq 0$, therefore we can conclude that $\{\nu(t),\ 0 \leq t < \infty\}$ is a Poisson process of density λ.

Theorem 5. Let $\{\nu(t),\ 0 \leq t < \infty\}$ be a Poisson process of density λ. Independently of each other let us mark each event in the process by one of the numbers $1,2,\dots,r$. Let p_i $(i = 1,2,\dots,r)$ be the probability that an event is marked by i where $p_i \geq 0$ and $p_1+p_2+\dots+p_r = 1$. Denote by $\nu_i(t)$ $(i = 1,2,\dots,r)$ the number of events marked by i and occurring in the interval $(0, t]$. Then $\{\nu_i(t),\ 0 \leq t < \infty\}$ is a Poisson process of density $\lambda_i = \lambda p_i$ and the processes $\{\nu_i(t),\ 0 \leq t < \infty\}$ $(i = 1,2,\dots,r)$ are mutually independent.

Proof. Obviously each point process $\{\nu_i(t),\ 0 \leq t < \infty\}$ satisfies conditions (a) and (b) and by Theorem 2 we obtain that

(51)
$$\mathbf{P}\{\nu_i(t) = k\} = \sum_{n=k}^{\infty} \mathbf{P}\{\nu(t) = n\} \binom{n}{k} p_i^k (1-p_i)^{n-k} =$$
$$= \frac{e^{-\lambda t}}{k!} \sum_{n=k}^{\infty} \frac{(\lambda p_i)^k (\lambda-\lambda p_i)^{n-k}}{(n-k)!} = e^{-\lambda_i t}\frac{(\lambda_i t)^k}{k!}$$

for $k = 0,1,2,\dots$ and $t \geq 0$. Consequently $\{\nu_i(t),\ 0 \leq t < \infty\}$ is a Poisson process of density λ_i for each $i = 1,2,\dots,r$.

If $0 = t_0 < t_1 < \ldots < t_n$ where $n = 2,3,\ldots$, then for $j = 1,2,\ldots,n$ the n sets of random variables $\{\nu_i(t_j) - \nu_i(t_{j-1})$ for $i = 1,2,\ldots,r\}$ are clearly mutually independent. Furthermore, within each set all the r random variables are mutually independent because for $u \geq 0$, $t \geq 0$ and $k_i = 0,1,2,\ldots$ $(i = 1,2,\ldots,r)$ we have

$$\underset{\sim}{P}\{\nu_i(u+t) - \nu_i(u) = k_i \text{ for } i = 1,2,\ldots,r\} =$$

(52)
$$= \underset{\sim}{P}\{\nu(u+t) - \nu(u) = k_1+k_2+\ldots+k_r\} \frac{(k_1+k_2+\ldots+k_r)!}{k_1!\,k_2!\ldots k_r!} p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} =$$
$$= \prod_{i=1}^{r} e^{-\lambda_i t} \frac{(\lambda_i t)^{k_i}}{k_i!} = \prod_{i=1}^{r} \underset{\sim}{P}\{\nu_i(u+t) - \nu_i(u) = k_i\} .$$

From the above facts it follows easily that the processes $\{\nu_i(t),\ 0 \leq t < \infty\}$ $(i = 1,2,\ldots,r)$ are mutually independent.

The following simple combinatorial result for Poisson processes has many important applications.

<u>Theorem 6.</u> <u>Let</u> $\{\nu(t),\ 0 \leq t < \infty\}$ <u>be a separable Poisson process of density</u> λ. <u>Then we have</u>

(53)
$$\underset{\sim}{P}\{\nu(u) \leq u \text{ for } 0 \leq u \leq t \mid \nu(t) = k\} = [1 - \frac{k}{t}]^+$$

<u>for</u> $k = 0,1,2,\ldots$ <u>and</u> $t > 0$ <u>where</u> $[x]^+ = \max(0, x)$.

<u>Proof.</u> Let us define $\nu_i = \nu(i) - \nu(i-1)$ for $i = 1,2,\ldots$. Then $\{\nu_i\}$ are mutually independent and identically distributed random variables taking on

nonnegative integers only.

If $k > t$, then (53) is obviously 0. If $k \leqq t$, then we have

$$\text{(54)}\quad \underset{\sim}{P}\{\nu(u) \leqq u \text{ for } 0 \leqq u \leqq t \mid \nu(t) = k\} = \underset{\sim}{P}\{\nu_1+\dots+\nu_r < r \text{ for } r = 1,2,\dots,k \mid \nu(t) = k\}.$$

By Lemma 20.2 we have

$$\text{(55)}\quad \underset{\sim}{P}\{\nu_1+\dots+\nu_r < r \text{ for } r = 1,2,\dots,k \mid \nu_1+\dots+\nu_k = j\} = [1-\tfrac{j}{k}]^+$$

for $j = 0,1,2,\dots$. Hence if $k \leqq t$, then

$$\text{(56)}\quad \begin{aligned} \underset{\sim}{P}\{\nu(u) \leqq u \text{ for } 0 \leqq u \leqq t \mid \nu(t) = k\} &= \sum_{j=0}^{k} (1-\tfrac{j}{k}) \underset{\sim}{P}\{\nu(k) = j \mid \nu(t) = k\} \\ &= \sum_{j=0}^{k} (1-\tfrac{j}{k}) \binom{k}{j} (\tfrac{k}{t})^j (1-\tfrac{k}{t})^{k-j} = 1-\tfrac{k}{t}. \end{aligned}$$

This proves (53) for $0 \leqq k \leqq t$ and $t > 0$.

We note that

$$\text{(57)}\quad \underset{\sim}{P}\{\nu(u) \leqq u \text{ for } 0 \leqq u \leqq t\} = \underset{\sim}{P}\{\nu(t) \leqq t\} - \lambda \underset{\sim}{P}\{\nu(t) \leqq t-1\}$$

for $t > 0$. For by (53)

$$\text{(58)}\quad \underset{\sim}{P}\{\nu(u) \leqq u \text{ for } 0 \leqq u \leqq t\} = \sum_{k=0}^{[t]} (1-\tfrac{k}{t}) \underset{\sim}{P}\{\nu(t) = k\}.$$

If we take into consideration that

$$(59) \qquad \underset{\sim}{P}\{\nu(t) = k\} = \frac{\lambda t}{k} \underset{\sim}{P}\{\nu(t) = k-1\}$$

for $k = 1,2,\ldots,$ then (58) reduces to (57).

We can define more general Poisson processes than the homogeneous Poisson process discussed previously. In what follows we shall mention briefly nonhomogeneous and abstract Poisson processes.

First, let us consider nonhomogeneous Poisson processes. (see A. Rényi [161] and C. Ryll-Nardzewski [168].) We can prove that if $\{\nu(t),\ 0 \leq t < \infty\}$ is the most general point process which satisfies the conditions (a) and (c) and furthermore

$$(60) \qquad \underset{\sim}{P}\{\nu(t) - \nu(t-0) = 0\} = 1$$

for all $t > 0$, then there exists a continuous, nondecreasing function $\Lambda(t)$ $(0 \leq t < \infty)$ with $\Lambda(0) = 0$ such that

$$(61) \qquad \underset{\sim}{P}\{\nu(t) - \nu(u) = k\} = e^{-[\Lambda(t)-\Lambda(u)]} \frac{[\Lambda(t)-\Lambda(u)]^k}{k!}$$

for $0 \leq u \leq t$ and $k = 0,1,2,\ldots$. Then $\underset{\sim}{E}\{\nu(t)\} = \Lambda(t)$ for $t \geq 0$.

If $\{\nu(t),\ 0 \leq t < \infty\}$ is a point process which satisfies the condition (a) and (61) with a function $\Lambda(t)$ $(0 \leq t < \infty)$ specified above, then we say that $\{\nu(t),\ 0 \leq t < \infty\}$ is a Poisson process for which $\underset{\sim}{E}\{\nu(t)\} = \Lambda(t)$ for $t \geq 0$. If $\Lambda(t)$ $(0 \leq t < \infty)$ is absolutely continuous, that is, if it can be represented in the form

(62) $$\Lambda(t) = \int_0^t \lambda(u)du$$

for $t \geq 0$, where $\lambda(u)$ is a nonnegative and integrable function of u, then we say that $\{\nu(t), 0 \leq t < \infty\}$ is a Poisson process with density $\lambda(t)$ for $t \geq 0$.

If $\Lambda(t) = \lambda t$ for $t \geq 0$ where λ is a positive constant, then $\{\nu(t), 0 \leq t < \infty\}$ reduces to a homogeneous Poisson process of density λ. If $\Lambda(t)$ is not a linear function of t, then we say that $\{\nu(t), 0 \leq t < \infty\}$ is a nonhomogeneous Poisson process.

Most of the results proved for homogeneous Poisson processes can easily be extended to the general case which includes both homogeneous and non-homogeneous Poisson processes.

Theorem 7. Let $\{\nu(t), 0 \leq t < \infty\}$ be a general Poisson process for which $E\{\nu(t)\} = \Lambda(t)$ for $t \geq 0$. Under the conditions that $\Lambda(t) > 0$ and $\nu(t) = n$ $(n = 1,2,\ldots)$ the joint distribution of the coordinates of the n random points in $(0, t]$ is the same as the joint distribution of the coordinates arranged in increasing order of n random points distributed independently of each other in the interval $(0, t]$ in such a way that for each point $\Lambda(x)/\Lambda(t)$ is the probability that it lies in the interval $(0, x]$ where $0 \leq x \leq t$.

Proof. If we replace the uniform distribution function by $\Lambda(x)/\Lambda(t)$ in the interval $x \in (0, t]$ in the proof of Theorem 2, then we obtain Theorem 7. See also the author [178].

Theorem 3 has an essentially different form for nonhomogeneous Poisson processes. See J. Mycielski [147].

Theorem 8. *Let* $\{\nu_i(t), 0 \leq t < \infty\}$ $(i = 1,2,\dots,r)$ *be independent general Poisson processes for which* $E\{\nu_i(t)\} = \Lambda_i(t)$ for $t \geq 0$. *Let* $\nu(t) = \nu_1(t) + \nu_2(t) + \dots + \nu_r(t)$ *for* $t \geq 0$ *and* $\Lambda(t) = \Lambda_1(t) + \Lambda_2(t) + \dots + \Lambda_r(t)$ *for* $t \geq 0$. *Then* $\{\nu(t), 0 \leq t < \infty\}$ *is a general Poisson process for which* $E\{\nu(t)\} = \Lambda(t)$ *for* $t \geq 0$.

Proof. The proof of Theorem 4 can easily be extended to cover this more general case.

Theorem 9. *Let* $\{\nu(t), 0 \leq t < \infty\}$ *be a general Poisson process for which* $E\{\nu(t)\} = \Lambda(t)$ *if* $t \geq 0$. *Independently of each other let us mark each event in the process by one of the numbers* $1,2,\dots,r$. *Denote by* $p_i(t)$ $(i = 1,2,\dots,r)$ *the probability that an event is marked by* i *if it occurs at time* t. *We suppose that* $p_i(t) \geq 0$ *and* $p_1(t) + p_2(t) + \dots + p_r(t) = 1$ *for* $t \geq 0$. *Denote by* $\nu_i(t)$ *the number of events marked by* i *and occurring in the interval* $(0, t]$. *Then* $\{\nu_i(t), 0 \leq t < \infty\}$ $(i = 1,2,\dots, r)$ *are independent Poisson processes for which*

$$E\{\nu_i(t)\} = \Lambda_i(t) = \int_0^t p_i(u)\,d\Lambda(u) \tag{63}$$

for $t \geq 0$ *provided that the integral* (63) *exists*.

Proof. If instead of Theorem 2 we use Theorem 7 then the proof of this theorem follows on the same lines as the proof of Theorem 5. The only difference in the proofs is that p_i in (51) and (52) should be replaced by $\Lambda_i(t)/\Lambda(t)$. In particular, now we have

$$(64) \qquad \underset{\sim}{P}\{\nu_i(t) = k \mid \nu(t) = n\} = \binom{n}{k}\left[\frac{\Lambda_i(t)}{\Lambda(t)}\right]^k \left[1 - \frac{\Lambda_i(t)}{\Lambda(t)}\right]^{n-k}$$

for $0 \leq k \leq n$ and $n \geq 1$. Thus it follows that

$$(65) \qquad \underset{\sim}{P}\{\nu_i(t) = k\} = e^{-\Lambda_i(t)} \frac{[\Lambda_i(t)]^k}{k!}$$

for $k = 0,1,2,\ldots$ and $t \geq 0$.

Both homogeneous and nonhomogeneous Poisson processes can be defined for more general spaces than the real line. Instead of the real line we can consider Euclidean spaces, metric spaces or general abstract spaces. See A. Blanc-Lapierre and R. Fortet [100], and the author [179].

Let us consider a random point distribution in a metric space X . Denote by F the class of Borel subsets of X . For each $S \varepsilon F$ denote by $\rho(S)$ the number of random points in the set S . Then $\{\rho(S) , S \varepsilon F\}$ determines a point process on X .

If $\mu(S)$ is a measure, that is, a nonnegative and σ-additive set function, defined on F , then there exists a point process $\{\rho(S) , S \varepsilon F\}$ such that if $S \varepsilon F$ and $\mu(S) < \infty$, then $\rho(S)$ is a random variable with distribution

$$\mathop{P}_{\sim}\{\rho(S) = k\} = e^{-\mu(S)} \frac{[\mu(s)]^k}{k!} \tag{66}$$

where $k = 0,1,2.,,,$ and for any n $(n = 2,3,\ldots)$ disjoint sets $S_1, S_2,\ldots,S_n$ having finite measures and belonging to F, the random variables $\rho(S_1)$, $\rho(S_2),\ldots,$ $\rho(S_n)$ are independent. We say that $\{\rho(S)\ , S \in F\}$ is a Poisson point peocess on X. This process is completely characterized by the set function $\mathop{E}_{\sim}\{\rho(S)\} = \mu(S)$ defined for $S \in F$.

Theorems 2, 4, 5 or Theorems 7, 8, 9 have natural analogues also for the stochastic process $\{\rho(S), S \in F\}$.

Our next subject is the definition of compound Poisson processes. Before defining the notion of a general compound Poisson process we shall consider a simple but important particular case which can be obtained from the definition of a Poisson process by removing condition (c). The definition of this particular compound Poisson process is based on the following result.

<u>Theorem 10.</u> <u>If</u> $\nu(t)$ <u>denotes the number of events occurring in the time interval</u> $(0, t]$ <u>in a random point process and if</u> $\{\nu(t), 0 \leq t < \infty\}$ <u>satisfies</u> (a) <u>and</u> (b), <u>then there exist nonnegative constants</u> $\lambda_1, \lambda_2,\ldots,$ <u>and</u> λ <u>such that</u> $\lambda_1 + \lambda_2 + \ldots = \lambda$ <u>and</u>

$$\mathop{P}_{\sim}\{\nu(u+t)-\nu(u) = k\} = e^{-\lambda t} \sum_{j_1+2j_2+\ldots+kj_k=k} \frac{(\lambda_1 t)^{j_1}(\lambda_2 t)^{j_2}\ldots(\lambda_k t)^{j_k}}{j_1!\ j_2!\ \ldots\ j_k!} \tag{67}$$

<u>for</u> $u \geq 0$, $t \geq 0$ <u>and</u> $k = 0,1,2,\ldots$ <u>where the summation is extended to</u>

all those $k_i = 0,1,2,\ldots$ for which $j_1 + 2j_2 + \ldots + kj_k = k$.

Proof. This theorem is a direct generalization of Theorem 1 and in the proof we shall use the same notation as in the proof of Theorem 1. We can easily see that indeed there exists a probability space $(\Omega, \mathcal{B}, \mathbf{P})$ and a family of random variables $\{\nu(t), 0 \leq t < \infty\}$ for which conditions (a) and (b) and (67) are satisfied.

Now we shall prove that if

$$\mathbf{P}\{\nu(t) = k\} = P_k(t) , \tag{68}$$

then

$$P_k(t) = e^{-\lambda t} \sum_{j_1+2j_2+\ldots+kj_k=k} \frac{(\lambda_1 t)^{j_1} (\lambda_2 t)^{j_2} \ldots (\lambda_k t)^{j_k}}{j_1!\, j_2! \ldots j_k!} \tag{69}$$

for $t \geq 0$ and $k = 0,1,2,\ldots$ where $\lambda_1, \lambda_2, \ldots,$ and λ are nonnegative constants and $\lambda_1 + \lambda_2 + \ldots = \lambda$.

As we have seen in the proof of Theorem 1 the probabilities $\{P_k(t)\}$ satisfy the following equation

$$P_k(t+u) = \sum_{j=0}^{k} P_{k-j}(t) P_j(u) \tag{70}$$

for $t \geq 0$, $u \geq 0$ and $k = 0,1,\ldots$.

From (70) it follows that either $P_0(t) = 1$ for all $t \geq 0$, or $P_0(t) = 0$ for all $t > 0$, or

$$P_0(t) = e^{-\lambda t} \tag{71}$$

for $t \geqq 0$ where λ is a finite positive number.

If $P_0(t) = 1$ for all $t \geqq 0$, then by (70) $P_k(t) = 0$ for all $t \geqq 0$ and $k = 1,2,\ldots$. This corresponds to (67) with $\lambda = 0$.

If $P_0(t) = 0$ for all $t \geqq 0$, then by (70) $P_k(t) = 0$ for all $t \geqq 0$ and $k = 1,2,\ldots$. This case is meaningless and should be excluded. This corresponds to (67) with $\lambda = \infty$.

It remains to prove (67) in the case where $P_0(t)$ is given by (71) with a finite positive λ.

Now we shall prove by mathematical induction that the probability $P_k(t)$ is given by (69) for $k = 0,1,2,\ldots$ where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are non-negative constants for which $\lambda_1 + \lambda_2 + \ldots + \lambda_k \leqq \lambda$.

Let us suppose that (69) is true for $0,1,\ldots,k$ where $k \geqq 1$. Then we have

(72) $$\lim_{t \to 0} \frac{1-P_0(t)}{t} = \lambda$$

and

(73) $$\lim_{t \to 0} \frac{P_i(t)}{t} = \lambda_i$$

for $i = 1,2,\ldots,k$. Define

$$f_{k+1}(t) = e^{\lambda t}P_{k+1}(t) - \sum_{j_1+2j_2+\dots+kj_k=k+1} \frac{(\lambda_1 t)^{j_1}(\lambda_2 t)^{j_2} \dots (\lambda_k t)^{j_k}}{j_1!\, j_2! \dots j_k!} \tag{74}$$

for $k = 0,1,\dots$. Then by (70) we obtain that

$$f_{k+1}(t+u) = f_k(t) + f_k(u) \tag{75}$$

for $t \geqq 0$ and $u \geqq 0$. Since $f_{k+1}(t)$ is bounded in the interval $[0,1]$, it follows that

$$f_{k+1}(t) = \lambda_{k+1}\, t \tag{76}$$

for $t \geqq 0$ where

$$\lambda_{k+1} = \lim_{t \to 0} \frac{P_{k+1}(t)}{t} . \tag{77}$$

The constant λ_{k+1} is nonnegative, and since $P_1(t)+\dots+ P_{k+1}(t) \leqq 1-P_0(t)$, it follows that $\lambda_1+\dots+ \lambda_{k+1} \leqq \lambda$.

Since (69) is true for $k = 0$, it follows by mathematical induction that (69) is true for every $k = 0,1,2,\dots$ and

$$\lim_{t \to 0} \frac{P_k(t)}{t} = \lambda_k \tag{78}$$

for $k = 1,2,\dots$. If we divide the equation

$$\sum_{k=1}^{\infty} P_k(t) = 1-P_0(t) \tag{79}$$

by t and let $t \to 0$, then we obtain that

$$\sum_{k=1}^{\infty} \lambda_k \leq \lambda . \tag{80}$$

Since (79) holds for all $t \geq 0$, therefore in (80) we have equality. This completes the proof of the theorem.

We note that if $\lambda_1 = \lambda$, then necessarily $\lambda_k = 0$ for $k > 1$, and in this case Theorem 10 reduces to Theorem 1.

We say that a family of real random variables $\{\nu(t), 0 \leq t < \infty\}$ forms a homogeneous compound Poisson point process if $P\{\nu(0) = 0\} = 1$, for any $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$ $(n = 2,3,\dots)$ the random variables $\nu(t_1)-\nu(t_0)$, $\nu(t_2)-\nu(t_1),\dots,\nu(t_n)-\nu(t_{n-1})$ are mutually independent and $P\{\nu(u+t)-\nu(u) = k\} = P_k(t)$ is given by (67) for $u \geq 0$, $t \geq 0$ and $k = 0,1,2,\dots$ where $\lambda_1, \lambda_2,\dots$ are nonnegative constants, and $\lambda = \lambda_1 + \lambda_2 + \dots$ is a finite positive constant.

In the case of Poisson processes we assumed that in any finite interval events occur singly with probability one. In the case of compound Poisson processes we allow the occurrence of multiple events too.

For the definition of compound Poisson point processes we refer to M. Fujiwara [121], J. M. Whittaker [183], and L. Jánossy, A. Rényi and J. Aczél [126].

We note that if

$$\nu(t) = \sum_{r=1}^{\infty} r\nu_r(t) \tag{81}$$

for $t \geqq 0$ where $\{\nu_r(t), 0 \leqq t < \infty\}$ $(r = 1,2,\ldots)$ are mutually independent Poisson processes with densities λ_r $(r = 1,2,\ldots)$ where λ_r $(r = 1,2,\ldots)$ are nonnegative constants with sum $\lambda_1 + \lambda_2 + \ldots = \lambda$ where λ is a finite positive number, then $\{\nu(t), 0 \leqq t < \infty\}$ is a homogeneous compound Poisson point process for which (67) holds.

The converse of the above statement is also true. This is the content of the next theorem.

Theorem 11. *If $\{\nu(t), 0 \leqq t < \infty\}$ is a homogeneous compound Poisson point process for which* (67) *holds with a finite positive* λ *and* $\nu_r(t)$ *denotes the number of jumps of magnitude* r *occurring in the interval* $(0,t]$ *in the process* $\{\nu(t), 0 \leqq t < \infty\}$, *then* $\{\nu_r(t), 0 \leqq t < \infty\}$ $(r = 1,2,\ldots)$ *are mutually independent Poisson processes with densities* λ_r $(r = 1,2,\ldots)$.

Proof. If $0 = t_0 < t_1 < \ldots < t_n$ where $n = 2,3,\ldots$, then for $j = 1,2,\ldots, n$ the n sets of random variables $\{\nu_r(t_j) - \nu_r(t_{j-1})$ for $r = 1,2,\ldots\}$ are clearly mutually independent. Furthermore, within each set all the random variables are mutually independent because for $u \geqq 0$, $t \geqq 0$, $k_r = 0,1,2,\ldots$ $(r = 1,2,\ldots)$ and $m = 1,2,\ldots$ we have

$$P\{\nu_r(u+t)-\nu_r(u) = k_r \quad \text{for} \quad r = 1,2,\dots,m\} =$$

(82)

$$= \lim_{n\to\infty} \frac{n!}{k_1!k_2!\dots k_m!(n-k_1-\dots-k_m)!} [P_1(\tfrac{t}{n})]^{k_1} [P_2(\tfrac{t}{n})]^{k_2} \dots$$

$$\dots [P_m(\tfrac{t}{n})]^{k_m} [1-P_1(\tfrac{t}{n}) - \dots - P_m(\tfrac{t}{n})]^{n-k_1-\dots-k_m} =$$

$$= e^{-(\lambda_1+\dots+\lambda_m)t} \frac{(\lambda_1 t)^{k_1} \dots (\lambda_m t)^{k_m}}{k_1! \dots k_m!} .$$

From the above facts it follows easily that the processes $\{\nu_r(t) , 0 \leq t < \infty\}$ $(r = 1,2,\dots)$ are mutually independent Poisson processes with densities λ_r $(r = 1,2,\dots)$.

We note that Theorem 6 holds unchangeably for homogeneous compound Poisson point processes.

Similarly to the Poisson processes we can define more general compound Poisson point processes than the homogeneous compound Poisson point process discussed previously. Thus we can define nonhomogeneous and abstract compound Poisson point processes.

The notion of a compound Poisson point process leads in a natural way to the definition of a general compound Poisson process.

Definition 2. Let $\{\nu(t) , 0 \leq t < \infty\}$ be a Poisson process of density λ . Let $\chi_1, \chi_2,\dots, \chi_i,\dots$ be mutually independent and identically distributed real random variables which are independent of the process $\{\nu(t) , 0 \leq t < \infty\}$. Let us define

(83) $$\chi(t) = \sum_{1 \leq i \leq \nu(t)} \chi_i$$

<u>for</u> $t \geq 0$. <u>We say that</u> $\{\chi(t), 0 \leq t < \infty\}$ <u>is a homogeneous compound Poisson process.</u>

If, in particular, $P\{\chi_i = 1\} = 1$, then the above definition reduces to the definition of a homogeneous Poisson process, and if $P\{\chi_i = r\} = \lambda_r/\lambda$ $(r = 1,2,\ldots)$ where $\lambda_1 + \lambda_2 + \ldots = \lambda$ is a finite positive number, then the above definition reduces to the definition of a homogeneous compound Poisson point process.

A homogeneous compound Poisson process $\{\chi(t), 0 \leq t < \infty\}$ satisfies the following properties:

(i) Homogenity. The probability $P\{\chi(u+t)-\chi(u) \leq x\}$ where $u \geq 0$, $t \geq 0$ does not depend on u .

(ii) Independent increments. For any $0 \leq t_0 < t_1 < \ldots < t_n$ where $n = 2,3,\ldots,$ the random variables $\chi(t_j)-\chi(t_{j-1})$ for $j = 1,2,\ldots,n$ are mutually independent.

(iii) Finite jump density. With probability one the limits $\chi(u+0)$ and $\chi(u-0)$ exist for all $u \geq 0$. If $\nu^*(t)$ denotes the number of points u in the interval $(0, t]$ for which $\chi(u+0)-\chi(u-0) \neq 0$, then with probability one $\nu^*(t)$ is a finite random variable for every $t \geq 0$ and

(84) $$E\{\nu^*(t)\} = \lambda t[1-P\{\chi_1 = 0\}] < \infty .$$

(iv) We have $\underset{\sim}{P}\{\chi(0) = 0\} = 1$.

Conversely, if we suppose that $\{\chi(t), 0 \leqq t < \infty\}$ is a separable real stochastic process which satisfies conditions (i), (ii) and (iv), then with probability one the limits $\chi(u+0)$ and $\chi(u-0)$ exist for all $u \geqq 0$. Let us define $\nu^*(t)$ for $t \geqq 0$ as above. If in addition $\nu^*(t)$ is a finite random variable for which $\underset{\sim}{E}\{\nu^*(t)\} < \infty$, then $\{\chi(t), 0 \leqq t < \infty\}$ is a homogeneous compound Poisson process.

We note that if $\{\chi(t), 0 \leqq t < \infty\}$ is a separable compound Poisson process and $\nu^*(t, A)$ denotes the number of points u in the interval $(0, t]$ for which $\chi(u+0)-\chi(u-0) \in A$ where A is a linear Borel set, then $\{\nu^*(t, A), 0 \leqq t < \infty\}$ is a Poisson process. If $\chi(t)$ is defined by (83), then $\underset{\sim}{E}\{\nu(t, A)\} = \lambda\underset{\sim}{P}\{\chi_1 \in A\}$. If $A_1, A_2, \ldots, A_r$ are disjoint linear Borel sets, then $\{\nu^*(t, A_i), 0 \leqq t < \infty\}$ $(i = 1,2,\ldots,r)$ are mutually independent Poisson processes. These results can be deduced as particular cases of more general results of <u>I. I. Gikhman</u> and <u>A. V. Skorokhod</u> [44 pp. 255-282].

Let

$$\underset{\sim}{P}\{\chi_i \leqq x\} = H(x) \tag{85}$$

and denote by $H_n(x)$ $(n = 1,2,\ldots)$ the n-th iterated convolution of $H(x)$ with itself. Let $H_0(x) = 1$ for $x \geqq 0$ and $H_0(x) = 0$ for $x < 0$.

From (83) it follows that

(86) $$\underset{\sim}{P}\{\chi(t) \leq x\} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} H_n(x)$$

for $t \geq 0$ and all x .

Let

(87) $$\Psi(s) = \int_{-\infty}^{\infty} e^{-sx} dH(x)$$

for $\mathrm{Re}(s) = 0$. Then

(88) $$\underset{\sim}{E}\{e^{-s\chi(t)}\} = e^{-\lambda t[1-\Psi(s)]}$$

for $t \geq 0$ and $\mathrm{Re}(s) = 0$.

Compound Poisson processes were encountered as early as in 1903 by F. Lundberg [134], in 1929 by B. De Finetti [413] and in 1933 by A. Ya. Khintchine [128].

Nonhomogeneous and abstract compound Poisson processes can also be introduced in a natural way.

We shall close this section by mentioning two useful theorems for homogeneous compound Poisson processes.

Theorem 12. *Let* $\{\chi(t), 0 \leq t < \infty\}$ *be a compound Poisson process defined by* (83). *If* $\underset{\sim}{E}\{\chi_i\} = a$ *exists and if* $\underset{\sim}{\mathrm{Var}}\{\chi_i\} = \sigma^2$ *is a finite positive number, then*

(89) $$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\chi(t) - \lambda a t}{\sqrt{\lambda(a^2 + \sigma^2)t}} \leq x \right\} = \Phi(x)$$

where $\Phi(x)$ is the normal distribution function.

Proof. Let

$$(90) \qquad \chi^*(t) = \frac{\chi(t) - \lambda at}{\sqrt{\lambda(a^2+\sigma^2)t}}$$

for $t > 0$. If we take into consideration that $\psi(s) = 1-sa+s^2(a^2+\sigma^2)/2+o(s)$ as $s \to 0$, then by (88) we get that

$$(91) \qquad \lim_{t\to\infty} E\{e^{-s\chi^*(t)}\} = e^{s^2/2}$$

for $Re(s) = 0$. Hence (89) follows by Theorem 41.9.

We can also use Theorem 45.2 in proving (89).

If $H(x)$ belongs to the domain of attraction of a nondegenerate stable distribution function, then by suitable normalization $\chi(t)$ also has a nondegenerate limiting distribution which can be found either by Theorem 45.2 or by using the same method which we used in proving Theorem 45.2.

The next theorem is concerned with a homogeneous compound Poisson process which has only nonnegative jumps with probability one.

We need the following auxiliary theorem.

Lemma 1. Let $\chi_1, \chi_2, \ldots, \chi_n$ be mutually independent nonnegative real random variables. Let $\tau_1, \tau_2, \ldots, \tau_n$ be the coordinates arranged in increasing order of magnitude of n points distributed uniformly and independently of each other in the interval $(0, t]$. If $\{\chi_i\}$ and $\{\tau_i\}$ are also independent, then

$$(92)\qquad \underset{\sim}{P}\{\chi_1+\dots+\chi_i \leqq \tau_i \text{ for } i=1,2,\dots,n \mid \chi_1+\dots+\chi_n = y\} = \begin{cases} 1-\frac{y}{t} & \underline{\text{for}}\ 0\leqq y \leqq t\,, \\ 0 & \underline{\text{for}}\ y > t\,, \end{cases}$$

<u>where the conditional probability is defined up to an equivalence.</u>

<u>Proof</u>. We prove (92) by mathematical induction. If $n = 1$, then (92) is obviously true. Let us suppose that (92) is true for $n-1$ where $n = 2,3,\dots$. We shall prove that it is true for n too. Thus it follows that (92) is true for every $n = 1,2,\dots$.

If $y > t$, then (92) is trivially true. Let $0 \leqq y \leqq t$. If $\tau_n = u$ where $0 \leqq u \leqq t$, then under this condition the random variables $\tau_1, \tau_2, \dots, \tau_{n-1}$ can be considered as the coordinates arranged in increasing order of $n-1$ points distributed uniformly and independently of each other in the interval $(0, u]$. Now by assumption

$$(93)\qquad \underset{\sim}{P}\{\chi_1+\dots+\chi_i \leqq \tau_i \text{ for } i=1,\dots,n \mid \chi_1+\dots+\chi_{n-1} = z,\ \chi_1+\dots+\chi_n = y,\ \tau_n = u\} = \begin{cases} 1-\frac{z}{u} & \text{for } 0\leqq z\leqq u \text{ and } y \leqq u \leqq t\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Since

$$(94)\qquad \underset{\sim}{E}\{\chi_1+\dots+\chi_{n-1} \mid \chi_1+\dots+\chi_n = y\} = \frac{(n-1)y}{n}\,,$$

therefore by (93) we obtain that

$$\underset{\sim}{P}\{\chi_1+\dots+\chi_i \leq \tau_i \quad \text{for} \quad i = 1,\dots,n \mid \chi_1+\dots+\chi_n = y \ , \ \tau_n = u\} =$$

$$(95) \qquad = \begin{cases} 1 - \dfrac{(n-1)y}{nu} & \text{for} \quad 0 \leq y \leq u \leq t \ , \\ \text{otherwise} \ . \end{cases}$$

Since

$$(96) \qquad \underset{\sim}{P}\{\tau_n \leq u\} = \left(\frac{u}{t}\right)^n \quad \text{for} \quad 0 \leq u \leq t \ ,$$

by (95) we get finally that

$$\underset{\sim}{P}\{\chi_1+\dots+\chi_i \leq \tau_i \quad \text{for} \quad i = 1,\dots,n \mid \chi_1+\dots+\chi_n = y\} =$$

$$(97) \qquad = n \int_0^t \left(1 - \frac{(n-1)y}{nu}\right)\left(\frac{u}{t}\right)^{n-1} \frac{du}{t} = 1 - \frac{y}{t}$$

for $0 \leq y \leq t$. Hence we can conclude that (92) is valid for all $n = 1,2,\dots$.

We note that Lemma 1 remains valid unchangeably if assume only that $\chi_1, \chi_2, \dots, \chi_n$ are interchangeable nonnegative real random variables which are independent of $\{\tau_i\}$. For the proof see reference [83] .

<u>Theorem 13</u>. <u>Let</u> $\{\chi(t) \ , \ 0 \leq t < \infty\}$ <u>be a separable homogeneous compound Poisson process which has only nonnegative jumps with probability one. Then we have</u>

$$(98) \qquad \underset{\sim}{P}\{\chi(u) \leq u \quad \text{for} \quad 0 \leq u \leq t \mid \chi(t) = y\} = \begin{cases} 1 - \dfrac{y}{t} & \underline{\text{for}} \quad 0 \leq y \leq t \ , \\ 0 & \underline{\text{otherwise}}, \end{cases}$$

<u>where the conditional probability is defined up to an equivalence.</u>

Proof. Denote by $\nu(t)$ the number of jumps occurring in the interval $(0, t]$ in the process $\{\chi(t), 0 \leq t < \infty\}$. Then $\{\nu(t), 0 \leq t < \infty\}$ is a Poisson process. Denote by $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ the times when an event occurs in the Poisson process. If $n = 1,2,\ldots$, then by Theorem 2 and by Lemma 1 we can write that

$$P\{\chi(u) \leq u \text{ for } 0 \leq u \leq t \mid \chi(t) = y, \nu(t) = n\} =$$

$$(99) \quad = P\{\chi_1+\ldots+\chi_i \leq \tau_i \text{ for } i = 1,\ldots,n \mid \chi(t) = y, \nu(t) = n\} =$$

$$= \begin{cases} 1-\frac{y}{t} & \text{for } 0 \leq y \leq t, \\ 0 & \text{otherwise}. \end{cases}$$

If $n = 0$, then (99) is obvious. Since (99) does not depend on n, (98) follows immediately.

From (98) it follows that

$$(100) \quad P\{\chi(u) \leq u \text{ for } 0 \leq u \leq t\} = E\{[1-\frac{\chi(t)}{t}]^+\}$$

for $t > 0$.

49. RECURRENT AND COMPOUND RECURRENT PROCESSES.

Theorem 48.3 made it possible to give a constructive definition of a homogeneous Poisson process. This definition is given after the proof of Theorem 48.3, and it suggests the following generalization.

Definition 1. Let us suppose that $\theta_1, \theta_2, \ldots, \theta_k, \ldots$ is a sequence of mutually independent and identically distributed positive random variables with distribution function $P\{\theta_k \leq x\} = F(x)$. Define $\tau_0 = 0$ and $\tau_k = \theta_1 + \theta_2 + \ldots + \theta_k$ for $k = 1,2,\ldots$. For every $t \geq 0$ let $\nu(t)$ be a random variable which takes on only nonnegative integers and satisfies the relation

$$\{\nu(t) \geq k\} \equiv \{\tau_k \leq t\} \tag{1}$$

for all $t \geq 0$ and $k = 0,1,2,\ldots$. We say that $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ is a recurrent stochastic process. That is if in the time interval $(0, \infty)$ events occur at random, if $\nu(t)$ denotes the number of events occurring in the time interval $(0, t]$, and if the time differences between successive events are mutually independent and identically distributed positive random variables, then we say that $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ is a recurrent process.

If, in particular,

$$F(x) = \begin{cases} 1-e^{-\lambda x} & \text{for } x \geq 0\ , \\ 0 & \text{for } x < 0\ , \end{cases} \tag{2}$$

where λ is a positive constant, in the previous definition, then $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ reduces to a homogeneous Poisson process with density λ .

Let us introduce the following notation

$$\phi(s) = \int_0^\infty e^{-sx} dF(x) \tag{3}$$

for $\mathrm{Re}(s) \geq 0$,

$$a = \int_0^\infty x dF(x) \tag{4}$$

and if $a < \infty$, then let

$$\sigma^2 = \int_0^\infty (x-a)^2 dF(x)\ . \tag{5}$$

Denote by $F_n(x)$ the n-th iterated convolution of $F(x)$ with itself and let $F_0(x) = 1$ for $x \geq 0$ and $F_0(x) = 0$ for $x < 0$.

The distribution of $\nu(t)$ can be obtained by the following formula

$$\mathbf{P}\{\nu(t) \leq n\} = 1 - F_{n+1}(t) \tag{6}$$

for $t \geq 0$ and $n = 0,1,2,\ldots$. For we have

$$\mathbf{P}\{\nu(t) \leq n\} = \mathbf{P}\{\tau_{n+1} > t\} = \mathbf{P}\{\theta_1 + \ldots + \theta_{n+1} > t\} \tag{7}$$

for $t \geq 0$ and $n = 0,1,2,\ldots$.

The Laplace transform of $\mathbf{P}\{\nu(t) \leq n\}$ is given by

$$\int_0^\infty e^{-st}\, \mathbf{P}\{\nu(t) \leq n\}dt = \frac{1 - [\phi(s)]^{n+1}}{s} \tag{8}$$

for $\mathrm{Re}(s) > 0$. Knowing $\phi(s)$ we can obtain $\mathbf{P}\{\nu(t) \leq n\}$ by inversion from (8).

Let

$$b_r(t) = \mathbf{E}\{\binom{\nu(t)}{r}\} \tag{9}$$

be the r-th binomial moment of $\nu(t)$ for $r = 0,1,2,\ldots$.

The r-th binomial moment $b_r(t)$ $(r = 0,1,2,\ldots)$ is a nondecreasing function of t and is finite for every t . We have $b_0(t) \equiv 1$ and

$$b_r(t) = \sum_{n=r}^{\infty} \binom{n-1}{r-1} F_n(t) \tag{10}$$

for $r = 1,2,\ldots$. For if $r = 1,2,\ldots$, then

$$(11) \qquad b_r(t) = \sum_{n=r}^{\infty} \binom{n}{r} P\{\nu(t) = n\} = \sum_{n=r}^{\infty} \binom{n-1}{r-1} P\{\nu(t) \geq n\}$$

and (10) follows by (6).

If we take into consideration that for every $t > 0$ there is an s $(s = 1,2,\ldots)$ such that $F_s(t) < 1$ and further that $F_{s+n}(t) \leq F_s(t)F_n(t)$ for all $n = 0,1,2,\ldots$, then we obtain easily from (10) that $b_r(t) < \infty$ for all $t \geq 0$. Furthermore, we can easily see that for every $t \geq 0$ there exists a finite $C(t)$ such that

$$(12) \qquad b_r(t) \leq [C(t)]^r$$

for $r = 0,1,2,\ldots$.

Since

$$(13) \qquad \sum_{n=r}^{\infty} \binom{n-1}{r-1} z^n = \left(\frac{z}{1-z}\right)^r$$

for $|z| < 1$, therefore by (10) we obtain that

$$(14) \qquad \int_0^{\infty} e^{-st} db_r(t) = \left[\frac{\phi(s)}{1-\phi(s)}\right]^r$$

for $\mathrm{Re}(s) > 0$ and $r = 1,2,\ldots$. If $r = 0$, then (14) is trivially true. By (14) we can write also that

$$(15) \qquad \int_0^{\infty} e^{-st}\, db_r(t) = \left[\int_0^{\infty} e^{-st}\, db_1(t)\right]^r$$

for $r = 0,1,2,\ldots$.

From (15) we can draw an interesting conclusion. If $b_1(t) = \underset{\sim}{E}\{\nu(t)\}$ is known for all $t \geq 0$, then by (15) $b_r(t)$ is uniquely determined for all $t \geq 0$ and $r = 1,2,\ldots$. If $c(t) < 1$ in (12), then we can write down that

$$(16) \qquad \underset{\sim}{P}\{\nu(t) = k\} = \sum_{r=k}^{\infty} (-1)^{r-k}\binom{r}{k} b_r(t)$$

for $k = 0,1,2,\ldots$. If $C(t) < \infty$, then $\underset{\sim}{P}\{\nu(t) = k\}$ can be obtained by a similar formula given in reference [84] . That is, in the case of a recurrent process, the function $b_1(t) = \underset{\sim}{E}\{\nu(t)\}$ completely determines the distribution of $\nu(t)$ for all $t \geq 0$. This can also be seen by (8) and (14). If $r = 1$ in (14), then we obtain that

$$(17) \qquad \phi(s) = \frac{\int_0^\infty e^{-st}\, db_1(t)}{1 + \int_0^\infty e^{-st} db_1(t)}$$

for $\mathrm{Re}(s) > 0$, and knowing $\phi(s)$ the distribution of $\nu(t)$ can be obtained by (8). There are many examples for recurrent processes where it is easier to determine $\underset{\sim}{E}\{\nu(t)\}$ than $F(x)$, and in this case the above observations are very useful.

Let

$$(18) \qquad m_r(t) = \underset{\sim}{E}\{[\nu(t)]^r\}$$

for $r = 0,1,2,\ldots$, that is, $m_r(t)$ is the r-th moment of $\nu(t)$. We have

(19)
$$m_r(t) = \sum_{j=0}^{r} \mathfrak{S}_r^j \, j! \, b_j(t)$$

for $r = 0,1,2,\ldots$ where the numbers $\mathfrak{S}_r^j$ $(j = 0,1,\ldots,r)$ are Stirling numbers of the second kind. We have $\mathfrak{S}_0^0 = 1$, $\mathfrak{S}_r^0 = 0$ for $r = 1,2,\ldots$, and

(20)
$$\mathfrak{S}_r^j = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^r$$

for $1 \leq j \leq r$. (See Ch. Jordan [49 pp. 168–173].) Formula (19) follows immediately from the identity

(21)
$$x^r = \sum_{j=0}^{r} \mathfrak{S}_r^j \, j! \binom{x}{j}$$

which holds for $r = 0,1,2,\ldots$ and for all x.

Let us introduce the notation

(22)
$$m(t) = \mathbf{E}\{\nu(t)\} ,$$

that is $m(t) = m_1(t) = b_1(t)$ and

(23)
$$d(t) = \mathbf{Var}\{\nu(t)\} ,$$

that is, $d(t) = m_2(t) - [m_1(t)]^2 = 2b_2(t) + b_1(t) - [b_1(t)]^2$.

In what follows we are interested in studying the asymptotic distribution of $\nu(t)$ as $t \to \infty$ and the limiting behavior of $m(t)$ and $d(t)$ as $t \to \infty$.

If $F(x)$ belongs to the domain of attraction of a nondegenerate stable distribution function, then by suitable normalization τ_n has a nondegenerate limiting distribution as $n \to \infty$. In this case by (1) we can conclude that by suitable normalization $\nu(t)$ also has a nondegenerate limiting distribution as $t \to \infty$.

In finding the asymptotic distribution of $\nu(t)$ as $t \to \infty$ it will be convenient to extend the definition of τ_n $(n = 0,1,2,\ldots)$ to a continuous parameter in the following way

(24) $$\tau_u = \tau_n \quad \text{for } n-1 < u \leq n \qquad (n = 0,1,2,\ldots) .$$

Then by Theorem 44.6 and Theorem 44.8 we can conclude that if $F(x)$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(\alpha,1,c,0)$ where $0 < \alpha \nless 2$ and $c > 0$, then there exist two functions A_u and $B_u > 0$ where $\lim_{u \to \infty} B_u = \infty$ such that ≤ /

(25) $$\lim_{u \to \infty} \mathbf{P}\left\{ \frac{\tau_u - A_u}{B_u} \leq x \right\} = R(x) .$$

By Problem 46.12 we have

(26) $$B_u = u^{1/\alpha} \rho(u)$$

where $\lim_{u \to \infty} \frac{\rho(\omega u)}{\rho(u)} = 1$ for every $\omega > 0$.

If $a < \infty$ and $0 < \sigma^2 < \infty$, then by Theorem 44.6 $F(x)$ belongs to the domain of attraction of the normal distribution function $\Phi(x)$, and (25) holds with $R(x) = \Phi(x)$ $(\alpha=2,\ c=1/2)$, $A_u = au$, and $B_u = \sigma\sqrt{u}$.

If

$$\lim_{x \to \infty} \frac{x^2 \int_{|u| \geq x} dF(u)}{\int_{|u|<x} u^2 dF(u)} = 0 , \tag{27}$$

then by Theorem 44.6 $F(x)$ belongs to the domain of attraction of the normal distribution function $\Phi(x)$, and (25) holds with $R(x) = \Phi(x)$ $(\alpha=2, c=1/2)$, $A_u = au$, and if $\sigma^2 = \infty$, then $B_u > 0$ can be chosen in such a way that

$$\lim_{u \to \infty} \frac{u}{B_u^2} \int_{|x|<\varepsilon B_u} x^2 dF(x) = 1 \tag{28}$$

for some $\varepsilon > 0$.

If

$$\lim_{x \to \infty} \frac{1 - F(x)}{1-F(\omega x)} = \omega^\alpha \tag{29}$$

for every $\omega > 0$ where $0 < \alpha < 2$, then $F(x)$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(\alpha,1,c,0)$ where $c > 0$, and in (25) we can choose $B_u > 0$ in such a way that

$$\lim_{u \to \infty} u[1-F(B_u x)] = \begin{cases} \dfrac{2c\Gamma(\alpha)}{\pi x^\alpha} \sin \dfrac{\alpha\pi}{2} & \text{for } \alpha \neq 1 , \\ \dfrac{2c}{\pi x} & \text{for } \alpha = 1 \end{cases} \tag{30}$$

for $x > 0$, and $A_u = 0$ for $0 < \alpha < 1$, $A_u = au$ for $1 < \alpha < 2$, and

$$A_u = u \int_{|x|<\tau B_u} x dF(x) - \frac{2cB_u}{\pi} [\log \tau - (1-C)] \tag{31}$$

for $\alpha = 1$ where τ is an arbitrary positive number and $C = 0.577215\ldots$ is Euler's constant. We note that by Problem 46.12 we have

$$(32) \qquad \lim_{u \to \infty} \frac{A_{\omega u} - \omega A_u}{B_{\omega u}} = \frac{2c}{\pi} \log \omega$$

if $\alpha = 1$ for any $\omega > 0$.

By using the above results we can find the asymptotic distribution of $\nu(t)$ as $t \to \infty$ in each case.

By (1) we have

$$(33) \qquad \{\nu(t) \geqq u\} \equiv \{\tau_u \leqq t\}$$

for all $t \geqq 0$ and $u \geqq 0$.

<u>Theorem 1.</u> <u>If</u> $0 < \sigma^2 < \infty$, <u>then we have</u>

$$(34) \qquad \lim_{t \to \infty} \underset{\sim}{P}\left\{\frac{\nu(t) - \frac{t}{a}}{\sqrt{\frac{\sigma^2 t}{a^3}}} \leqq x\right\} = \Phi(x) .$$

<u>Proof.</u> In this case by the central limit theorem we have

$$(35) \qquad \lim_{u \to \infty} \underset{\sim}{P}\left\{\frac{\tau_u - au}{\sigma\sqrt{u}} \leqq x\right\} = \Phi(x)$$

for every x. If we write

$$(36) \qquad t = au + x\sigma\sqrt{u} ,$$

then by (33) and (35) it follows that

(37) $$\lim_{t \to \infty} P\{\nu(t) \geq u\} = \Phi(x)$$

where u can be determined by (36). For if $u \to \infty$, then $t \to \infty$ for any x. By (36) we can easily prove that

(38) $$\lim_{t \to \infty} \frac{u - \frac{t}{a}}{\frac{\sigma}{a}\sqrt{\frac{t}{a}}} = -x .$$

Thus by (37) and (38) we obtain that

(39) $$\lim_{t \to \infty} P\left\{ \frac{\nu(t) - \frac{t}{a}}{\frac{\sigma}{a}\sqrt{\frac{t}{a}}} \geq -x \right\} = \Phi(x)$$

for any x. Since $\Phi(-x) = 1-\Phi(x)$, therefore (39) implies (34).

Theorem 2. If (29) holds with $0 < \alpha < 1$, then

(40) $$\lim_{t \to \infty} P\{\nu(t)[1-F(t)] \leq x\} = 1-R\left(\left[\frac{2c\Gamma(\alpha)}{\pi x} \sin \frac{\alpha\pi}{2}\right]^{\frac{1}{\alpha}}\right)$$

for $x > 0$ where $R(x)$ is a stable distribution function of type $S(\alpha,1,c,0)$.

Proof. In this case we have

(41) $$\lim_{u \to \infty} P\left\{ \frac{\tau_u}{B_u} \leq x \right\} = R(x)$$

where $R(x) = 0$ for $x \leq 0$ and $B_u > 0$ satisfies (30) for any $x > 0$. If $x > 0$ and if we write

(42) $$t = B_u x ,$$

then by (33) and (41) it follows that

(43) $$\lim_{t \to \infty} P\{\nu(t) \geq u\} = R(x)$$

for $x > 0$ where u can be determined by (42). Now from (30) it follows that

(44) $$\lim_{t \to \infty} u[1-F(t)] = \frac{2c\Gamma(\alpha)}{\pi x^{\alpha}} \sin \frac{\alpha\pi}{2} .$$

Thus by (43) and (44) we obtain that

(45) $$\lim_{t \to \infty} P\{\nu(t)[1-F(t)] \geq \frac{2c\Gamma(\alpha)}{\pi x^{\alpha}} \sin \frac{\alpha\pi}{2}\} = R(x)$$

for $x > 0$. Hence (40) follows immediately. In (40) the dependence on c is only apparent.

<u>Note</u>. If

(46) $$\lim_{x \to \infty} x^{\alpha}[1-F(x)] = q$$

where $0 < \alpha < 1$ and $q > 0$, then Theorem 2 is applicable and by (40) we have

(47) $$\lim_{t \to \infty} P\{\frac{q\nu(t)}{t^{\alpha}} \leq x\} = 1-R([\frac{2c\Gamma(\alpha)}{\pi x} \sin \frac{\alpha\pi}{2}]^{\frac{1}{\alpha}})$$

for $x > 0$.

<u>Theorem 3</u>. <u>If</u> (29) <u>holds with</u> $1 < \alpha < 2$, <u>then</u>

(48) $$\lim_{t \to \infty} P\{ \frac{\nu(t) - \frac{t}{a}}{B_t a^{-(\alpha+1)/\alpha}} \leq x\} = 1-R(-x)$$

<u>for every</u> x <u>where</u> $R(x)$ <u>is a stable distribution function of type</u> $S(\alpha,1,c,0)$ <u>and</u> $B_t > 0$ <u>can be obtained by</u>

(49) $$\lim_{t\to\infty} t[1-F(B_t)] = \frac{2c\Gamma(\alpha)}{\pi}\sin\frac{\alpha\pi}{2} .$$

<u>Proof</u>. In this case we have

(50) $$\lim_{u\to\infty} \underset{\sim}{P}\{\frac{\tau_u - au}{B_u} \leq x\} = R(x)$$

for every x where $B_u > 0$ satisfies (30) for any $x > 0$. If we write

(51) $$t = au + x B_u ,$$

then by (33) and (50) it follows that

(52) $$\lim_{t\to\infty} \underset{\sim}{P}\{\nu(t) \geq u\} = R(x)$$

where u can be determined by (51). For if $u\to\infty$, then $t\to\infty$ for any x. If we make use of the fact that B_u has the form (25), then we can prove that

(53) $$\lim_{t\to\infty} \frac{u - \frac{t}{a}}{B_t a^{-(\alpha+1)/\alpha}} = -x .$$

Thus by (52) and (53) we obtain that

(54) $$\lim_{t\to\infty} \underset{\sim}{P}\{\frac{\nu(t) - \frac{t}{a}}{B_t a^{-(\alpha+1)/\alpha}} \geq -x\} = R(x)$$

for any x. Hence (48) follows. Again the dependence on c is only apparent in (48).

<u>Note.</u> If

(55) $$\lim_{x \to \infty} x^{\alpha}[1-F(x)] = q$$

where $1 < \alpha < 2$ and $q > 0$, then Theorem 3 is applicable and by (48) we have

(56) $$\lim_{t \to \infty} P\{ \frac{\nu(t) - \frac{t}{a}}{B_t a^{-(1+\alpha)/\alpha}} \leqq x\} = 1-R(-x)$$

where

(57) $$B_t = [\frac{q\pi t}{2c\Gamma(\alpha)\sin\frac{\alpha\pi}{2}}]^{\frac{1}{\alpha}} .$$

This follows from (30) and (55).

If (28) holds and $\sigma^2 = \infty$, then in a similar way as (48) we obtain that

(58) $$\lim_{t \to \infty} P\{ \frac{\nu(t) - \frac{t}{a}}{B_t\, a^{-3/2}} \leqq x\} = \Phi(x)$$

where $B_t > 0$ can be obtained by (28).

The case where $\alpha = 1$ is somewhat more complicated, but in a similar way as above we can also obtain the asymptotic distribution of $\nu(t)$ as $t \to \infty$. For this case we mention only an example. Let

(59) $$F(x) = \begin{cases} 1 - \frac{1}{x} & \text{for } x \geqq 1 , \\ 0 & \text{for } x < 1 . \end{cases}$$

Then by Theorem 44.8 we can prove that

(60) $$\lim_{u \to \infty} P\{ \frac{\tau_u - u \log u}{u} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(1,1,\frac{\pi}{2}, 1-C)$ where $C = 0.577215 \ldots$ is Euler's constant. (See Problem 46.19 .)

If we write

(61) $$t = u \log u + xu ,$$

then

(62) $$\lim_{t \to \infty} \frac{u - \frac{t}{\log t}}{\frac{t}{(\log t)^2}} = -x ,$$

and since by (33) and (60) we have

(63) $$\lim_{t \to \infty} P\{\nu(t) \geq u\} = R(x) ,$$

therefore it follows that

(64) $$\lim_{t \to \infty} P\{ \frac{\nu(t) - \frac{t}{\log t}}{\frac{t}{(\log t)^2}} \geq -x\} = R(x)$$

or

(65) $$\lim_{t \to \infty} P\{ \frac{\nu(t) - \frac{t}{\log t}}{\frac{t}{(\log t)^2}} \leq x\} = 1-R(-x)$$

for every x.

The limit distributions (34), (40), and (48) were found for a lattice distribution function $F(x)$ in 1940 by W. Feller [206]. For the general case see the author [263], [264] .

The theory of recurrent processes has attracted much attention in connection with industrial replacement problems. See for example H. Hadwiger [214] and A. Lotka [225]. In industrial replacement problems we assume that a machine works continuously in the time interval $(0, \infty)$ and if a part of the machine breaks down, then we replace it immediately by a similar part. Denote by $\theta_1, \theta_2, \ldots, \theta_k, \ldots$ the lifetimes of the successive parts used in the machine in the time interval $(0, \infty)$, and denote by $\nu(t)$ the number of replacements in the time interval $(0, t]$. If we suppose that $\{\theta_k\}$ is a sequence of mutually independent and identically distributed positive random variables with distribution function $\underset{\sim}{P}\{\theta_k \leq x\} = F(x)$, then $\{\nu(t), 0 \leq t < \infty\}$ is a recurrent process as defined previously. It is important to know the stochastic behavior of $\{\nu(t), 0 \leq t < \infty\}$, for example, if we want to decide how large the stock of the spare parts should be in order to satisfy the demand in a given time interval with high probability.

The first results were concerned with the asymptotic behavior of the expectation

$$m(t) = \underset{\sim}{E}\{\nu(t)\} = \sum_{n=1}^{\infty} F_n(t) \quad (66)$$

We can easily see that $m(t)$ satisfies the following integral equation

$$m(t) = F(t) + \int_0^t m(t-x)\,dF(x) \quad (67)$$

for $t \geq 0$.

If

$$\sum_{j=0}^{\infty} \underset{\sim}{P}\{\theta_k = jd\} = 1 \quad (68)$$

for some $d > 0$, then we say that $F(x)$ is a lattice distribution function and $d > 0$ is called the step of $F(x)$ if d is the largest positive number which satisfies (68). If $d > 0$ is the step of a lattice distribution function $F(x)$, then the g.c.d. $\{j : \underset{\sim}{P}\{\theta_k = jd\} > 0\} = 1$. If $F(x)$ is a lattice distribution function with step d , then by introducing a new time scale we can achieve that d becomes 1 .

If $F(x)$ is a lattice distribution function with step 1, then let us write

(69) $$f_j = F(j) - F(j-0)$$

for $j = 0,1,2,\ldots$ and

(70) $$u_n = m(n) - m(n-1)$$

for $n = 1,2,\ldots$ and $u_0 = 1$. In this case

(71) $$m(t) = u_1 + u_2 + \ldots + u_n$$

for $n \leq t < n+1$ and (67) can be expressed in the following equivalent form

(72) $$u_n = \sum_{j=1}^{n} f_j u_{n-j}$$

for $n = 1,2,\ldots$. If we define

(73) $$r_n = \sum_{j=n+1}^{\infty} f_j$$

for $n = 0,1,2,\ldots,$ then by (72) we can prove that

(74)
$$\sum_{j=0}^{n} r_j u_{n-j} = 1$$

for $n = 0,1,2,\ldots$.

In the theory of recurrent processes it has been first conjectured that

(75)
$$\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{a}$$

where a is defined by (4).

In 1940 H. Richter [234] demonstrated that if $\sigma^2 < \infty$ and $F(x)$ is an absolutely continuous distribution function or a lattice distribution function, then (75) is true. Richter proved also that if $d(t) = \mathrm{Var}\{\nu(t)\}$, then under some restrictions on $F(x)$ we have

(76)
$$\lim_{t \to \infty} \frac{d^2(t)}{m(t)} = \frac{\sigma^2}{a^2} .$$

In 1941 W. Feller [205] proved that (75) is generally true without making any restriction on $F(x)$. Feller used a Tauberian theorem. (See Theorem 9.13 in the Appendix.) However, we can prove this result in an elementary way, which we shall demonstrate soon. Feller also proved that if

(77)
$$a_r = \int_0^\infty x^r dF(x)$$

is finite for some $r \geq 2$ and if some other conditions are satisfied too, then

(78)
$$\lim_{t \to \infty} t^{r-2} [m(t) - \frac{t}{a}] = 0 .$$

In 1942 H. Schwarz [236] too proved that (75) is true if $a < \infty$ and if $F(x)$ is either a lattice distribution function or an absolutely continuous distribution function. He used a Tauberian theorem. (See Theorem 9.13 in the Appendix.) Schwartz also proved that $\lim_{t \to \infty} d(t)/t = 0$.

In 1944 S. Täcklind [271] proved in an elementary way that

$$(79) \qquad m(t) - \frac{t}{a} = \begin{cases} o(t) & \text{if } a < \infty , \\ o(t^{2-r}) & \text{if } a_r < \infty \text{ for some } r \in (1,2), \\ O(1) & \text{if } a_2 < \infty, \end{cases}$$

and in 1945 S. Täcklind [272] proved that if $a_r < \infty$ for some $r > 2$, and if $F(x)$ is not a lattice distribution function, then

$$(80) \qquad \lim_{t \to \infty} [m(t) - \frac{t}{a}] = \frac{\sigma^2}{2a^2} - \frac{1}{2} .$$

Furthermore, if $a_r < \infty$ for some $r > 2$, and if $F(x)$ is a lattice distribution function with step 1 , then

$$(81) \qquad \lim_{t \to \infty} [m(t) - \frac{[t] + \frac{1}{2}}{a}] = \frac{\sigma^2}{2a^2} - \frac{1}{2} .$$

In (80) and (81) the condition $a_r < \infty$ for some $r > 2$ can be replaced by the condition $\sigma^2 < \infty$. This was proved in 1949 by W. Feller [206] for (81) and in 1954 by W. L. Smith [553] for (80). These authors demonstrated also that (76) is valid if we assume only that $\sigma^2 < \infty$.

In the case when $F(x)$ is an absolutely continuous distribution function, then $m'(t)$ exists almost everywhere and it is interesting to find conditions under which

$$(82) \qquad \lim_{t \to \infty} m'(t) = \frac{1}{a}$$

exists. Such conditions were given in 1941 by W. Feller [205], in 1945 by S. Täcklind [271], in 1953 by D. R. Cox and W. L. Smith [196] and in 1954 by W. L. Smith [553],[554].

Now we shall prove that (75) is generally true. First we shall consider the lattice case, and then the general case. The following proofs are entirely elementary.

Theorem 4. *If* $F(x)$ *is a lattice distribution function, and if*

$$(83) \qquad a = \int_0^\infty x dF(x) ,$$

then

$$(84) \qquad \lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{a} .$$

If $a = \infty$, *then* $1/a = 0$.

Proof. We may assume without loss of generality that $F(x)$ has step 1 . In this case we shall prove that

$$(85) \qquad \lim_{n \to \infty} \frac{u_0 + u_1 + \dots + u_n}{n+1} = \frac{1}{a}$$

where u_n is defined (70). This implies (84).

Now by (74) we have the inequality

$$(86) \qquad \mathbf{n+1} = \sum_{j+k \leqq n} u_j r_k \leqq \left(\sum_{j=0}^{n} u_j\right)\left(\sum_{k=0}^{n} r_k\right) .$$

Hence

$$(87) \qquad \frac{1}{\sum_{k=0}^{n} r_k} \leqq \frac{\sum_{j=0}^{n} u_j}{\mathbf{n+1}} .$$

If $a < \infty$, then $\sum_{k=0}^{\infty} r_k = a$ and if $a = \infty$, then $\sum_{k=0}^{\infty} r_k = \infty$. If $n \to \infty$ in (87), then we obtain that

$$(88) \qquad \frac{1}{a} \leqq \liminf_{n \to \infty} \frac{\sum_{j=0}^{n} u_j}{\mathbf{n+1}} .$$

On the other hand, if $0 \leqq s \leqq n$, then by (74) we have

$$(89) \qquad \mathbf{n+1} = \sum_{j+k \leqq n} u_j r_k \geqq \left(\sum_{j=0}^{n-s} u_j\right)\left(\sum_{k=0}^{s} r_k\right) .$$

Hence it follows that

$$(90) \qquad \limsup_{n \to \infty} \frac{\sum_{j=0}^{n} u_j}{\mathbf{n+1}} \leqq \frac{1}{\sum_{k=0}^{s} r_k}$$

for $s = 0,1,2,\ldots$. If $s \to \infty$ in (90), then we get

$$(91) \qquad \limsup_{n \to \infty} \frac{\sum_{j=0}^{n} u_j}{\mathbf{n+1}} \leqq \frac{1}{a} .$$

By (88) and (91) we obtain (85) where $1/a = 0$ if $a = \infty$. This proves the theorem.

Theorem 5. If

$$a = \int_0^\infty x dF(x) , \tag{92}$$

then

$$\lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{a} \tag{93}$$

where $1/a = 0$ for $a = \infty$.

Proof. First, let $a = \infty$. In this case let us associate a new recurrent process $\{\bar{\nu}(t), 0 \leqq t < \infty\}$ with the process $\{\nu(t), 0 \leqq t < \infty\}$ by assuming that the recurrence times are $\bar{\theta}_k = [\theta_k] + 1$ $(k = 1,2,\ldots)$ where $[x]$ denotes the integral part of x. Let $\bar{m}(t) = \underset{\sim}{E}\{\bar{\nu}(t)\}$. Obviously we have $\bar{m}(t) \leqq m(t)$. If $a = \infty$, then $\underset{\sim}{E}\{\bar{\theta}_k\} = \infty$ and by Theorem 4 it follows that $\lim_{t \to \infty} \bar{m}(t)/t = 0$. This implies (93) for $a = \infty$.

Second, let $a < \infty$. Then we have the inequality

$$\frac{1}{a} - \frac{1}{t} \leqq \frac{m(t)}{t} \leqq \frac{m(h)+1}{h} + \frac{m(h)}{t} \tag{94}$$

for $t > 0$ and $h > 0$. Since the event $\{\nu(t)+1 = n\}$ and the random variables $\theta_{n+1}, \theta_{n+2}, \ldots$ are independent for $n = 1,2,\ldots,$ it follows by Theorem **6.1** of the Appendix that

$$\underset{\sim}{E}\{\tau_{\nu(t)+1}\} = [m(t)+1]a \geqq t . \tag{95}$$

The last inequality follows from the fact that $\tau_{\nu(t)+1} \geqq t$. By (95) we obtain the first inequality in (94). To prove the second inequality in (94), let us observe that

(96) $$m(u+h) - m(u) \leq m(h)+1$$

holds for all $u \geq 0$ and $h \geq 0$. Let $nh \leq t < (n+1)h$. If we add (96) for $u = t-h, t-2h, \ldots, t-nh$ and if we take into consideration that $m(t-nh) \leq m(h)$, then we get the inequality

(97) $$m(t) \leq (n+1)m(h) + n \leq \frac{t}{h}[m(h)+1] + m(h)$$

which proves the second half of (94).

From (94) it follows that

(98) $$\frac{1}{a} \leq \liminf_{t \to \infty} \frac{m(t)}{t} \leq \limsup_{t \to \infty} \frac{m(t)}{t} \leq \frac{m(h)+1}{h}$$

for all $h > 0$. Now we shall prove that

(99) $$\limsup_{h \to \infty} \frac{m(h)+1}{h} \leq \frac{1}{a-\varepsilon}$$

where ε is any positive number. By (98) and (99) we get (93).

To prove (99) for every $\varepsilon > 0$ let us associate a new recurrent process $\{\bar{\nu}(t), 0 \leq t < \infty\}$ with the process $\{\nu(t), 0 \leq t < \infty\}$ by assuming that the recurrence times are $\bar{\theta}_k = \varepsilon[\theta_k/\varepsilon]$ $(k = 1,2,\ldots)$. Let $\bar{m}(t) = E\{\bar{\nu}(t)\}$. Since $a-\varepsilon \leq \bar{a} = E\{\bar{\theta}_k\} \leq a$, it follows from Theorem 4 that

(100) $$\lim_{t \to \infty} \frac{\bar{m}(t)}{t} = \frac{1}{\bar{a}} \leq \frac{1}{a-\varepsilon}.$$

Finally, the inequality $m(t) \leq \bar{m}(t)$ and (100) imply (99). This completes the proof of theorem.

The next two theorems give more information about the asymptotic behavior of $m(t)$ as $t \to \infty$. These theorems have many important applications in the theories of Markov chains and stochastic processes.

The following theorem can be deduced from a more general theorem of A. N. Kolmogorov [221]. In 1949 P. Erdős, W. Feller and H. Pollard [202] provided an elementary proof of this theorem.

Theorem 6. *If* $F(x)$ *is a lattice distribution function with step* d, *then*

$$\lim_{n\to\infty} [m(nd+d) - m(nd)] = \frac{d}{a} \tag{101}$$

where a *is defined by (83). If* $a = \infty$, *then* $1/a = 0$.

Proof. We shall use the same notation as in the proof of Theorem 4. We may assume without loss of generality that $F(x)$ has step 1, that is, $d = 1$. We shall prove that

$$\lim_{n\to\infty} u_n = \frac{1}{a} \tag{102}$$

which implies (101).

We shall use the relations (72) and (74) and that $\text{g.c.d}\{j : f_j > 0\} = 1$.

Since $0 \leqq u_n \leqq 1$, therefore there exists a number $\lambda = \limsup_{n\to\infty} u_n$ and there exists a sequence $n_1, n_2, \ldots$ such that $\lim_{\nu\to\infty} u_{n_\nu} = \lambda$.

Now we shall prove that if $f_j > 0$, then

(103) $$\lim_{\nu \to \infty} u_{n_\nu - j} = \lambda .$$

By (72) we can write that

(104) $$\lambda = \lim_{\nu \to \infty} u_{n_\nu} = \lim_{\nu \to \infty} \inf\{f_j u_{n_\nu - j} + \sum_{\substack{i=1 \\ i \neq j}}^{n_\nu} f_i \, u_{n_\nu - i}\} \leqq$$
$$\leqq f_j \liminf_{\nu \to \infty} u_{n_\nu - j} + \lambda \sum_{\substack{i=1 \\ i \neq j}}^{m} f_i + \sum_{i=m+1}^{\infty} f_i$$

for any $m = 1,2,\ldots$. If $m \to \infty$ in (104), then we get

(105) $$\lambda \leqq f_j \liminf_{\nu \to \infty} u_{n_\nu - j} + \lambda(1-f_j) .$$

$\longleftarrow$ By (105) we have $\liminf_{\nu \to \infty} u_{n_\nu - j} \geqq \lambda$. By definition, we have $\limsup_{\nu \to \infty} u_{n_\nu - j} \leqq \lambda$. Thus (103) follows.

Accordingly, we have proved that if $\lim_{\nu \to \infty} u_{n_\nu} = \lambda$ and $f_j > 0$, then $\lim_{\nu \to \infty} u_{n_\nu - j} = \lambda$.

Since g.c.d$\{j : f_j > 0\} = 1$, we can find a finite number of positive integers $j_1, j_2, \ldots, j_s$ such that $f_{j_1} > 0, f_{j_2} > 0, \ldots, f_{j_s} > 0$ and g.c.d$\{j_1, j_2, \ldots, j_s\} = 1$. By the repeated applications of the previous result we can conclude that if $\lim_{\nu \to \infty} u_{n_\nu} = \lambda$, then $\lim_{\nu \to \infty} u_{n_\nu - k} = \lambda$ where

(106) $$k = r_1 j_1 + r_2 j_2 + \ldots + r_s j_s$$

and $r_1, r_2, \ldots, r_s$ are nonnegative integers. Every integer $k \geqq j_1 j_2 \ldots j_s$ can be represented in the form (106). Therefore $\lim_{\nu \to \infty} u_{n_\nu - k} = \lambda$ whenever

$k \geqq q = j_1 j_2 \cdots j_s$.

If we put $n = n_\nu - q$ in (74), then we obtain that

(107) $$\sum_{j=0}^{m} r_j u_{n_\nu - q - j} \leqq 1$$

for $0 \leqq m \leqq n_\nu - q$. If $\nu \to \infty$ in (107), then for any $m = 0,1,2,\ldots$ we get

(108) $$\lambda \sum_{j=0}^{m} r_j \leqq 1 .$$

If $a = \infty$, then $\sum_{j=0}^{\infty} r_j = \infty$, and it follows from (108) that $\lambda = 0$. This proves that (102) holds with $1/a = 0$.

If $a < \infty$, then $\sum_{j=0}^{\infty} r_j = a$, and by (108) it follows that

(109) $$\lambda = \limsup_{n \to \infty} u_n \leqq \frac{1}{a} .$$

Finally, we shall prove that if $a < \infty$, then

(110) $$\gamma = \liminf_{n \to \infty} u_n \geqq \frac{1}{a} .$$

From (109) and (110) it follows that $\lambda = \gamma = 1/a$ which proves (102).

We can prove (110) in a similar way as (109). If $\gamma = \liminf_{n \to \infty} u_n$, then there is a sequence $n_1, n_2, \ldots$ such that $\lim_{\nu \to \infty} u_{n_\nu} = \gamma$. By using (72) we can prove that if $f_j > 0$, then $\lim_{\nu \to \infty} u_{n_\nu - j} = \gamma$ also holds. In exactly the same way as before this implies that $\lim_{\nu \to \infty} u_{n_\nu - k} = \gamma$ for $k \geqq q$. If $a < \infty$, then for any $\varepsilon > 0$ and for sufficiently large m we have $r_{m+1} + r_{m+2} + \ldots < \varepsilon$. Thus by (64) we have

$$\sum_{j=0}^{m} r_j u_{n_\nu-q-j} \geqq 1-\varepsilon \tag{111}$$

for $n_\nu-q \geqq m$ if m is large enough. If $\nu \to \infty$ in (111), then we get

$$\gamma \sum_{j=0}^{m} r_j \geqq 1-\varepsilon . \tag{112}$$

If $m \to \infty$ in (112), then we get $\gamma a \geqq 1-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, therefore $\gamma a \geqq 1$. This proves (110), and (109) and (110) imply (102) for $a < \infty$.

In 1945 S. Täcklind [271] found the result (80) which implies that

$$\lim_{t \to \infty} [m(t+h) - m(t)] = \frac{h}{a} \tag{113}$$

for any $h > 0$ if $F(x)$ is not a lattice distribution function and $a_r < \infty$ for some $r > 2$ where a_r is defined by (77). In 1948 J. L. Doob [199] proved that (113) holds if $F_k(x)$ is not a singular distribution function for some k . In 1948 D. Blackwell [187] proved that (113) is valid if $F(x)$ is not a lattice distribution function. New proofs for this result of D. Blackwell were found in 1961 by W. Feller and S. Orey [208], and W. Feller [207]. In what follows we shall present the proof of W. Feller [207]. This proof is based on the following auxiliary theorem found in 1960 by G. Choquet and J. Deny [194].

Lemma 1. *Let $F(x)$ be a nonlattice distribution function of a postivie random variable. If $u(x)$ is a continuous bounded solution of*

$$u(x) = \int_0^\infty u(x-y)dF(y) \tag{114}$$

then $u(x) \equiv$ constant.

Proof. First, we shall prove that if $u(x)$ is a uniformly continuous bounded solution of (114), then $u(x) \equiv$ constant.

Denote by S the set of points of increase of $F(x)$, that is,

$$S = \{x : F(x+\varepsilon) - F(x-\varepsilon) > 0 \text{ for all } \varepsilon > 0\} . \tag{115}$$

Denote by S^* the smallest set which contains S and which has the following property: If $x \varepsilon S^*$ and $y \varepsilon S^*$, then $x+y \varepsilon S^*$ and $x-y \varepsilon S^*$. Since $F(0) < 1$, it follows by Theorem 43.5 that $S^* = (-\infty, \infty)$.

In what follows we shall prove that if $a \varepsilon S$, then $u(x) = u(x-a)$ for every x. Then by the previous remark we can conclude that $u(x) = u(x-a)$ holds for every x and every a, that is, $u(x) \equiv$ constant.

Let $a \varepsilon S^{\cdot}$ and define $v(x) = u(x) - u(x-a)$.

For every a the function is uniformly continuous and bounded and satisfies

$$v(x) = \int_0^\infty v(x-y)dF(y) . \tag{116}$$

Let $\sup\limits_{-\infty<x<\infty} v(x) = q$. Then there is a sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $\lim\limits_{n\to\infty} v(x_n) = q$. Define $w_n(x) = v(x+x_n)$ for $n = 1,2,\ldots$. Since $u(x)$ is uniformly continuous, the sequence $\{w_n(x)\}$ is equicontinuous and by a theorem of C. Arzelà (cf. A. N. Kolmogorov and S. V. Fomin[56 p. 54]) it contains a subsequence $\{w_{n_k}(x)\}$ which converges uniformly in every finite interval. Let $\lim_{k\to\infty} w_{n_k}(x) = w(x)$. The function $w(x)$ is uniformly continuous, bounded, $w(x) \leq q$, and satisfies

$$(117) \qquad w(x) = \int_0^\infty w(x-y)dF(y) .$$

By definition $w(0) = q$. If $w(x) = q$ for some x , then $w(x-a) = q$ also holds because $w(x)$ is the weighted average of $w(x-y)$ for $0 \leq y < \infty$ and a is a point of increase of $F(y)$. Thus it follows that $w(-ja) = q$ for $j = 0,1,2,\ldots$. Since $w(x) = \lim_{k\to\infty} v(x+x_{n_k})$ for every x , therefore if $z = x_{n_k}$ where k is sufficiently large we have the inequality

$$(118) \qquad v(z-ja) = u(z-ja) - u(z-ja-a) > \frac{q}{2}$$

for $j = 0,1,\ldots, r$ where r is any integer. If we add (118) for $j = 0,1,\ldots, r-1$, then we obtain that

$$(119) \qquad u(z) - u(z-ra) > \frac{rq}{2} .$$

Since $u(x)$ is bounded and r is arbitrary, we can conclude that $q = \sup_{-\infty<x<\infty} v(x) \leq 0$. But the same argument applies to the function $-v(x)$, and therefore $\sup_{-\infty<x<\infty} [-v(x)] \leq 0$ also holds. Consequently $v(x) \equiv 0$. This proves that $u(x) = u(x-a)$ for every x and therefore $u(x) \equiv$ constant.

Now suppose that $u(x)$ is a continuous bounded solution of (114). Let us define

$$(120) \qquad u_\varepsilon(x) = \int_{-\infty}^{\infty} u(x-y) \frac{\varepsilon}{\varepsilon^2 + y^2} dy$$

for $\varepsilon > 0$. Then $u_\varepsilon(x)$ is a uniformly continuous bounded function of x and satisfies

(121) $$u_\varepsilon(x) = \int_0^\infty u_\varepsilon(x-y)dF(y) .$$

By the previous result we can conclude that $u_\varepsilon(x) \equiv$ constant for every $\varepsilon > 0$. If $\varepsilon \to 0$, then by (120) $u_\varepsilon(x) \to u(x)$, and therefore $u(x) \equiv$ constant. This completes the proof of the lemma.

Now we are going to prove the following theorem of D. Blackwell [187].

Theorem 7. *If* $F(x)$ *is not a lattice distribution function and*

(122) $$a = \int_0^\infty x dF(x) ,$$

then

(123) $$\lim_{t \to \infty} [m(t+u) - m(t)] = \frac{u}{a}$$

for any $u > 0$. *If* $a = \infty$, *then* $1/a = 0$.

Proof. Let

(124) $$H_t(u) = m(t+u) - m(t)$$

for $t \geqq 0$ and $-\infty < u < \infty$. For every t the function $H_t(u)$ is nondecreasing and bounded in every finite interval. For $H_t(u) \leqq m(u)+1 < \infty$ for all $t \geqq 0$ and u. By Theorem 41.7 it follows that the family of functions $\{H_t(u) , 0 \leqq t < \infty\}$ is weakly compact in any finite interval $[-U, U]$. Thus there exist a nondecreasing function $H(u)$ and a sequence $t_1, t_2, \ldots, t_n, \ldots$ such that $t_n \to \infty$ as $n \to \infty$ and

(125) $$\lim_{n \to \infty} H_{t_n}(u) = H(u)$$

in every continuity point of $H(u)$ in any finite interval $[-U, U]$. Furthermore, by the Note after Theorem 41.8, it follows that if $g(u)$ is a continuous function of u and if $g(u) = 0$ for $|u| \geqq U$, then

$$\text{(126)} \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} g(x-u)d_u H_{t_n}(u) = \int_{-\infty}^{\infty} g(x-u)d\,H(u) \,.$$

Let

$$\text{(127)} \qquad u(x) = \int_{-\infty}^{\infty} g(x-u)dH(u) \,,$$

and

$$\text{(128)} \qquad h(t) = g(t) + \int_{0}^{\infty} g(t-u)dm(u) \,.$$

By (128) we obtain that

$$\text{(129)} \qquad h(t) = g(t) + \int_{0}^{\infty} h(t-y)dF(y) \,.$$

If we put $t = t_n + x$ in (128), then we get

$$\text{(130)} \qquad h(t_n + x) = g(t_n + x) + \int_{-t_n}^{\infty} g(x-u)d_u H_{t_n}(u) + g(x-t_n)m(t_n) \,.$$

If we let $n \to \infty$ in (130) then obtain that

$$\text{(131)} \qquad \lim_{n\to\infty} h(t_n + x) = u(x)$$

defined by (127). If we put $t = t_n + x$ in (129) and let $n \to \infty$, then by (131) we obtain that

$$\text{(132)} \qquad u(x) = \int_{0}^{\infty} u(x-y)dF(y)$$

for all x. Since $u(x)$ is continuous and bounded, by Lemma 1 it follows that $u(x) \equiv$ constant, that is,

$$\int_{-\infty}^{\infty} g(x-u)dH(u) \equiv \text{constant} \tag{133}$$

for every continuous function $g(u)$ such that $g(u) = 0$ for $|u| \geq U$ where U is a finite positive number. We observe that $H(0) = 0$ if $u = 0$ is a continuity point of $H(u)$. For $H_t(0) = 0$ for $t \geq 0$. Thus by (133) it follows that

$$H(u) = Cu \tag{134}$$

where C is a constant. By Theorem 5 it follows immediately that $C = 1/a$ if $a < \infty$ and $C = 0$ if $a = \infty$. However, we can prove this directly by using (67). By (67) it follows that

$$\int_0^t [1-F(t-u)]dm(u) = F(t) \tag{135}$$

for $t \geq 0$. If we use (134), that is, that

$$\lim_{t_n \to \infty} [m(t_n + u) - m(t_n)] = Cu \tag{136}$$

for every u, and if we put $t = t_n$ in (135) and let $t_n \to \infty$, then we obtain that

$$C \int_0^{\infty} [1-F(u)]du = 1 . \tag{137}$$

Thus, we have $Ca = 1$.

Since in (136) the limit does not depend on the particular sequence $\{t_n\}$, it follows that

$$\lim_{t \to \infty} [m(t+u) - m(t)] = \frac{u}{a} \tag{138}$$

also holds. In (138) $1/a = 0$ if $a = \infty$. This completes the proof of the theorem.

Theorem 7 has many useful applications in the theory of regenerative stochastic processes. A stochastic process is said to be regenerative if it has the property that every time some given pattern appears the future stochastic behavior of the process is the same independently of the past. Theorem 7 can be used in finding the limiting distribution of such processes.

In several cases we can use Theorem 7 in the following form. (See W. L. Smith [553], [240] and the author [261], [262], [269] .

Theorem 8. Let us assume that $Q(x)$ is of bounded variation in the interval $[0, \infty)$ and

(139) $$Q = \int_0^\infty Q(x)dx$$

exists. Furthermore, let $F(x)$ be a nonlattice distribution function of a positive random variable for which

(140) $$a = \int_0^\infty x dF(x) .$$

Then we have

(141) $$\lim_{t\to\infty} \int_0^t Q(t-u)dm(u) = \frac{Q}{a}$$

where $1/a = 0$ if $a = \infty$.

Proof. Every function of bounded variation can be expressed as the difference of two nonincreasing functions. Thus in proving the theorem we can restrict ourself to the case where $Q(x)$ is a nonnegative and nonincreasing function of x for $0 \leq x < \infty$. If $Q(x) \equiv 0$, then (141) is obviously true. Thus we may assume that $Q(0) > 0$.

Let

$$Q_1(t) = \int_0^{t/2} Q(t-u)\,dm(u) \tag{142}$$

and

$$Q_2(t) = \int_{t/2}^{t} Q(t-u)\,dm(u) \,. \tag{143}$$

We have evidently

$$0 \leqq Q_1(t) \leqq Q(\tfrac{t}{2})m(\tfrac{t}{2}) \,. \tag{144}$$

Since

$$\lim_{t \to \infty} \frac{t}{2} Q(\tfrac{t}{2}) = 0 \tag{145}$$

and since by Theorem 5

$$\lim_{t \to \infty} m(\tfrac{t}{2})/\tfrac{t}{2} = \frac{1}{a} \,, \tag{146}$$

we obtain that

$$\lim_{t \to \infty} Q_1(t) = 0 \,. \tag{147}$$

Now we shall prove that

$$\lim_{t \to \infty} Q_2(t) = \frac{Q}{a} \,. \tag{148}$$

For any $\varepsilon > 0$ let us choose an h such that $0 < h < \varepsilon/Q(0)$. Then we have

$$0 < Q - h \sum_{n=1}^{\infty} Q(nh) < hQ(0) < \varepsilon \,. \tag{149}$$

If we choose t so large that

$$h \sum_{n=[t/2h]}^{\infty} Q(nh) < \varepsilon \tag{150}$$

and

$$(151) \qquad \left| \frac{m(u+h) - m(u)}{h} - \frac{1}{a} \right| < \varepsilon$$

for $u \geq t/2$, then we have

$$(152) \qquad (\frac{1}{a} - \varepsilon)[h \sum_{n=1}^{\infty} Q(nh) - \varepsilon] < Q_2(t) < (\frac{1}{a} + \varepsilon)h \sum_{n=1}^{\infty} Q(nh) .$$

Hence it follows that

$$(153) \qquad (\frac{1}{a} - \varepsilon)(Q-2\varepsilon) < Q_2(t) < (\frac{1}{a} + \varepsilon)(Q+\varepsilon)$$

if t is large enough. Since $\varepsilon > 0$ is arbitrary, (153) proves (148). By (147) and (148) we obtain (141). This completes the proof of the theorem.

It is interesting to study the asymptotic behavior of $m(t)$ as $t \to \infty$ in the case when $a = \infty$. By Theorem 5 it follows that

$$(154) \qquad \lim_{t \to \infty} \frac{m(t)}{t} = 0$$

if $a = \infty$. If we know the asymptotic behavior of $1-F(x)$ as $x \to \infty$, then we can obtain more precise results for the asymptotic behavior of $m(t)$ as $t \to \infty$, We shall prove the following result.

Theorem 9. *If*

$$(155) \qquad 1-F(x) \sim \frac{h(x)}{x^{\alpha}}$$

as $x \to \infty$ *where* $0 < \alpha < 1$ *and* $h(x)$ *is a slowly variyng function of* x *at* $x \to \infty$, *that is,*

$$(156) \qquad \lim_{x \to \infty} \frac{h(\omega x)}{h(x)} = 1$$

<u>for any</u> $\omega > 0$, <u>then</u>

$$m(t) \sim \frac{\sin \alpha\pi}{\alpha\pi} \frac{t^{\alpha}}{h(t)} \tag{157}$$

<u>as</u> $t \to \infty$.

<u>Proof.</u> In formulas (155) and (157) the symbol $\sim$ means that the two sides are asymptotically equal, that is, their ratio tends to 1 as $x \to \infty$ or $t \to \infty$.

Let

$$\phi(s) = \int_0^\infty e^{-sx} dF(x) \tag{158}$$

for $\operatorname{Re} \geq 0$. Then we have

$$\int_0^\infty e^{-st} dm(t) = \frac{\phi(s)}{1-\phi(s)} = \frac{1}{1-\phi(s)} - 1 \tag{159}$$

for $\operatorname{Re}(s) > 0$. If $s \to +0$, then by an Abelian theorem (Theorem 9.12 in the Appendix) we obtain that

$$1-\phi(s) = s \int_0^\infty e^{-sx} [1-F(x)] dx \sim \Gamma(1-\alpha) s^{\alpha} h(\tfrac{1}{s}) \ . \tag{160}$$

Hence

$$\int_0^\infty e^{-st}\, dm(t) \sim \frac{1}{\Gamma(1-\alpha) s^{\alpha} h(\frac{1}{s})} \tag{161}$$

as $s \to +0$ and by a Tauberian theorem (Theorem 9.14 in the Appendix) we obtain (157).

In the case where $F(x)$ is a lattice distribution function the result

(157) was found in 1949 by W. Feller [206]. Actually, Feller considered the particular case when $h(x) \equiv$ constant. For the case where $h(x)$ satisfies (156) see A. Garsia and J. Lamperti [209]. In the general case, the result (157) was proved in 1955 by E. B. Dynkin [200]. See also W. L. Smith [238]. Dynkin also proved that (157) implies (155).

In 1961 W. L. Smith [243] proved that if (155) holds with $\alpha = 0$, then

$$m(t) \sim \frac{1}{1-F(t)} \tag{162}$$

as $t \to \infty$, and if (155) holds with $\alpha = 1$, then

$$m(t) \sim \frac{t}{\int_0^t [1-F(u)]du} \tag{163}$$

as $t \to \infty$, and the converse statements are also true.

In a similar way as Theorem 9 we can prove that if

$$d(t) = \underset{\sim}{\text{Var}}\{\nu(t)\} \tag{164}$$

and if $F(x)$ satisfies (155) with $0 < \alpha < 1$, then

$$d(t) \sim \left[\frac{\Gamma(\alpha+1)\pi^{1/2}\, 2^{1-2\alpha}}{\Gamma(\alpha+\frac{1}{2})} - 1\right] \frac{\sin^2\alpha\pi}{\alpha^2\pi^2} \frac{t^{2\alpha}}{(h(t))^2} \tag{165}$$

as $t \to \infty$. For the proof of (165) we refer to W. Feller [206] and J. L. Teugels [273].

If $F(x)$ satisfies (155) with $1 < \alpha < 2$, then the expectation of $F(x)$ is a finite positive number a and we have

$$m(t) - \frac{t}{a} \sim \frac{t^{2-\alpha} h(t)}{(\alpha-1)(2-\alpha)a^2} \tag{166}$$

and

$$d(t) \sim \frac{2t^{3-\alpha} h(t)}{(2-\alpha)(3-\alpha)a^3} \tag{167}$$

as $t \to \infty$. See W. Feller [206] and J. L. Teugels [273], [274].

For the recurrent process $\{\nu(t), 0 \leq t < \infty\}$ denote by χ_t the distance between t and the occurrence time of the first event occurring after time t. The distribution function of χ_t is given by

$$\mathbf{P}\{\chi_t \leq x\} = \int_t^{t+x} [1-F(t+x-u)]dm(u) \tag{168}$$

for $x \geq 0$. For the event $\{\chi_t \leq x\}$ occurs if and only if at least one event occurs in the interval $(t, t+x]$ in the recurrent process. This event can occur in several mutually exclusive ways: the last event occurring in the interval $(t, t+x]$ is the n-th event $(n = 1,2,\ldots)$ in the recurrent process. Thus by the theorem of total probability we obtain that

$$\begin{aligned} \mathbf{P}\{\chi_t \leq x\} &= \sum_{n=1}^{\infty} \mathbf{P}\{t < \tau_n \leq t+x < \tau_{n+1}\} = \\ &= \sum_{n=1}^{\infty} \int_t^{t+x} [1-F(t+x-u)]d\mathbf{P}\{\tau_n \leq u\} . \end{aligned} \tag{169}$$

Since

$$m(u) = \sum_{n=1}^{\infty} \mathbf{P}\{\tau_n \leq u\} , \tag{170}$$

we get (168) from (169).

<u>Theorem 9</u>. <u>If</u> $F(x)$ <u>is not a lattice distribution function and if</u> $a < \infty$, <u>then the limiting distribution</u>

$$\lim_{t \to \infty} \mathbf{P}\{\chi_t \leq x\} = F^*(x) \tag{171}$$

<u>exists and we have</u>

$$F^*(x) = \begin{cases} \frac{1}{a}\int_0^x [1-F(y)]dy & \underline{\text{for}} \quad x \geq 0 , \\ 0 & \underline{\text{for}} \quad x < 0 . \end{cases} \tag{172}$$

<u>Proof</u>. This theorem follows immediately from Theorem 8 if we apply it to the function

$$Q(u) = \begin{cases} 1-F(u) & \text{for} \quad u \leq x , \\ 0 & \text{for} \quad u > x . \end{cases} \tag{173}$$

If $F(x)$ is a lattice distribution function, then the limiting behavior of $\mathbf{P}\{\chi_t \leq x\}$ can easily be obtained by Theorem 6 .

We note that if we suppose that $F(x)$ is not a lattice distribution function and if $F(x)$ has a finite variance of σ^2, then we have

$$\lim_{t \to \infty} \mathbf{E}\{\chi_t\} = \int_0^\infty x dF^*(x) = \frac{\sigma^2 + a^2}{2a} . \tag{174}$$

For $\tau_{\nu(t)+1} = t+\chi_t$ and therefore by (95) we have

$$\mathbf{E}\{\chi_t\} = [m(t) + 1]a - t . \tag{175}$$

If $t \to \infty$ in (175), then by (80) we obtain that the limit of the right-hand side is $(\sigma^2 + a^2)/2a$. This proves (174).

If $F(x)$ satisfies (155) with $0 < \alpha < 1$, then

(176) $$\lim_{t \to \infty} \underset{\sim}{P}\{ \frac{\chi_t}{t} \leq x\} = H_\alpha(x)$$

where

(177) $$H_\alpha(x) = \begin{cases} \frac{\sin \alpha\pi}{\pi} \int_0^x \frac{du}{u^\alpha(1+u)} & \text{for } 0 < x < \infty, \\ 0 & \text{for } x < 0 . \end{cases}$$

This result was found in 1955 by E. B. Dynkin [200]. See also J. Lamperti [222].

It is interesting to observe that the limiting distribution (176) depends on $F(x)$ only through the parameter α.

Let us define η_t as the distance between t and the occurrence time of the last event occurring before time t, and $\eta_t = t$ if no events occur in the interval $(0, t]$. For η_t we have the obvious relations

(178) $$\underset{\sim}{P}\{\eta_t > y\} = \underset{\sim}{P}\{\chi_{t-y} > y\}$$

and

(179) $$\underset{\sim}{P}\{\chi_t > x, \eta_t > y\} = \underset{\sim}{P}\{\chi_{t-y} > x+y\}$$

for $x \geq 0$ and $0 \leq y \leq t$.

If we know the asymptotic distribution of χ_t as $t \to \infty$, then by (178) and (179) we can determine the asymptotic distributions of η_t and (η_t, χ_t)

as $t \to \infty$. Also we can determine the asymptotic distribution of $\theta_t^* = \eta_t + \chi_t$ for $t \to \infty$. The random variable θ_t^* is the time difference between the occurrence time of the first event occurring after t and the occurrence time of the last event occurring before t.

If $F(x)$ is not a lattice distribution function and if $a < \infty$, then by (171) and (178) we obtain that

(180) $$\lim_{t \to \infty} P\{\eta_t \leq x\} = F^*(x)$$

where $F^*(x)$ is given by (172). Furthermore, by (179) we obtain that

(181) $$\lim_{t \to \infty} P\{\theta_t^* \leq x\} = \frac{1}{a} \int_0^x y dF(y)$$

for $x \geq 0$. If, in addition, $\sigma^2 < \infty$, then we have

(182) $$\lim_{t \to \infty} E\{\theta_t^*\} = a + \frac{\sigma^2}{a} .$$

If $F(x)$ satisfies (155) with $0 < \alpha < 1$, then by (179) we obtain that

(183) $$\lim_{t \to \infty} P\{\frac{\chi_t}{t} > x , \frac{\eta_t}{t} > y\} = 1-H_\alpha(\frac{x+y}{1-y})$$

for $0 < y < 1$ and $x > 0$ where $H_\alpha(x)$ is given by (177). From (183) it follows that

(184) $$\lim_{t \to \infty} P\{\frac{\theta_t^*}{t} \leq x\} = \frac{\sin \alpha\pi}{\pi} \int_0^x \frac{q(u)}{u^{\alpha+1}} du$$

for $x \geq 0$ where

(185)
$$q(u) = \begin{cases} 1-(1-u)^{\alpha} & \text{for } 0 \leq u \leq 1 , \\ 1 & \text{for } u \geq 1 . \end{cases}$$

Note 1. If we suppose in Definition 1 that θ_1 is a positive random variable with distribution function $\underset{\sim}{P}\{\theta_1 \leq x\} = \hat{F}(x)$ whereas $\underset{\sim}{P}\{\theta_n \leq x\} = F(x)$ for $n = 2,3,\ldots,$ and if every other assumption remains unchanged, then we arrive at the notion of a general recurrent process. For a general recurrent process we have

(186)
$$\underset{\sim}{P}\{\nu(t) \leq n\} = 1-\hat{F}(t) * F_n(t)$$

for $n = 0,1,2,\ldots$ where $*$ means convolution. By (186) we have

(187)
$$\underset{\sim}{E}\{\nu(t)\} = \sum_{n=0}^{\infty} \hat{F}(t) * F_n(t) .$$

If we use the definition (3) and if

(188)
$$\hat{\phi}(s) = \int_0^{\infty} e^{-sx}\, d\hat{F}(x)$$

for $\text{Re}(s) \geq 0$, then by (187) we obtain that

(189)
$$\int_0^{\infty} e^{-st} d\underset{\sim}{E}\{\nu(t)\} = \frac{\hat{\phi}(s)}{1-\phi(s)}$$

for $\text{Re}(s) > 0$.

Most of the limit theorems which we proved for ordinary recurrent processes remain valid for general recurrent processes too.

Let us suppose that $F(x)$ has a finite expectation a and let $\hat{F}(x) = F^{*}(x)$ defined by (172). In this case we say that the recurrent process is homogeneous. For a homogeneous recurrent process we have

$$E\{\nu(t)\} = \frac{t}{a} \tag{190}$$

for every $t \geq 0$ and

$$P\{\chi_t \leq x\} = F^{*}(x) \tag{191}$$

for every $t \geq 0$.

Note 2. Recurrent processes have useful applications in the investigations of the fluctuations of sums of mutually independent and identically distributed random variables.

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of mutually independent and identically distributed random variables. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$.

Let $\tau_0 = 0$. Denote by τ_1 the smallest $n = 1,2,\ldots$ for which $\zeta_n > \zeta_0 = 0$. Denote by τ_2 the smallest $n = 1,2,\ldots$ for which $\zeta_n > \zeta_{\tau_1}$ and so on for $k = 2,3,\ldots$ denote by τ_k the smallest $n = 1,2,\ldots$ for which $\zeta_n > \zeta_{\tau_{k-1}}$. For every $t \geq 0$ let $\nu(t)$ be a random variable which takes on nonnegative integers only and satisfies the relation

$$\{\nu(t) \geq k\} \equiv \{\tau_k \leq t\} \tag{192}$$

for all $t \geq 0$ and $k = 0,1,2,\ldots$.

In this case the family of random variables $\{\nu(t), 0 \leq t < \infty\}$ forms a recurrent process and the recurrence times $\theta_k = \tau_k - \tau_{k-1}$ $(k = 1,2,\ldots)$ are mutually independent and identically distributed discrete random variables taking on positive integers only. The random variables $\tau_1, \tau_2, \ldots, \tau_k, \ldots$ are the ladder indices of the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$ as we defined in Section 19. By Theorem 19.3 we have

$$(193) \qquad \sum_{n=1}^{\infty} \underset{\sim}{P}\{\theta_k = n\} z^n = 1 - e^{\sum\limits_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n > 0\}}$$

for $|z| < 1$.

If we define $\chi_k = \zeta_{\tau_k} - \zeta_{\tau_{k-1}}$ for $k = 1,2,\ldots$, then $\chi_1, \chi_2, \ldots, \chi_k, \ldots$ is a sequence of mutually independent and identically distributed positive random variables. If we consider the random variables $\chi_1, \chi_2, \ldots, \chi_k, \ldots$ as recurrence times, then by Definition 1 they too determine a recurrent process. By Theorem 19.4 we have

$$(194) \qquad \underset{\sim}{E}\{e^{-s\chi_k}\} = 1 - e^{-\sum\limits_{n=1}^{\infty} \frac{1}{n} \underset{\sim}{E}\{e^{-s\zeta_n} \delta(\zeta_n > 0)\}}$$

for $\operatorname{Re}(s) \geq 0$ where $\delta(\zeta_n > 0)$ is the indicator variable of the event $\zeta_n > 0$.

Finally, we note that Theorem 6 and Theorem 7 can be extended for an infinite sequence of mutually independent and identically distributed real random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ which are not necessarily positive. Let $\underset{\sim}{P}\{\xi_n \leq x\} = F(x)$ and define $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$.

Denote by $M(x, h)$ the expected number of integers $n = 1,2,\ldots$ for which $x < \zeta_n \leq x+h$, that is

(195) $$M(x, h) = \sum_{n=1}^{\infty} P\{x < \zeta_n \leq x+h\} .$$

Let

(196) $$a = \int_{-\infty}^{\infty} x dF(x)$$

where $a = +\infty$ or $a = -\infty$ is allowed.

If $F(x)$ is a lattice distribution function with step d and if $a > 0$, then

(197) $$\lim_{x \to \infty} M(x, d) = \frac{d}{a}$$

where $1/a = 0$ for $a = +\infty$, and

(198) $$\lim_{x \to -\infty} M(x,d) = 0 .$$

The case $a < 0$ can be obtained by symmetry. This result generalizes Theorem 6.

If $F(x)$ is not a lattice distribution function and if $a > 0$, then

(199) $$\lim_{x \to \infty} M(x, h) = \frac{h}{a}$$

for any $h > 0$ where $1/a = 0$ for $a = +\infty$, and

(200) $$\lim_{x \to -\infty} M(x, h) = 0$$

for any $h > 0$. The case $a < 0$ can be obtained by symmetry. This result generalizes Theorem 7.

The above extensions of Theorem 6 and Theorem 7 were given in 1952 and in 1953 by K. L. Chung and H. Pollard [192], K. L. Chung and J. Wolfowitz [193] and D. Blackwell [188].

In conclusion of this section we shall define the notion of a compound recurrent process.

Definition 2. Let $\{\nu(t), 0 \leq t < \infty\}$ be a recurrent process as we defined in Definition 1. Let $\chi_1, \chi_2, \ldots, \chi_i, \ldots$ be a sequence of mutually independent and identically distributed real random variables which are independent of the process $\{\nu(t), 0 \leq t < \infty\}$. Let us define

$$\chi(t) = \sum_{1 \leq i \leq \nu(t)} \chi_i \tag{201}$$

for $t \geq 0$. We say that $\{\chi(t), 0 \leq t < \infty\}$ is a compound recurrent process.

Denote by $\theta_1, \theta_2, \ldots, \theta_n, \ldots$ the successive recurrence times in the process. Let $\underset{\sim}{P}\{\theta_n \leq x\} = F(x)$ and $\underset{\sim}{P}\{\chi_i \leq x\} = H(x)$.

If we know $F(x)$ and $H(x)$, then the distribution function of $\chi(t)$ can be obtained by the following formula

$$\underset{\sim}{P}\{\chi(t) \leq x\} = \sum_{n=0}^{\infty} [F_n(t) - F_{n+1}(t)]H_n(x) \tag{202}$$

where $F_n(x)$ and $H_n(x)$ denote the n-th iterated convolutions of $F(x)$ and $H(x)$ respectively, and $F_0(x) = H_0(x) = 1$ for $x \geq 0$ and $F_0(x) = H_0(x) = 0$ for $x < 0$.

If both $F(x)$ and $H(x)$ belong to the domain of attraction of a stable distribution function, then by suitable normalization $\chi(t)$ has a limiting distribution as $t \to \infty$.

Let us suppose that

$$(203) \qquad \lim_{n \to \infty} P\left\{\frac{\theta_1 + \ldots + \theta_n - A_1(n)}{A_2(n)} \leq x\right\} = P\{\theta \leq x\}$$

and

$$(204) \qquad \lim_{n \to \infty} P\left\{\frac{\chi_1 + \ldots + \chi_n - B_1(n)}{B_2(n)} \leq x\right\} = P\{\chi \leq x\}$$

where $\lim_{n \to \infty} A_2(n) = \infty$ and $\lim_{n \to \infty} B_2(n) = \infty$, and θ and χ are independent random variables. If $F(x)$ and $H(x)$ belong to the domain of attraction of a stable distribution function, then the limiting distributions (203) and (204) can be obtained by Theorem 44.6 and by Theorem 44.8. If (203) is satisfied, then we can find normalizing functions $C_1(t)$ and $C_2(t)$ such that $C_2(t) \to \infty$ as $t \to \infty$ and

$$(205) \qquad \lim_{t \to \infty} P\left\{\frac{\nu(t) - C_1(t)}{C_2(t)} \leq x\right\} = P\{\nu \leq x\}$$

where the random variable ν depends on θ. The limiting distribution (205) can be obtained by Theorems 1, 2 and 3 in this section. Finally, by Theorem 45.2 or by using the same method which we used in proving Theorem 45.2 we can conclude that there are normalizing functions $D_1(t)$ and $D_2(t)$ such that $D_2(t) \to \infty$ as $t \to \infty$ and a distribution function $Q(x)$ such that

$$(206) \qquad \lim_{t \to \infty} P\{ \frac{\chi(t) - D_1(t)}{D_2(t)} \leq x\} = Q(x) .$$

Let us assume that in (203) $A_1(n) = A_1 n$ and $A_2(n) = A_2 n^a$ where $A_2 > 0$, and $a > 0$ for $A_1 = 0$ and $0 < a < 1$ for $A_1 > 0$. Furthermore, in (204) let $B_1(n) = B_1 n$ and $B_2(n) = B_2 n^b$ where $B_2 > 0$, and $b > 0$ for $B_1 = 0$ and $0 < b < 1$ for $B_1 > 0$. In this case, in (205) we have $C_1(t) = C_1 t$ and $C_2(t) = C_2 t^c$ where the constants C_1, C_2 and c and the random variable ν are given in Table I.

TABLE I

A_1	C_1	C_2	c	ν
0	0	$1/A_2^{1/a}$	$1/a$	$\theta^{-1/a}$
> 0	$1/A_1$	A_2/A_1^{1+a}	a	$-\theta$

Now by Theorem 45.2 we can conclude that in (206) $D_1(t) = D_1 t$ and $D_2(t) = D_2 t^d$ where the constants D_1, D_2, d and the distribution function $Q(x)$ are given in Table II.

TABLE II

B_1	C_1	(b,c)	D_1	D_2	d	$Q(x)$
0	0	–	0	$B_2C_2^b$	bc	$P\{\chi\nu^b \leq x\}$
> 0	0	–	0	B_1C_2	c	$P\{\nu \leq x\}$
0	> 0	–	0	$B_2C_1^b$	b	$P\{\chi \leq x\}$
> 0	> 0	$b<c$	B_1C_1	B_1C_2	c	$P\{\nu \leq x\}$
> 0	> 0	$b=c$	B_1C_1	1	b	$P\{B_1C_2\nu + B_2C_1^b\chi \leq x\}$
> 0	> 0	$b>c$	B_1C_1	$B_2C_1^b$	b	$P\{\chi \leq x\}$

In the particular case when $E\{\theta_n\} = a$, $E\{\chi_n\} = b$, and $Var\{\theta_n\} = \sigma_a^2$ and $Var\{\chi_n\} = \sigma_b^2$ are finite positive numbers we have

$$\lim_{n \to \infty} P\left\{ \frac{\theta_1 + \ldots + \theta_n - na}{\sigma_a \sqrt{n}} \leq x \right\} = \Phi(x) \tag{207}$$

and

$$\lim_{n \to \infty} P\left\{ \frac{\chi_1 + \ldots + \chi_n - nb}{\sigma_b \sqrt{n}} \leq x \right\} = \Phi(x) \tag{208}$$

where $\Phi(x)$ is the normal distribution function. Now by the 5-th statement of Table II we can conclude that

$$\lim_{t \to \infty} P\left\{ \frac{\chi(t) - \frac{bt}{a}}{\sqrt{t}} \leq x \right\} = P\left\{ \frac{a\sigma_b \chi - b\sigma_\alpha \theta}{a^{3/2}} \leq x \right\} \tag{209}$$

where χ and θ are independent random variables with distribution functions $P\{\chi \leq x\} \doteq P\{\theta \leq x\} = \Phi(x)$. Hence it follows that

(210) $$\lim_{t \to \infty} P\{ \frac{\chi(t) - (bt/a)}{[(a^2\sigma_b^2 + b^2\sigma_a^2)t/a^3]^{1/2}} \leq x\} \doteq \Phi(x) .$$

As another example, let us suppose that $\{\theta_n\}$ and $\{\chi_n\}$ are positive random variables for which

(211) $$\lim_{x \to \infty} P\{\theta_n > x\}x^{\alpha_1} = a_1$$

where $0 < \alpha_1 < 1$ and $a_1 > 0$, and

(212) $$\lim_{x \to \infty} P\{\chi_n > x\}x^{\alpha_2} = a_2$$

where $0 < \alpha_2 < 1$ and $a_2 > 0$. Then

(213) $$\lim_{n \to \infty} P\{ \frac{\theta_1+\ldots+\theta_n}{(na_1)^{1/\alpha_1}} \leq x\} = R_1(x)$$

where $R_1(x)$ is a stable distribution function of type $S(\alpha_1,1,\Gamma(1-\alpha_1)\cos \frac{\alpha_1\pi}{2},0)$ and

(214) $$\lim_{n \to \infty} P\{ \frac{\chi_1+\ldots+\chi_n}{(na_2)^{1/\alpha_2}} \leq x\} = R_2(x)$$

where $R_2(x)$ is a stable distribution function of type $S(\alpha_2,1,\Gamma(1-\alpha_2)\cos \frac{\alpha_2\pi}{2},0)$. Then by the first statement of Table II we obtain that

(215) $$\lim_{t \to \infty} P\{ \frac{\chi(t)}{(a_2 t^{\alpha_1}/a_1)^{1/\alpha_2}} \leq x\} = Q(x)$$

where

$$Q(x) = \underset{\sim}{P}\{\chi\,\theta^{-\alpha_1/\alpha_2} \leqq x\} \tag{216}$$

and θ and χ are independent random variables for which $\underset{\sim}{P}\{\theta \leqq x\} = R_1(x)$ and $\underset{\sim}{P}\{\chi \leqq x\} = R_2(x)$.

It is instructive to deduce (215) directly. Let $\underset{\sim}{E}\{e^{-s\theta_n}\} = \phi(s)$ and $\underset{\sim}{E}\{e^{-s\chi_n}\} = \psi(s)$ for $Re(s) \geqq 0$. Then by (202) we have

$$q\int_0^\infty e^{-qt}\,\underset{\sim}{E}\{e^{-s\chi(t)}\}dt = \frac{1-\phi(q)}{1-\phi(q)\psi(s)} \tag{217}$$

for $Re(q) > 0$ and $Re(s) \geqq 0$. Now let us define a random variable ν in such a way that ν and $\{\chi(t)\}$ are independent and

$$\underset{\sim}{P}\{\nu \leqq x\} = \begin{cases} 1 - e^{-x} & \text{for } x \geqq 0 , \\ 0 & \text{for } x < 0 , \end{cases} \tag{218}$$

Then by (217) we have

$$\underset{\sim}{E}\{e^{-s\chi(\nu/q)}\} = \frac{1-\phi(q)}{1-\phi(q)\psi(s)} \tag{219}$$

for $q > 0$ and $Re(s) \geqq 0$. Since

$$1 - \phi(s) = a_1\Gamma(1-\alpha_1)s^{\alpha_1} + o(s^{\alpha_1}) \tag{220}$$

and

$$1 - \psi(s) = a_2\Gamma(1-\alpha_2)s^{\alpha_2} + o(s^{\alpha_2}) \tag{221}$$

as $s \to +0$, it follows from (219) that

$$(222) \qquad \lim_{q \to +0} \underset{\sim}{E}\{e^{-sq^{\alpha_1/\alpha_2}\chi(\nu/q)}\} = \frac{a_1\Gamma(1-\alpha_1)}{a_1\Gamma(1-\alpha_1) + a_2\Gamma(1-\alpha_2)s^{\alpha_2}}$$

for $\mathrm{Re}(s) \geq 0$. From (217) we can deduce that (215) exists, and if we write

$$(223) \qquad \Omega(s) = \int_0^\infty e^{-sx} dQ(x)$$

for $\mathrm{Re}(s) \geq 0$, then we have

$$(224) \qquad \int_0^\infty \Omega(sx^{\alpha_1/\alpha_2})e^{-x}\,dx = \frac{\Gamma(1-\alpha_1)}{\Gamma(1-\alpha_1) + \Gamma(1-\alpha_2)s^{\alpha_2}}$$

for $\mathrm{Re}(s) \geq 0$. From (224) by inversion we obtain that

$$(225) \qquad \Omega(s) = E_{\alpha_1}\left(-\frac{\Gamma(1-\alpha_2)s^{\alpha_2}}{\Gamma(1-\alpha_1)}\right)$$

for $\mathrm{Re}(s) \geq 0$ where $E_\alpha(z)$ is the Mittag-Leffler function defined by

$$(226) \qquad E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+1)}$$

for $0 < \alpha < 1$.

If θ and χ are independent random variables for which $\underset{\sim}{P}\{\theta \leq x\} = R_1(x)$ and $\underset{\sim}{P}\{\chi \leq x\} = R_2(x)$, then by (42.171) we have

$$(227) \qquad \underset{\sim}{E}\{e^{-s\chi}\} = e^{-s^{\alpha_2}\Gamma(1-\alpha_2)}$$

for $\mathrm{Re}(s) \geq 0$, and by (42.181)

$$(228) \qquad \underset{\sim}{E}\{e^{-s\theta^{-\alpha_1}}\} = E_{\alpha_1}\left(-\frac{s}{\Gamma(1-\alpha_1)}\right)$$

for $\mathrm{Re}(s) \geq 0$. Thus (225) can also be expressed as

$$\Omega(s) = \underset{\sim}{E}\{e^{-s\chi\theta^{-\alpha_1/\alpha_2}}\} \tag{229}$$

for $\operatorname{Re}(s) \geqq 0$. This is in agreement with (216).

Note 3. If in Definition 2 we do not require that the sequences $\{\theta_n\}$ and $\{\chi_n\}$ be independent, then we arrive at the notion of a generalized compound recurrent process $\{\chi(t)\ ,\ 0 \leqq t < \infty\}$. If (θ_n, χ_n) $(n = 1,2,\ldots)$ are independent and identically distributed vector variables and if $\underset{\sim}{P}\{\theta_n > 0\} = 1$ and

$$\underset{\sim}{E}\{e^{-q\theta_n - s\chi_n}\} = \psi(q,\ s) \tag{230}$$

for $\operatorname{Re}(q) \geqq 0$ and $\operatorname{Re}(s) = 0$, then we have

$$q\int_0^\infty e^{-qt}\ \underset{\sim}{E}\{e^{-s\chi(t)}\}dt = \frac{1 - \psi(q,\ 0)}{1 - \psi(q,\ s)} \tag{231}$$

for $\operatorname{Re}(q) > 0$ and $\operatorname{Re}(s) = 0$. If $\underset{\sim}{P}\{\chi_n > 0\} = 1$, then (230) and (231) hold for $\operatorname{Re}(s) \geqq 0$ too.

In several cases we can easily determine the asymptotic distribution of $\chi(t)$ as $t \to \infty$ by using (231). As an example let us suppose that $\underset{\sim}{P}\{\chi_n > 0\} = 1$ and

$$\lim_{n\to\infty} \underset{\sim}{P}\left\{\frac{\theta_1+\ldots+\theta_n}{n^a} \leqq x\ ,\ \frac{\chi_1+\ldots+\chi_n}{n^b} \leqq y\right\} = F(x,\ y) \tag{232}$$

where $a > 1$ and $b > 1$. Let

$$\Phi(q,\ s) = \int_0^\infty\int_0^\infty e^{-qx-sy}\ d_x d_y\ F(x,\ y) \tag{233}$$

for $\operatorname{Re}(q) \geqq 0$ and $\operatorname{Re}(s) \geqq 0$. By (232) we have

(234) $$\lim_{n \to \infty} [\psi(\frac{q}{n^a}, \frac{s}{n^b})]^n = \Phi(q, s)$$

and

(235) $$\lim_{n \to \infty} n[\psi(\frac{q}{n^a}, \frac{s}{n^b}) - 1] = \log \Phi(q, s)$$

for $\operatorname{Re}(q) \geqq 0$ and $\operatorname{Re}(s) \geqq 0$.

If ν is a random variable which has the distribution (218) and which is independent of $\{\chi(t), 0 \leqq t < \infty\}$, then by (231) we have

(236) $$\underset{\sim}{E}\{e^{-s\chi(\nu/q)}\} = \frac{1 - \psi(q, 0)}{1 - \psi(q, s)}$$

for $q > 0$ and $\operatorname{Re}(s) \geqq 0$. Hence

(237) $$\lim_{q \to 0} \underset{\sim}{E}\{e^{-sq^{b/a}\chi(\nu/q)}\} = \lim_{q \to 0} \frac{[1-\psi(q,0)]q^{-1/a}}{[1-\psi(q,sq^{b/a})]q^{-1/a}} = \frac{\log \Phi(1, 0)}{\log \Phi(1, s)}$$

for $\operatorname{Re}(s) \geqq 0$. From (231) we can deduce that

(238) $$\lim_{t \to \infty} \underset{\sim}{P}\{ \frac{\chi(t)}{t^{b/a}} \leqq x\} = Q(x)$$

exists, and if

(239) $$\Omega(s) = \int_0^\infty e^{-sx}\, dQ(x)$$

for $\operatorname{Re}(s) \geqq 0$, then

(240) $$\int_0^\infty \Omega(sx^{b/a}) e^{-x}\, dx = \frac{\log \Phi(1, 0)}{\log \Phi(1, s)}$$

for $\operatorname{Re}(s) \geq 0$. From (240) $\Omega(s)$ can be obtained by inversion.

If we suppose that ξ, η_1, η_2 are mutually independent random variables for which $\mathbf{P}\{\xi \leq x\} = Q(x)$ and $\mathbf{P}\{\eta_1 \leq x\} = \mathbf{P}\{\eta_2 \leq x\} = 1 - e^{-x}$ for $x \geq 0$, then by (240) we obtain that

$$\mathbf{P}\{\xi\eta_1^{b/a}\eta_2^{-1} \leq x\} = \frac{\log \Phi(1, 0)}{\log \Phi(1, 1/x)} \tag{241}$$

for $x > 0$. By (241) we have

$$\mathbf{E}\{\xi^s\}\mathbf{E}\{\eta_1^{bs/a}\}\mathbf{E}\{\eta_2^{-s}\} = \int_0^\infty x^s d\, \frac{\log \Phi(1, 0)}{\log \Phi(1, 1/x)} , \tag{242}$$

or

$$\mathbf{E}\{\xi^s\} = \frac{1}{\Gamma(1-s)\Gamma(1 + \frac{bs}{a})} \int_0^\infty x^s d\, \frac{\log \Phi(1, 0)}{\log \Phi(1, 1/x)} \tag{243}$$

for sufficiently small $|\operatorname{Re}(s)|$ and hence $\mathbf{P}\{\xi \leq x\} = Q(x)$ can be obtained by Mellin's inversion formula.

We note that if $\mathbf{P}\{\theta \leq x , \chi \leq y\} = F(x, y)$, $\mathbf{P}\{\eta_1 \leq x\} = \mathbf{P}\{\eta_2 \leq x\} = 1 - e^{-x}$ for $x \geq 0$, and (θ, χ) , η_1, η_2 are mutually independent, then by (233) we have

$$\mathbf{P}\{\theta\eta_1^{-1} \leq x, \chi\eta_2^{-1} \leq y\} = \int_0^\infty\int_0^\infty \mathbf{P}\{\theta \leq xu, \chi \leq yv\}e^{-(u+v)}du dv = \Phi(\tfrac{1}{x}, \tfrac{1}{y}) \tag{244}$$

for $x > 0$ and $y > 0$. If we introduce the notation

$$\mho(s) = \frac{\log \Phi(1, s)}{\log \Phi(1, 0)} \tag{245}$$

for $\operatorname{Re}(s) \geq 0$ and if we take into consideration that

$$\frac{\log \Phi(z, sz^{b/a})}{\log \Phi(z, 0)} = U(s) \tag{246}$$

for $Re(s) \geqq 0$ and $Re(z) > 0$, $\log \Phi(s, 0) = -As^{1/a}$ and $\log \Phi(0, s) = -Bs^{1/b}$ for $Re(s) \geqq 0$ where $A > 0$ and $B > 0$, then we can prove that

$$\begin{aligned} \underset{\sim}{P}\{\chi\theta^{-b/a}\eta_1^{b/a}\eta_2^{-1} \leqq x\} &= \iint_{u^{-b/a}v \leqq x} d_u\, d_v\, \Phi(\tfrac{1}{u}, \tfrac{1}{v}) = \\ &= 1 - \frac{bU'(1/x)}{xU(1/x)} \end{aligned} \tag{247}$$

for $x > 0$.

By (241) and (247) we can conclude that $Q(x) = \underset{\sim}{P}\{\xi \leqq x\} = \underset{\sim}{P}\{\chi\theta^{-b/a} \leqq x\}$ if and only if

$$U(x) - bxU'(x) = 1 \tag{248}$$

for $x > 0$ and $\lim_{x \to \infty} U(x)x^{-b/a} = B/A$. These conditions are satisfied if and only if

$$U(x) = 1 + \frac{B}{A}x^{1/b}, \tag{249}$$

or

$$\Phi(q, s) = e^{-Aq^{1/a} - Bs^{1/b}}. \tag{250}$$

50. Brownian Motion and Gaussian Processes.

The notion of the Brownian motion process is based on the definition of the normal distribution. We say that a random variable ξ has a normal distribution of type $N(a, \sigma^2)$ where σ is a positive number, if

$$(1) \qquad \mathbf{P}\{\xi \leq x\} = \Phi\left(\frac{x-a}{\sigma}\right)$$

where

$$(2) \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \ .$$

The parameters a and σ^2 have simple probability interpretations. We have $\mathbf{E}\{\xi\} = a$ and $\mathbf{Var}\{\xi\} = \sigma^2$.

The normal distribution has its origin in the investigations of A. De Moivre [325], P. S. Laplace [351] and C. F. Gauss [336] . See the discussion at the beginning of Section 39.

Definition 1. *We say that a family of real random variables* $\{\xi(u) , 0 \leq u < \infty\}$ *forms a Brownian motion process if the following conditions are satisfied:*

(i) *For* $k = 2,3,\ldots$ *and for any* $0 \leq t_0 < t_1 < \ldots < t_k$ *the random variables* $\xi(t_1) - \xi(t_0) , \xi(t_2) - \xi(t_1),\ldots, \xi(t_k) - \xi(t_{k-1})$ *are mutually independent.*

(ii) $\mathbf{P}\{\xi(0) = 0\} = 1$.

(iii) For $0 \leq u < u + t$ we have

$$P\{\xi(u+t) - \xi(u) \leq x\} = \Phi(\frac{x}{\sqrt{t}}) \tag{3}$$

where $\Phi(x)$ is given by (2).

By Theorem 47.1 we can conclude that the above defined process $\{\xi(u), 0 \leq u < \infty\}$ indeed exists. The conditions (i), (ii), (iii) uniquely determine the finite dimensional distribution functions of the process and these distribution functions are consistent.

By Theorem 47.2 we may assume without loss of generality that the process $\{\xi(t), 0 \leq t < \infty\}$ is separable.

The stochastic process $\{\xi(u), 0 \leq u < \infty\}$ was introduced in 1900 by L. Bachelier [321] in studying the fluctuations of prices in a stock exchange. The process $\{\xi(u), 0 \leq u < \infty\}$ also appears in the theory of random walks and in studying the phenomenon of Brownian motion. (See Section 37.) The first rigorous mathematical description of the Brownian motion was given in 1923 by N. Wiener [370]. See also P. Lévy [352], K. Itô and H. P. McKean [342], and D. Freedman [334].

Let us define

$$\zeta(u) = au + \sigma\xi(u) \tag{4}$$

for $0 \leq u < \infty$ where $\{\xi(u), 0 \leq u < \infty\}$ satisfies the conditions (i), (ii), (iii) and a is a real number and σ is a positive real number. Then $\{\zeta(u), 0 \leq u < \infty\}$ too satisfies conditions (i) and (ii) and we have

$$(5) \qquad P\{\zeta(u+t) - \zeta(u) \leq x\} = \Phi(\frac{x-at}{\sigma\sqrt{t}})$$

for $0 \leq u < u + t$. The process $\{\zeta(u) , 0 \leq u < \infty\}$ is called a general Brownian motion process.

The following theorem was essentially found in 1923 by N. Wiener [370]. See also J. L. Doob [30 p. 393].

Theorem 1. Almost all sample functions of a separable Brownian motion process are continuous.

Proof. Let $(\Omega, \mathcal{B}, P)$ be a probability space and $\xi(u) = \xi(u, \omega)$ $(0 \leq u < \infty, \ \omega \in \Omega)$ a family of random variables which satisfies conditions (i), (ii), (iii). If we suppose that $\{\xi(u) , 0 \leq u < \infty\}$ is a separable process, then $\sup_{0 \leq u \leq t} \xi(u)$ is a random variable for every $t \geq 0$, and we have the inequality

$$(6) \qquad P\{\sup_{0 \leq u \leq t} \xi(u) > x\} \leq 2P\{\xi(t) > x\}$$

for every x . We shall prove that for any $k = 2,3,\ldots$ and for any $t_0 = 0 < t_1 < \ldots < t_k = t$ we have

$$(7) \qquad P\{\max_{0 \leq j \leq n} \xi(t_j) > x\} \leq 2P\{\xi(t) > x\} .$$

Since the process $\{\xi(u) , 0 \leq u < \infty\}$ is separable, therefore (7) implies (6).

The inequality (7) follows from the following two inequalities. First, evidently we have

$$(8) \qquad \mathbf{P}\{\max_{0\leq j\leq n} \xi(t_j) > x\ ,\ \xi(t) > x\} = \mathbf{P}\{\xi(t) > x\}$$

for every x. Second, if we define ν as the smallest $j = 0,1,\ldots,n$ (if any) for which $\xi(t_j) > x$, then we can write that

$$\mathbf{P}\{\max_{0\leq j\leq n} \xi(t_j) > x\ ,\ \xi(t) \leq x\} = \sum_{j=0}^{n-1} \mathbf{P}\{\nu = j\ ,\ \xi(t) \leq x\} \leq$$

$$(9) \quad \begin{cases} \leq \displaystyle\sum_{j=0}^{n-1} \mathbf{P}\{\nu = j\ ,\ \xi(t) - \xi(t_j) < 0\} = \sum_{j=0}^{n-1} \mathbf{P}\{\nu = j\}\mathbf{P}\{\xi(t) - \xi(t_j) < 0\} = \\ = \displaystyle\sum_{j=0}^{n-1} \mathbf{P}\{\nu = j\}\mathbf{P}\{\xi(t) - \xi(t_j) > 0\} = \sum_{j=0}^{n-1} \mathbf{P}\{\nu = j\ ,\ \xi(t) - \xi(t_j) > 0\} \end{cases}$$

$$\leq \sum_{j=0}^{n-1} \mathbf{P}\{\nu = j\ ,\ \xi(t) > x\} \leq \mathbf{P}\{\xi(t) > x\}$$

for every x. If we add (8) and (9), then we obtain (7).

From (6) it follows that

$$\mathbf{P}\{\sup_{0\leq u\leq t} |\xi(u)| > x\} \leq 4\mathbf{P}\{\xi(t) > x\} = 4\,[1 - \Phi(x)] = \qquad \frac{x}{\sqrt{t}}$$

$$(10) \qquad = \frac{4}{\sqrt{2\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-u^2/2}\,du < \frac{4\sqrt{t}}{\sqrt{2\pi}\,x} \int_{x/\sqrt{t}}^{\infty} u\,e^{-u^2/2}\,du = \frac{4\sqrt{t}\,e^{-x^2/2t}}{\sqrt{2\pi}\,x}$$

for $x > 0$.

Let

$$(11) \qquad A_n = \{\omega : \sup|\xi(u,\omega)-\xi(\tfrac{j}{n},\omega)| > \frac{1}{n^{1/4}} \text{ for } |u-\tfrac{j}{n}| \leqq \tfrac{1}{n} \text{ and } j=1,2,\dots,n^2\}$$

for $n = 1,2,\dots$ and denote by A^* the event that infinitely many events occur in the sequence $A_1, A_2,\dots, A_n,\dots$.

Now by (6) we can write that

$$(12) \qquad \begin{aligned} \underset{\sim}{P}\{A_n\} &\leqq n^2\underset{\sim}{P}\{\sup|\xi(u)-\xi(\tfrac{j}{n})| > \frac{1}{n^{1/4}} \text{ for } |u-\tfrac{j}{n}| \leqq \tfrac{1}{n}\} \leqq \\ &\leqq 2n^2\underset{\sim}{P}\{\sup_{0\leqq u\leqq \frac{1}{n}}|\xi(u)| > \frac{1}{n^{1/4}}\} \leqq \frac{8n^{7/4}}{\sqrt{2\pi}} e^{-\sqrt{n}/2} . \end{aligned}$$

Since

$$(13) \qquad \sum_{n=1}^{\infty} \underset{\sim}{P}\{A_n\} < \infty ,$$

therefore by Theorem 41.1 it follows that $\underset{\sim}{P}\{A^*\} = 0$.

Accordingly, if $\omega \notin A^*$, then

$$(14) \qquad |\xi(u,\omega)-\xi(\tfrac{j}{n},\omega)| \leqq \frac{1}{n^{1/4}} \text{ for } |u-\tfrac{j}{n}| \leqq \tfrac{1}{n} \text{ and } j = 1,2,\dots, n^2$$

for every $n = 1,2,\dots$ except a finite number of n's .

Thus if $\omega \notin A^*$, then for any $\varepsilon > 0$ and $t > 0$ there exists a $\delta = \delta(\varepsilon,t,\omega)$ such that $|\xi(u,\omega)-\xi(v,\omega)| < \varepsilon$ whenever $|u-v| < \delta$ and $u \in [0,t]$, $v \in [0,t]$. For each $\omega \notin A^*$ let us choose an $n = n(\omega)$ such that $n > (2/\varepsilon)^4$ and $n > t$ and (14) is satisfied and let $\delta = 1/n$. If $|u-v| < \delta$, then there is a $j = 1,2,\dots, n^2$ such that $|u-\frac{j}{n}| \leqq \delta$ and $|v-\frac{j}{n}| \leqq \delta$, and thus by (14) we have $|\xi(u,\omega)-\xi(v,\omega)| \leqq 2/n^{1/4} < \varepsilon$. This completes the proof of the theorem.

Theorem 1 makes it possible to define a Brownian motion process in the following way. Let Ω, the sample space, be the set of all those continuous functions $\omega(u)$ defined on the interval $[0,\infty)$ for which $\omega(0) = 0$. Let $\mathcal{B}$ be the smallest σ-algebra which contains the sets $A(t,x) = \{\omega(u):\omega(t) \leq x\}$ for all $t \geq 0$ and x. Let $\underset{\sim}{P}$ be the probability measure which satisfies

$$(15)\quad \underset{\sim}{P}\{A(t_1,x_1)\ldots A(t_k,x_k)\} = \underset{\substack{y_1+\ldots+y_r\leq x_r \\ (r=1,2,\ldots,k)}}{\iint\ldots\int} \prod_{i=1}^{k} \phi\left(\frac{y_i}{t_i-t_{i-1}}\right)\frac{1}{\sqrt{(t_i-t_{i-1})}}\, dy_1 dy_2 \ldots dy_k$$

for all $t_0 = 0 < t_1 < \ldots < t_k$ and $x_1, x_2, \ldots, x_k$ where

$$(16)\qquad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The probability measure $\underset{\sim}{P}$ is uniquely determined for $\mathcal{B}$ by (15).

If we define $\xi(u,\omega) = \omega(u)$ for $0 \leq u < \infty$ and $\omega \in \Omega$ whenever $\omega = \{\omega(u),\ 0 \leq u < \infty\}$, then $\{\xi(u,\omega),\ 0 \leq u < \infty,\ \omega \in \Omega\}$ is a Brownian motion process for which the sample functions are continuous for every $\omega \in \Omega$.

In what follows if we speak about a Brownian motion process then we may assume without loss of generality that all the sample functions are continuous functions of u.

In 1956 G. A. Hunt [339] demonstrated that a separable Brownian motion process has an important property, the so-called strong Markov property. This property is based on the notion of stopping time. Let

$\{\xi(u)\ ,\ 0 \leq u < \infty\}$ be a Brownian motion process. A nonnegative random variable τ is called a stopping time if for every $u \geq 0$ we have

$$\{\tau \leq u\} \in \mathcal{B}_u \tag{17}$$

where $\mathcal{B}_u$ is the σ-algebra generated by the random variables $\{\xi(s)\ ,\ 0 \leq s \leq u\}$.

Let us denote by $\mathcal{A}$ the σ-algebra which consists of all those events $A \in \mathcal{B}$ for which $A \cap \{\tau \leq u\} \in \mathcal{B}_u$ for every u .

Theorem 2. *Let τ be a stopping time of a separable Brownian motion process $\{\xi(u)\ ,\ 0 \leq u < \infty\}$. Let*

$$\xi^*(u) = \xi(\tau+u)-\xi(\tau) \tag{18}$$

for $u \geq 0$. Then $\{\xi^*(u)\ ,\ 0 \leq u < \infty\}$ is also a separable Brownian motion process and $\{\xi(u)\ ,\ 0 \leq u \leq \tau\}$ and $\{\xi^*(u)\ ,\ 0 \leq u < \infty\}$ are independent processes, that is, if $A \in \mathcal{A}$ and $B \in \mathcal{B}^*$ where $\mathcal{B}^*$ is the σ-algebra generated by the random variables $\{\xi^*(u)\ ,\ 0 \leq u < \infty\}$, then A and B are independent.

Proof. Let

$$B = \{\xi^*(u_i) \leq x_i \quad \text{for} \quad i = 1,2,\dots,r\} \tag{19}$$

and

$$B(s) = \{\xi(s+u_i)-\xi(s) \leq x_i \quad \text{for} \quad i = 1,2,\dots,r\} \tag{20}$$

where $0 \leq u_1 < u_2 \dots < u_r$ and $x_1, x_2, \dots, x_r$ are real numbers.

For each $n = 1,2,\ldots$ let us define

(21) $$\tau_n = \frac{k}{n} \text{ if } \frac{k-1}{n} < \tau \leqq \frac{k}{n} \text{ and } k = 0,1,2,\ldots .$$

We can easily see that τ_n is a stopping time for each $n = 1,2,\ldots$.

If in (18) we replace τ by τ_n, then let B_n the event which corresponds to B given by (19) .

If $A \in \mathcal{A}$ and B_n is given by (19) with $\tau = \tau_n$, then we have

(22) $$\begin{aligned} \underset{\sim}{P}\{AB_n\} &= \sum_{k=0}^{\infty} \underset{\sim}{P}\{AB_n \text{ and } \tau_n = \tfrac{k}{n}\} = \sum_{k=0}^{\infty} \underset{\sim}{P}\{AB(\tfrac{k}{n}) \text{ and } \tau_n = \tfrac{k}{n}\} = \\ &= \sum_{k=0}^{\infty} \underset{\sim}{P}\{A \text{ and } \tau_n = \tfrac{k}{n}\}\underset{\sim}{P}\{B(\tfrac{k}{n})\} = \underset{\sim}{P}\{A\}\underset{\sim}{P}\{B(0)\} \end{aligned}$$

because $\underset{\sim}{P}\{B(s)\} = \underset{\sim}{P}\{B(0)\}$ for all $s \geqq 0$. Since the sample functions are continuous with probability 1 it follows that $\lim_{n\to\infty} \underset{\sim}{P}\{AB_n\} = \underset{\sim}{P}\{AB\}$ and thus $\underset{\sim}{P}\{AB\} = \underset{\sim}{P}\{A\}\underset{\sim}{P}\{B(0)\}$ for every $A \in \mathcal{A}$ and B defined by (19). Consequently $\underset{\sim}{P}\{B\} = \underset{\sim}{P}\{B(0)\}$, and A and B are independent. This completes the proof of the theorem.

We note that if $\{\xi(u) , 0 \leqq u < \infty\}$ is a Brownian motion process and s is any positive number, then $\{\xi(us)/\sqrt{s} , 0 \leqq u < \infty\}$ is also a Brownian motion process. Furthermore, $\{u\xi(1/u) , 0 \leqq u < \infty\}$ is also a Brownian motion process.

If $\xi_0, \xi_1,\ldots, \xi_k,\ldots$ are mutually independent and identically distributed random variables with distribution function $\underset{\sim}{P}\{\xi_k \leqq x\} = \Phi(x)$ defined by (2), then

$$\xi(t) = t\xi_0 + \sqrt{2} \sum_{k=1}^{\infty} \frac{\sin k\pi t}{k\pi} \xi_k \tag{23}$$

is a Brownian motion process on the interval $[0, 1]$. Furthermore,

$$\xi(t) = \sqrt{2} \sum_{k=0}^{\infty} \frac{\sin(k+\frac{1}{2})\pi t}{(k+\frac{1}{2})\pi} \xi_k \tag{24}$$

is also a Brownian motion process on the interval $[0, 1]$.

Both in (23) and (24) the sums converge with probability 1 for every $t \varepsilon [0, 1]$ and thus $\xi(t)$ is a random variable for every $t \varepsilon [0, t]$.

The representations (23) and (24) can be obtained from some results of N. Wiener [370] and R. E. A. C. Paley and N. Wiener [69] on the harmonic analysis of random functions.

From a more general result of J. L. Doob [27] we can conclude that the law of large numbers is valid for a Brownian motion process.

Theorem 3. If $\{\xi(u), 0 \leq u < \infty\}$ is a separable Brownian motion process, then

$$\mathbf{P}\{\lim_{t\to\infty} \frac{\xi(t)}{t} = 0\} = 1 . \tag{25}$$

Proof. Since $\xi(n) - \xi(n-1)$ $(n = 1,2,\ldots)$ are mutually independent and identically distributed random variables with $\mathbf{E}\{\xi(n) - \xi(n-1)\} = 0$, it follows from Theorem 43.3 that

$$\lim_{n\to\infty} \frac{\xi(n)}{n} = 0 \tag{26}$$

with probability. On the other hand

(27) $$\sup_{n \leq u < n+1} |\xi(u) - \xi(n)| \qquad (n = 1,2,\ldots)$$

are also mutually independent and identically distributed random variables with expectation

(28) $$E\{\sup_{n \leq u < n+1} |\xi(u)-\xi(n)|\} \leq 2E\{|\xi(1)|\} = \frac{4}{\sqrt{2\pi}} \int_0^\infty x\, e^{-x^2/2}\, dx = \frac{4}{\sqrt{2\pi}} .$$

The last inequality follows from (6). Thus by Theorem 43.3 we obtain that

(29) $$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \sup_{j \leq u < j+1} |\xi(u) - \xi(j)| = E\{\sup_{0 \leq u < 1} |\xi(u)|\}$$

with probability 1 , and therefore

(30) $$\lim_{n \to \infty} \frac{1}{n} \sup_{n \leq u < n+1} |\xi(u) - \xi(n)| = 0$$

with probability 1 . If $n \leq t < n+1$, then

(31) $$\left| \frac{\xi(t)}{t} - \frac{\xi(n)}{n} \right| \leq \frac{1}{n} \sup_{n \leq t < n+1} |\xi(t) - \xi(n)| + \frac{1}{n^2}|\xi(n)|$$

and by (26) and (30) we obtain that

(32) $$\lim_{t \to \infty} \frac{\xi(t)}{t} = \lim_{n \to \infty} \frac{\xi(n)}{n} = 0$$

with probability 1 . This proves (25).

For a Brownian motion process $\{\xi(u) , 0 \leq u < \infty\}$ the law of iterated logarithm is also valid and we have

$$(33) \qquad \underset{\sim}{P}\{\limsup_{t\to\infty} \frac{\xi(t)}{\sqrt{2t \log\log t}} = 1\} = 1 .$$

See A. Ya. Khintchine [128].

Next we shall define a more general class of stochastic processes which class contains the Brownian motion processes as a particular case. This more general class is the class of Gaussian processes. The definition of a Gaussian process is based on the notion of the multidimensional normal distribution. Multidimensional normal distributions were studied as early as in 1846 by A. Bravais [11].

We say that the real random variables $\xi_1, \xi_2, \ldots, \xi_n$ have an n-dimensional normal distribution of type

$$(34) \qquad N\left(\left\|\begin{matrix} a_1 \\ \vdots \\ a_n \end{matrix}\right\|, \left\|\begin{matrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{matrix}\right\|\right)$$

where $a_1, a_2, \ldots, a_n$ are real numbers, $\sigma_{ij} = \sigma_{ji}$ and

$$(35) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

is a positive definite quadratic form, if $\xi_1, \xi_2, \ldots, \xi_n$ have the joint density function

$$(36) \qquad f(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{2\pi D^n}} e^{-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(x_i-a_i)(x_j-a_j)}$$

where

$$(37) \qquad D = \begin{vmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{vmatrix}$$

and

$$(38) \qquad \begin{Vmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{Vmatrix} = \begin{Vmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{Vmatrix}^{-1} .$$

The parameters $a_1, \dots, a_n$ and $\sigma_{11}, \dots, \sigma_{nn}$ have simple probability interpretation. We have

$$(39) \qquad \mathbf{E}\{\xi_i\} = a_i$$

for $i = 1, 2, \dots, n$ and

$$(40) \qquad \mathbf{E}\{(\xi_i - a_i)(\xi_j - a_j)\} = \sigma_{ij}$$

for $1 \leqq i \leqq n$ and $1 \leqq j \leqq n$.

Let T be a finite or infinite interval, say, $T = (0, 1)$ or $T = (0, \infty)$.

Definition 2. A real stochastic process $\{\xi(u) , u \in T\}$ is called Gaussian, if for any finite subset $(u_1, u_2, \dots, u_n)$ of the parameter set T the random variables $\xi(u_1), \xi(u_2), \dots, \xi(u_n)$ have a joint normal distribution.

If $\{\xi(u), u \in T\}$ is a Gaussian stochastic process and if we know the expectation

$$(41) \qquad \mathbf{E}\{\xi(u)\} = a(u)$$

for $u \in T$ and the covariance

$$\text{(42)} \qquad \operatorname{Cov}\{\xi(u),\ \xi(v)\} = E\{[\xi(u) - a(u)][\xi(v) - a(v)]\} = r(u,\ v)$$

for $u \in T$ and $v \in T$, then the finite dimensional distribution functions of the process are uniquely determined by (41) and (42).

Conversely, if $a(u)$ is any real function defined for $u \in T$ and $r(u, v)$ is a real function defined for $u \in T$ and $v \in T$ which satisfies the conditions: (i) $r(u, v) = r(v, u)$ for all $u \in T$ and $v \in T$ and (ii) for any finite subset $(u_1, u_2, \ldots, u_n)$ of T the quadratic form

$$\text{(43)} \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} r(u_i,\ u_j) x_i x_j$$

is positive definite, then there exists a Gaussian process $\{\xi(u),\ u \in T\}$ for which (41) and (42) hold. This follows from Theorem 47.1 .

If $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a Brownian motion process, then $\{\xi(u)\ ,\ 0 < u < \infty\}$ is a Gaussian process for which $E\{\xi(u)\} = 0$ and

$$\text{(44)} \qquad E\{\xi(u)\xi(v)\} = \min(u,\ v)\ .$$

We can obtain Gaussian processes from a Brownian motion process by suitable transformations. For example if $\{\xi(u)\ ,\ 0 \leq u < \infty\}$ is a Brownian motion process and

$$\text{(45)} \qquad \eta(u) = (1-u)\xi\left(\frac{u}{1-u}\right)$$

for $0 < u < 1$, then $\{\eta(u)\ ,\ 0 < u < 1\}$ is a Gaussian process for which

$$\text{(46)} \qquad E\{\eta(u)\} = 0$$

and

$$E\{\eta(u)\eta(v)\} = \min(u, v) - uv \; . \tag{47}$$

See J. L. Doob [328].

If we suppose that $\{\xi(u)\,,\, 0 \leqq u < \infty\}$ is a separable Brownian motion process and if $\eta(u)$ is defined by (45) for $0 < u < 1$, and $P\{\eta(0) = 0\} = 1$ and $P\{\eta(1) = 0\} = 1$, then the process $\{\eta(u)\,,\, 0 \leqq u \leqq 1\}$ has continuous sample functions with probability 1. The process $\{\eta(u)\,,\, 0 \leqq u \leqq 1\}$ can also be represented in the following form

$$\eta(u) = \sqrt{2} \sum_{k=1}^{\infty} \frac{\sin k\pi u}{k\pi} \eta_k \tag{48}$$

where $\eta_1, \eta_2, \ldots, \eta_k, \ldots$ is a sequence of mutually independent and identically distributed random variables with distribution function $P\{\eta_k \leqq x\} = \Phi(x)$ defined by (2). In (48) the sum converges with probability 1 and thus $\eta(u)$ is a random variable for every u.

We note that if $\{\eta(u)\,,\, 0 \leqq u \leqq 1\}$ is the process defined above and if η_0 is a random variable which is independent of $\{\eta(u)\,,\, 0 \leqq u \leqq 1\}$ and which has the distirbution function $P\{\eta_0 \leqq x\} = \Phi(x)$, then

$$\xi(u) = u\xi_0 + \eta(u) \tag{49}$$

defined for $0 \leqq u \leqq 1$ is a Brownian motion process.

51. Stochastic Processes with Independent Increments.

Definition. *We say that a family of arbitrary random variables* $\{\xi(u), 0 \leq u < \infty\}$ *forms a stochastic process with independent increments if for any* $k = 2,3,\ldots$ *and* $0 \leq t_0 < t_1 < \ldots < t_k$ *the random variables* $\xi(t_i) - \xi(t_{i-1})$ $(i = 1,2,\ldots, k)$ *are mutually independent.*

We say that a stochastic process $\{\xi(u), 0 \leq u < \infty\}$ is *homogeneous* if for $0 \leq u < u+t$ the distribution of $\xi(u+t) - \xi(u)$ does not depend on u.

In what follows we shall consider only real homogeneous stochastic processes with independent increments, or in other words, real stochastic processes with stationary independent increments.

The Poisson process and the Brownian motion process, discussed in the previous two sections, are examples for real homogeneous stochastic processes with independent increments. In fact these processes are the building blocks of a general real stochastic process with independent increments.

Theorem 1. *Let* $\{\xi(u), 0 \leq u < \infty\}$ *be a homogeneous real stochastic process with independent increments. If* $\mathbf{P}\{\xi(0) = 0\} = 1$, *then*

$$\mathbf{E}\{e^{-s\xi(u)}\} = e^{u\Psi(s)} \tag{1}$$

exists for $\mathrm{Re}(s) = 0$ *and the most general form of* $\Psi(s)$ *is given by*

$$\Psi(s) = -as + \frac{1}{2}\sigma^2 s^2 + \int_{-\infty}^{-0} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dM(x) +$$

(2)

$$+ \int_{+0}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dN(x)$$

where a *is a real constant,* σ^2 *is a nonnegative constant,* $M(x)$ $(-\infty < x < 0)$ *and* $N(x)$ $(0 < x < \infty)$ *are nondecreasing functions of* x *satisfying the conditions* $\lim_{x \to -\infty} M(x) = 0$, $\lim_{x \to \infty} N(x) = 0$ *and*

$$\int_{-\varepsilon}^{-0} x^2 dM(x) + \int_{+0}^{\varepsilon} x^2 dN(x) < \infty \tag{3}$$

for some $\varepsilon > 0$.

Proof. For every $n = 1,2,\ldots$ we can write that

$$\xi(1) = \sum_{i=1}^{n} \left[\xi\left(\frac{i}{n}\right) - \xi\left(\frac{i-1}{n}\right)\right] \tag{4}$$

where $\xi(\frac{i}{n}) - \xi(\frac{i-1}{n})$ $(i = 1,2,\ldots, n)$ are mutually independent and identically distributed random variables. Thus by Definition 41.1 the distribution function $\underset{\sim}{P}\{\xi(1) \leq x\}$ is infinitely divisible and by Theorem 41.2 we can conclude that

$$\underset{\sim}{E}\{e^{-s\xi(1)}\} = e^{\Psi(s)} \tag{5}$$

for $\mathrm{Re}(s) = 0$ where $\Psi(s)$ is given by (2). Since

$$\underset{\sim}{E}\{e^{-s\xi(u+v)}\} = \underset{\sim}{E}\{e^{-s\xi(u)}\}\underset{\sim}{E}\{e^{-s\xi(v)}\} \tag{6}$$

for $u \geq 0$ and $v \geq 0$ and $|E\{e^{-s\xi(u)}\}| \leq 1$ for $Re(s) = 0$ and $u \geq 0$, it follows that (1) holds for all $u \geq 0$.

The representation (2) was found in 1934 by P. Lévy [435], [436]. In some particular cases the representation (2) was earlier found by B. De Finetti [412], [413], [414], [415], [416], and A. N. Kolmogorov [432], [433].

From the representation (1) it follows that

(7) $$E\{\xi(u)\} = -u\Psi'(0) = u[a + \int_{-\infty}^{-0} \frac{x^3}{1+x^2}\, dM(x) + \int_{+0}^{\infty} \frac{x^3}{1+x^2}\, dN(x)]$$

provided that the integrals on the right-hand side are convergent. Furthermore, we have

(8) $$Var\{\xi(u)\} = u\Psi''(0) = u[\sigma^2 + \int_{-\infty}^{-0} x^2 dM(x) + \int_{+0}^{\infty} x^2 dN(x)]$$

provided that the integrals on the right-hand side are convergent. Both in (7) and (8) we form the derivatives of $\Psi(s)$ along the line $Re(s) = 0$.

Now we shall prove a few auxiliary theorems which will be useful in studying homogeneous stochastic processes with independent increments.

Lemma 1. Let ξ and η be real random variables having finite expectations. If $E\{\xi\} = 0$, then

(9) $$E\{|\eta|\} \leq E\{|\xi + \eta|\}.$$

Proof. Since $x = \underset{\sim}{E}\{\xi + x\}$, we have

(10) $$|x| = |\underset{\sim}{E}\{\xi + x\}| \leq \underset{\sim}{E}\{|\xi + x|\} .$$

If we integrate (10) with respect to $\underset{\sim}{P}\{\eta \leq x\}$, then we obtain (9).

Lemma 2. Let $\xi_1, \xi_2, \ldots, \xi_n$ be mutually independent random variables for which $\underset{\sim}{E}\{|\xi_k|\} < \infty$ $(k = 1,2,\ldots, n)$. Set $\zeta_k = \xi_1 + \xi_2 + \ldots + \xi_k$ for $k = 1,2,\ldots, n$. If the random variables $\xi_1, \xi_2, \ldots, \xi_n$ have a symmetric distribution, then

(11) $$\underset{\sim}{E}\{\max_{1\leq k\leq n} |\zeta_k|\} \leq 2\underset{\sim}{E}\{|\zeta_n|\} .$$

Proof. Define $\nu = k$ $(k = 1,2,\ldots, n)$ if ζ_k is the first partial sum for which $\zeta_k > x$. Let $x > 0$. Then we can write that

(12)
$$\begin{aligned}
&\underset{\sim}{P}\{\max_{1\leq k\leq n} \zeta_k > x \text{ and } \zeta_n \leq x\} = \sum_{k=1}^{n-1} \underset{\sim}{P}\{\nu = k \text{ and } \zeta_n \leq x\} \leq \\
&\leq \sum_{k=1}^{n-1} \underset{\sim}{P}\{\nu = k \text{ and } \zeta_n - \zeta_k < 0\} = \sum_{k=1}^{n-1} \underset{\sim}{P}\{\nu = k \text{ and } \zeta_n - \zeta_k > 0\} \leq \\
&\leq \sum_{k=1}^{n-1} \underset{\sim}{P}\{\nu = k \text{ and } \zeta_n > x\} \leq \underset{\sim}{P}\{\zeta_n > x\}
\end{aligned}$$

and evidently

(13) $$\underset{\sim}{P}\{\max_{1\leq k\leq n} \zeta_k > x \text{ and } \zeta_n > x\} = \underset{\sim}{P}\{\zeta_n > x\} .$$

If we add (12) and (13) then we get

$$(14) \qquad P\{\max_{1 \le k \le n} \zeta_k > x\} \le 2P\{\zeta_n > x\} = P\{|\zeta_n| > x\} .$$

Hence it follows that

$$(15) \qquad P\{\max_{1 \le k \le n} |\zeta_k| > x\} \le P\{\max_{1 \le k \le n} \zeta_k > x\} + P\{\max_{1 \le k \le n} (-\zeta_k) > x\} \le \le 2P\{|\zeta_n| > x\} .$$

If we integrate (15) with respect to x from 0 to ∞, then we obtain (11) which was to be proved.

Lemma 3. Let $\xi_1, \xi_2, \ldots, \xi_n$ be mutually independent random variables for which $E\{|\xi_k|\} < \infty$ $(k = 1,2,\ldots, n)$. Set $\zeta_k = \xi_1 + \xi_2 + \ldots + \xi_k$ for $k = 1,2,\ldots, n$. If $E\{\xi_k\} = 0$ for $k = 1,2,\ldots, n$, then

$$(16) \qquad E\{\max_{1 \le k \le n} |\zeta_k|\} \le 5E\{|\zeta_n|\} .$$

Proof. Let $\xi_1^*, \xi_2^*, \ldots, \xi_n^*$ be mutually independent random variables which are independent of the variables $\xi_1, \xi_2, \ldots, \xi_n$ and for which $P\{\xi_k^* \le x\} = P\{\xi_k \le x\}$ $(k = 1,2,\ldots, n)$. Define $\zeta_k^* = \xi_1^* + \xi_2^* + \ldots + \xi_k^*$ for $k = 1,2,\ldots, n$. Since the variables $\xi_k - \xi_k^*$ $(k = 1,2,\ldots, n)$ are symmetrically distributed, by Lemma 2 we have

$$(17) \qquad E\{\max_{1 \le k \le n} |\zeta_k - \zeta_k^*|\} \le 2E\{|\zeta_n - \zeta_n^*|\} \le 4E\{|\zeta_n|\} .$$

If we take into consideration that

$$|\zeta_k| \leqq \max_{1\leqq k\leqq n} |\zeta_k - \zeta_k^*| + |\zeta_k^*| \ , \tag{18}$$

if we integrate the right-hand side of (18) with respect to the joint distribution function of $(\xi_1^*, \xi_2^*, \ldots, \xi_n^*)$, if we use the inequality

$$\mathop{E}_{\sim}\{|\zeta_k^*|\} \leqq \mathop{E}_{\sim}\{|\zeta_n^*|\} = \mathop{E}_{\sim}\{|\zeta_n|\} \tag{19}$$

which follows from Lemma 1, if we form the maximum of the left-hand side of (18) with respect to k $(k = 1,2,\ldots, n)$, and if we integrate both sides with respect to the joint distribution function of $(\xi_1, \xi_2, \ldots, \xi_n)$, then we obtain that

$$\mathop{E}_{\sim}\{\max_{1\leqq k\leqq n} |\zeta_k|\} \leqq \mathop{E}_{\sim}\{\max_{1\leqq k\leqq n} |\zeta_k - \zeta_k^*|\} + \mathop{E}_{\sim}\{|\zeta_n|\} \ . \tag{20}$$

By (17) and (20) we obtain (16) which was to be proved.

In what follows we always suppose that $\{\xi(u) \ , 0 \leqq u < \infty\}$ is **a** homogeneous real stochastic process with stationary independent increments for which $\mathop{P}_{\sim}\{\xi(0) = 0\} = 1$.

By using Lemma 3 we can prove that the strong law of large numbers is valid for homogeneous processes with independent increments. The following result is due to <u>J. L. Doob</u> [27].

<u>Theorem 2</u>. <u>Let</u> $\{\xi(u) \ , 0 \leqq u < \infty\}$ <u>be a separable, real, homogeneous stochastic process with independent increments for which</u> $\mathop{P}_{\sim}\{\xi(0) = 0\} = 1$. <u>If</u> $\mathop{E}_{\sim}\{\xi(u)\}$ <u>exists, then</u>

(21) $$\lim_{t \to \infty} \frac{\xi(t)}{t} = E\{\xi(1)\}$$

<u>with probability one</u>.

<u>Proof</u>. We may assume without loss of generality that $E\{\xi(1)\} = 0$. If $E\{\xi(1)\} \neq 0$, then let us consider the process $\xi(u) - E\{\xi(u)\}$ $(0 \leq u < \infty)$ instead of $\{\xi(u), 0 \leq u < \infty\}$.

If $E\{\xi(u)\} = 0$ for $u \geq 0$, then for any $t_0 = 0 < t_1 < \ldots < t_n = t$ we have

(22) $$E\{\max_{1 \leq k \leq n} |\xi(t_k)|\} \leq 5E\{|\xi(t)|\}.$$

Since the process $\{\xi(u), 0 \leq u < \infty\}$ is separable, it follows from (22) that

(23) $$E\{\sup_{0 \leq u \leq t} |\xi(u)|\} \leq 5E\{|\xi(t)|\}$$

holds for every $t \geq 0$.

Now we can repeat word for word the proof of Theorem 50.3. The only difference is that (50.28) should be replaced by

(24) $$E\{\sup_{n \leq u < n+1} |\xi(u) - \xi(n)|\} \leq 5E\{|\xi(1)|\} < \infty.$$

<u>Theorem 3.</u> <u>If</u> $\{\xi(u), 0 \leq u < \infty\}$ <u>is a real, homogeneous stochastic process with independent increments and if</u> $E\{[\xi(u)]^2\}$ <u>exists, and</u> $Var\{\xi(u)\} > 0$, <u>then</u>

$$(25) \qquad \lim_{t \to \infty} P\left\{ \frac{\xi(t) - E\{\xi(t)\}}{\sqrt{\operatorname{Var}\{\xi(t)\}}} \leq x \right\} = \Phi(x)$$

where $\Phi(x)$ is the normal distribution function.

Proof. We can easily show that the Laplace-Stieltjes transform of $[\xi(t) - E\{\xi(t)\}]/\sqrt{\operatorname{Var}\{\xi(t)\}}$ tends to $e^{s^2/2}$ as $t \to \infty$ and $\operatorname{Re}(s) = 0$. Thus (25) follows by Theorem 41.10.

We note that by Theorem 47.3 we can conclude that any countable and everywhere dense subset S of $[0, \infty)$ is a separability set of a separable process $\{\xi(u), 0 \leq u < \infty\}$. Since obviously

$$(26) \qquad P\{|\xi(t)| > \varepsilon\} \leq 2\left[1 - \left|\frac{\varepsilon}{4} \int_{-2/\varepsilon}^{2/\varepsilon} e^{t\Psi(iy)}\, dy\right|\right]$$

for any $\varepsilon > 0$ and since $\Psi(iy)$ is a continuous function of y, it follows that

$$(27) \qquad \lim_{t \to 0} P\{|\xi(t)| > \varepsilon\} = 0$$

for any $\varepsilon > 0$. Thus Theorem 47.3 is applicable.

By the investigations of P. Lévy [437], J. L. Doob [30], A. V. Skorokhod [446] and I. I. Gikhman and A. V. Skorokhod [44] we can completely describe the behavior of the sample functions of a stochastic process with independent increments.

We shall mention only briefly the main results without giving complete proofs.

Theorem 4. _If_ $\{\xi(u), 0 \leq u < \infty\}$ _is a separable, homogeneous, real stochastic process with independent increments, then with probability_ 1 _the limits_ $\xi(u+0)$ _exist for all_ $u \geq 0$ _and the limits_ $\xi(u-0)$ _exist for all_ $u > 0$.

The proof of this theorem is based on the following observation. If for any $\varepsilon > 0$ a function $x(u)$ defined on the interval $[0, t]$ has only a finite number of oscillations greater than $\varepsilon > 0$, then $x(u+0)$ exists for $u \in [0, t)$ and $x(u-0)$ exists for $u \in (0, t]$. We say that a function $x(u)$ in $[0, t]$ has at least n oscillations greater than ε if there are $n+1$ points $t_0, t_1, \ldots, t_n$ in $[0, t]$ such that $0 \leq t_0 < t_1 < \ldots < t_n \leq t$ and $|x(t_k) - x(t_{k-1})| > \varepsilon$ holds for all $k = 1, 2, \ldots, n$.

We can prove that for any $\varepsilon > 0$ the sample functions $\xi(u)$ in any finite interval $[0, t]$ have only a finite number of oscillations greater than ε with probability 1. This implies the theorem.

Since the process $\{\xi(u), 0 \leq u < \infty\}$ is separable, it follows that if $u_1, u_2, \ldots, u_n, \ldots$ are elements of the separability set of the process and if $u_n \to u$ as $n \to \infty$, then $P\{\lim_{n \to \infty} \xi(u_n) = \xi(u)\} = 1$. Consequently, the process $\{\xi(u), 0 \leq u < \infty\}$ has the property that for every $u \geq 0$, either $\xi(u) = \xi(u+0)$ or $\xi(u) = \xi(u-0)$ with probability 1.

Theorem 5. _Let_ $\{\xi(u), 0 \leq u < \infty\}$ _be a homogeneous, real stochastic process with independent increments defined on a probability space_ $(\Omega, \mathcal{B}, P)$. _Then there exists a separable homogeneous, real stochastic process with independent increments_ $\{\xi^*(u), 0 \leq u < \infty\}$ _defined on the same probability space such that_

$$P\{\xi^*(u) = \xi(u)\} = 1 \tag{28}$$

for all $u \geq 0$ and with probability 1 the sample functions of $\{\xi^*(u), 0 \leq u < \infty\}$ have a right limit $\xi^*(u+0)$ for every $u \geq 0$, and a left limit $\xi^*(u-0)$ for every $u > 0$ and $\xi^*(u+0) = \xi(u)$ for $u \geq 0$.

It follows from Theorem 47.1 that there exists a separable process $\{\xi^*(u), 0 \leq u < \infty\}$ for which (28) holds for all $u \geq 0$ and by Theorem 4 we can prove the remaining statements.

Since the finite dimensional distributions of the two processes $\{\xi(u), 0 \leq u < \infty\}$ and $\{\xi^*(u), 0 \leq u < \infty\}$ are identical, therefore we can always choose such version of the process $\{\xi(u), 0 \leq u < \infty\}$ which has the same properties as the process $\{\xi^*(u), 0 \leq u < \infty\}$.

Theorem 6. Let $\{\xi(u), 0 \leq u < \infty\}$ be a separable, homogeneous, real stochastic process with independent increments for which $P\{\xi(0) = 0\} = 1$. Let $I_k = [a_k, b_k]$ $(k = 1,2,\ldots, m)$ be disjoint intervals not containing the point $x = 0$. Denote by $\nu(t, I_k)$ the number of points u in $[0, t]$ for which $\xi(u+0) - \xi(u-0) \in I_k$, then $\{\nu(t, I_k), 0 \leq t < \infty\}$ $(k = 1,2,\ldots, m)$ are mutually independent Poisson processes and

$$E\{\nu(t, I_k)\} = \begin{cases} t[\overline{M}(b_k + 0) - \overline{M}(a_k - 0)] & \text{for } a_k \leq b_k < 0 \\ \\ t[\overline{N}(b_k + 0) - \overline{N}(a_k - 0)] & \text{for } 0 < a_k \leq b_k \end{cases} \tag{29}$$

where $\overline{M}(x)$ $(-\infty < x < 0)$ and $\overline{N}(x)$ $(0 < x < \infty)$ are nondecreasing functions of x satisfying the conditions $\lim_{x \to -\infty} \overline{M}(x) = 0$, and $\lim_{x \to \infty} \overline{N}(x) = 0$.

Since the vector process $\{\nu(t, I_k), (k = 1,2,\ldots, m), 0 \leqq t < \infty\}$ is homogeneous and has independent increments, it is sufficient to prove that for each $t \geqq 0$ the random variables $\nu(t, I_k)$ $(k = 1,2,\ldots, m)$ are independent and $\nu(t, I_k)$ has a Poisson distribution.

Theorem 7. Let $\{\xi(u), 0 \leqq u < \infty\}$ be a separable, homogeneous, real stochastic process with independent increments for which $\mathbf{P}\{\xi(0) = 0\} = 1$. Let $I_k = [a_k, b_k]$ $(k = 1,2,\ldots, m)$ be disjoint intervals not containing the point $x = 0$. Denote by $\xi(t, I_k)$ the sum of jumps $\xi(u+0) - \xi(u-0)$ belonging to the interval I_k and occurring in the interval $[0, t]$. Then $\{\xi(t, I_k), 0 \leqq t < \infty\}$ $(k = 1,2,\ldots, m)$ are mutually independent compound Poisson processes and

$$(30) \qquad \mathbf{E}\{e^{-s\xi(t,I_k)}\} = \exp\{-t \int_{I_K} (e^{-sx} - 1)d\overline{M}(x)\}$$

for $a_k \leqq b_k < 0$ and

$$(31) \qquad \mathbf{E}\{e^{-s\xi(t,I_k)}\} = \exp\{-t \int_{I_k} (e^{-sx} - 1)d\overline{N}(x)\}$$

for $0 < a_k \leqq b_k$. We have

$$(32) \qquad \int_{-\varepsilon}^{-0} x^2 d\overline{M}(x) + \int_{+0}^{\varepsilon} x^2 d\overline{N}(x) < \infty$$

for any $\varepsilon > 0$.

The proof of this theorem is similar to the proof of the previous theorem. Since the vector process $\{\xi(t, I_k), (k = 1,2,\ldots, m), 0 \leqq t < \infty\}$ is homogeneous and has independent increments, it is sufficient to prove that for each $t \geq 0$ the random variables $\xi(t, I_k)$ $(k = 1,2,\ldots, m)$ are independent and $\xi(t, I_k)$ has a compound Poisson distribution.

We note that both Theorem 6 and Theorem 7 remain valid if we assume that each I_k is one of the intervals $[a_k, b_k]$, (a_k, b_k), $(a_k, b_k]$, $[a_k, b_k)$. Only (29) needs obvious changes.

Let $\varepsilon_1 = 1 > \varepsilon_2 > \ldots > \varepsilon_n > 0$ where $\varepsilon_n \to 0$ as $n \to \infty$. For each $n = 1,2,\ldots$ denote by $\xi_n(t)$ the sum of jumps $\xi(u+0) - \xi(u-0)$ having absolute value greater than or equal to ε_n and occurring in the interval $[0, t]$. We can prove that

$$\mathrm{Var}\{[\xi(t) - \xi_1(t)]\} < \infty . \tag{33}$$

This implies that (32) holds for any $\varepsilon > 0$.

Let us choose $\varepsilon_1 = 1, \varepsilon_2, \ldots, \varepsilon_n, \ldots$ in such a way that

$$\int_{-\varepsilon_n}^{-\varepsilon_{n+1}} x^2 d\overline{M}(x) + \int_{\varepsilon_{n+1}}^{\varepsilon_n} x^2 d\overline{N}(x) < \frac{1}{n^2} \tag{34}$$

for $n = 1,2,\ldots$.

Let us define

$$\chi(t) = \xi_1(t) + \lim_{n \to \infty} [\xi_n(t) - \xi_1(t) - \mathrm{E}\{\xi_n(t) - \xi_1(t)\}] \tag{35}$$

for $t \geq 0$. By (34) we can prove that on the right-hand side of (35) the limit exists with probability 1 and the convergence is uniform in t in any finite interval. Thus $\{\chi(t), 0 \leq t < \infty\}$ is a stochastic process.

Let $\zeta(t) = \xi(t) - \chi(t)$ for $t \geq 0$. The process $\{\zeta(t), 0 \leq t < \infty\}$ is independent of the process $\{\chi(t), 0 \leq t < \infty\}$.

We can prove that there are only two possibilities: either $\{\zeta(t), 0 \leq t < \infty\}$ is a stochastic process for which $\underset{\sim}{P}\{\zeta(t) = \bar{a}t\} = 1$ for $t \geq 0$ where $\bar{a}$ is a real constant or $\{\zeta(t), 0 \leq t < \infty\}$ is a general Brownian motion process for which

$$\underset{\sim}{P}\left\{\frac{\zeta(t) - \bar{a}t}{\bar{\sigma}\sqrt{t}} \leq x\right\} = \Phi(x) \tag{36}$$

if $t > 0$ where $\bar{a}$ is a real constant and $\bar{\sigma}$ is a positive real constant.

Accordingly, $\{\xi(u), 0 \leq u < \infty\}$ can be represented as the sum of two independent processes, $\{\zeta(u), 0 \leq u < \infty\}$ and $\{\chi(u), 0 \leq u < \infty\}$, where $\{\zeta(u), 0 \leq u < \infty\}$ is a general Brownian motion process (or a degenerate process) and $\{\chi(u), 0 \leq u < \infty\}$ is the limit of centered compound Poisson processes.

By (30), (31), (35) and (36) we can conclude that

$$\underset{\sim}{E}\{e^{-s\xi(u)}\} = e^{u\Psi(s)} \tag{37}$$

for $u \geq 0$ and $\mathrm{Re}(s) = 0$ where

$$\begin{aligned}\Psi(s) = -\bar{a}s + \frac{\bar{\sigma}^2 s^2}{2} &+ \int_{(-\infty,1]} (e^{-sx}-1)d\bar{M}(x) + \int_{(-1,0)} (e^{-sx}-1+sx)d\bar{M}(x) + \\ &+ \int_{(0,1)} (e^{-sx}-1+sx)d\bar{N}(x) + \int_{[1,\infty)} (e^{-sx}-1)d\bar{N}(x)\end{aligned} \tag{38}$$

and $\bar{a}$ is a real constant and $\bar{\sigma}^2$ is a nonnegative constant.

A comparison with (2) shows that necessarily $\overline{M}(x) = M(x)$ if $x < 0$ and x is a continuity point of $M(x)$, $\overline{N}(x) = N(x)$ if $x > 0$ and x is a continuity point of $N(x)$, and $\overline{\sigma}^2 = \sigma^2$. The constant $\overline{a}$ can easily be expressed with the aid of a, $M(x)$ and $N(x)$.

In what follows we assume that $\{\xi(u), 0 \leqq u < \infty\}$ is a homogeneous, real stochastic process with independent increments for which $\underset{\sim}{P}\{\xi(0) = 0\} = 1$. Then (1) holds with $\Psi(s)$ defined by (2). The finite dimensional distributions of the process $\{\xi(u), 0 \leqq u < \infty\}$ are completely determined by the parameters a and σ^2 and by the functions $M(x)$ $(-\infty < x < 0)$ and $N(x)$ $(0 < x < \infty)$. We can classify the processes $\{\xi(u), 0 \leqq u < \infty\}$ according to the properties of a, σ^2, $M(x)$ and $N(x)$.

If a is a real number, σ^2 is a positive real number, $M(x) \equiv 0$ for $x < 0$, and $N(x) \equiv 0$ for $x > 0$, then

(39) $$\Psi(s) = -as + \frac{\sigma^2 s^2}{2}$$

for all s, and $\{\xi(u), 0 \leqq u < \infty\}$ is a general Brownian motion process for which

(40) $$\underset{\sim}{P}\left\{\frac{\xi(u) - au}{\sigma\sqrt{u}} \leqq x\right\} = \Phi(x)$$

for $u > 0$. If the process $\{\xi(u), 0 \leqq u < \infty\}$ is separable, then the sample functions are continuous with probability 1.

If a is a real number, $\sigma^2 = 0$, $M(x) \equiv 0$ for $x < 0$ and $N(x) \equiv 0$ for $x > 0$, then

(41) $$\Psi(s) = -as$$

for all s , and $\underset{\sim}{P}\{\xi(u) = au\} = 1$ for all $u \geqq 0$. If the process is separable, then the sample functions are continuous with probability 1.

Conversely, if the sample functions of the process $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ are continuous with probability 1 , then $M(x) \equiv 0$ for $x < 0$ and $N(x) \equiv 0$ for $x > 0$, that is, $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ is either a general Brownian motion process or a degenerate process.

If $a = 0$, $\sigma^2 = 0$, and $\lambda = M(-0) + N(+0)$ is a finite positive constant, then there exists a distribution function $H(x)$ such that if x is a continuity point of $H(x)$, then

(42) $$M(x) = \lambda H(x)$$

for $x < 0$ and

(43) $$N(x) = \lambda[H(x)-1]$$

for $x > 0$. If

(44) $$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} dH(x)$$

for $\mathrm{Re}(s) = 0$, then

(45) $$\Psi(s) = \lambda[1-\psi(s)]$$

for $\mathrm{Re}(s) = 0$, and $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ is a compound Poisson process. If the process $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ is separable, then with probability 1 the sample functions are step functions having only a finite number of jumps in every finite interval $[0, t]$.

Conversely, if the sample functions of the process $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ are step functions having only a finite number of jumps in every finite interval

$[0, t]$ with probability 1 , then $\{\xi(u) , 0 \leq u < \infty\}$ is either a compound Poisson process or a degenerate process for which $P\{\xi(u) = 0\} = 1$ for all $u \geq 0$.

Let us suppose that $a \geq 0$, $\sigma^2 = 0$, $M(x) \equiv 0$ for $x < 0$, $N(+0) < \cancel{0}$, ∞ /- and

$$(46) \qquad \int_{+0}^{\varepsilon} x dN(x) < \infty$$

for some $\varepsilon > 0$. In this case

$$(47) \qquad \Psi(s) = -as + \int_{+0}^{\infty} (e^{-sx}-1) dN(x)$$

for $Re(s) \geq 0$, and if the process $\{\xi(u) , 0 \leq u < \infty\}$ is separable, then with probability 1 the sample functions are nondecreasing functions of u . Conversely, if with probability 1 the sample functions of the process $\{\xi(u) , 0 \leq u < \infty\}$ are nondecreasing functions of u , then $\Psi(s)$ has the form (47) where $a \geq 0$ and $N(x)$ satisfies (46). Furthermore, apart from a set of probability zero, each sample function can be expressed as the sum of the linear function au $(0 \leq u < \infty)$ and a step function. If $N(+0) = -\infty$, then the step function has infintely many jumps in every interval $[0, t]$ of positive length.

If in the above case

$$(48) \qquad \rho = \int_{+0}^{\infty} x dN(x)$$

is a finite positive number, then there exists a distribution function $H^*(x)$ of a positive random variable such that

(49) $$\frac{\int_0^x N(y)dy}{\int_0^\infty N(y)dy} = H^*(x)$$

for $x \geqq 0$. If

(50) $$\Psi^*(s) = \int_0^\infty e^{-sx} dH^*(x)$$

for $\mathrm{Re}(s) \geqq 0$, then (47) becomes

(51) $$\Psi(s) = \rho s \Psi^*(s) - as .$$

If in addition $\lambda = -N(+0) < \infty$, then there exists a distribution function $H(x)$ of a positive random variable such that

(52) $$\frac{N(+0) - N(x)}{N(+0)} = H(x)$$

for every continuity point of $H(x)$ in the interval $[0, \infty)$. If

(53) $$\Psi(s) = \int_0^\infty e^{-sx} dH(x)$$

for $\mathrm{Re}(s) \geqq 0$, then

(54) $$\psi^*(s) = \frac{\lambda[1 - \psi(s)}{\rho s}$$

for $\mathrm{Re}(s) \geqq 0$ and $s \neq 0$ in (51).

Let a be a real number, $\sigma^2 \geqq 0$, $M(x) \equiv 0$ for $x < 0$ and

(55) $$\int_{+0}^{\varepsilon} x^2 dN(x) < \infty$$

for some $\varepsilon > 0$. In this case

$$\Psi(s) = -as + \frac{\sigma^2 s^2}{2} + \int_{+0}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dN(x) \tag{56}$$

for $\mathrm{Re}(s) \geqq 0$ and if the process $\{\xi(u), 0 \leqq u < \infty\}$ is separable, then with probability 1 the sample functions have no negative jumps. Conversely, if with probability 1 the sample functions of the process have no negative jumps, then $\Psi(s)$ is given by (56) for $\mathrm{Re}(s) \geqq 0$.

We note that if in (56)

$$\int_{+0}^{\varepsilon} x\, dN(x) < \infty \tag{57}$$

for some $\varepsilon > 0$, then (56) can be reduced to the following form

$$\Psi(s) = -\bar{a}s + \frac{\sigma^2 s^2}{2} + \int_{+0}^{\infty} (e^{-sx} - 1)\, dN(x) \tag{58}$$

for $\mathrm{Re}(s) \geqq 0$ where $\bar{a}$ is a real number. If in (56) we have

$$\int_{\varepsilon}^{\infty} x\, dN(x) < \infty \tag{59}$$

for some $\varepsilon > 0$, then (56) can be reduced to the following form

$$\Psi(s) = -\bar{a}s + \frac{\sigma^2 s^2}{2} + \int_{+0}^{\infty} (e^{-sx} - 1 + sx)\, dN(x) \tag{60}$$

for $\mathrm{Re}(s) \geqq 0$ where $\bar{a}$ is a real number. The constant $\bar{a}$ is in general not the same as in (58).

We say that $\{\xi(u), 0 \leqq u < \infty\}$ is a stable process of type $S(\alpha,\beta,c,m)$ if $\xi(1)$ has a stable distribution of type $S(\alpha,\beta,c,m)$. In this case

$$\Psi(s) = -ms - c|s|^{\alpha} \left[1 + \beta \frac{s}{|s|} d(s, \alpha)\right] \tag{61}$$

for $Re(s) = 0$ where m is a real constant, $c > 0$, $0 < \alpha \leqq 2$, $-1 \leqq \beta \leqq 1$ and

$$(62) \qquad d(s,\alpha) = \begin{cases} \tan\frac{\alpha\pi}{2} & \text{for } \alpha \neq 1, \\ -\frac{2}{\pi}\log|s| & \text{for } \alpha = 1. \end{cases}$$

In (61) $s/|s| = 0$ if $s = 0$. See Theorem 42.4.

Finally, we shall prove a general result for separable, homogeneous, real stochastic processes $\{\xi(u), 0 \leqq u < \infty\}$ with independent increments in the case when the sample functions are nondecreasing step functions with probability one. If $P\{\xi(0) = 0\} = 1$, then for such processes we have

$$(63) \qquad E\{e^{-s\xi(u)}\} = e^{u\Psi(s)}$$

for $Re(s) \geqq 0$ where

$$(64) \qquad \Psi(s) = \int_{+0}^{\infty} (e^{-sx}-1)dN(x)$$

and $N(x)$ $(0 < x < \infty)$ is a nondecreasing function which satisfies the conditions $\lim_{x \to +\infty} N(x) = 0$ and

$$(65) \qquad \int_{+0}^{\varepsilon} x dN(x) < \infty$$

for some $\varepsilon > 0$. We note that if $-N(+0) < \infty$ then with probability 1, the sample functions have only a finite number of jumps in any finite interval $[0, t]$, whereas if $-N(+0) = \infty$, then with probability 1, the sample functions have infinitely many jumps in any finite interval $[0, t]$.

The general result mentioned above is based on the following auxiliary theorem. (See reference [86].)

Lemma 4. *Let* $x(u)$ *be a nondecreasing function of* u *in the interval* $[0, t]$ *for which* $x'(u) = 0$ *almost everywhere and* $x(0) = 0$. *Let us extend the definition of* $x(u)$ *for* $u \geq 0$ *by assuming that* $x(u+t) = x(u) + x(t)$ *for* $u \geq 0$. *Define*

$$(66) \qquad \delta(u) = \begin{cases} 1 & \text{if} \quad u-x(u) \leq v-x(v) \quad \text{for} \quad u \leq v , \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$(67) \qquad \int_0^t \delta(u)du = \begin{cases} t-x(t) & \text{if} \quad x(t) < t , \\ 0 & \text{if} \quad x(t) \geq t . \end{cases}$$

Proof. If $x(t) > t$, then $\delta(u) = 0$ for all $u \geq 0$ and thus the theorem is obviously true. Let $x(t) \leq t$ and define

$$(68) \qquad y(u) = \inf\{v-x(v) \quad \text{for} \quad v \geq u\}$$

for $u \geq 0$. Since $x(u+t) = x(u) + x(t)$ for $u \geq 0$, we have $y(u+t) = y(u) + t - x(t)$ for $u \geq 0$. Furthermore, we have $0 \leq y(v) - y(u) \leq v-u$ for $0 \leq u \leq v$. Thus $y(u)$ $(0 \leq u < \infty)$ is a nondecreasing and absolutely continuous function of u . Consequently, $y'(u)$ exists for almost all u , $0 \leq y'(u) \leq 1$, and

$$(69) \qquad \int_0^t y'(u)du = y(t) - y(0) = t-x(t) .$$

Now we shall prove that $y'(u) = \delta(u)$ for almost all u , which implies (67). We note that $\delta(u) = 1$ if and only if $y(u) = u-x(u)$.

The inequality $y(u) \leqq u-x(u)$ always holds. Furthermore, we have $x(u+0) = x(u)$ and $x'(u) = 0$ for almost all $u \geqq 0$.

First, we prove that

$$(70) \qquad y'(u) \leqq \delta(u) \quad \text{for almost all} \quad u \geqq 0 .$$

If $y'(u)$ exists, and if $y'(u) = 0$, then (70) is obviously true. Now we shall prove that if $y'(u)$ exists, if $y'(u) > 0$ and $x(u+0) = x(u)$, then $\delta(u) = 1$. If $y'(u) > 0$, then $y(v) > y(u)$ for $v > u$ and therefore $y(u) = \inf\{s-x(s)$ for $u \leqq s \leqq v\}$ holds for all $v > u$. Thus $u-x(v) \leqq y(u) \leqq u-x(u)$ for all $v > u$, and consequently $u-x(u+0) \leqq y(u) \leqq u-x(u)$. If $x(u+0) = x(u)$, then $y(u) = u-x(u)$ which implies that $\delta(u) = 1$. Since $y'(u) \leqq 1$ always holds, therefore (70) follows.

Second, we prove that

$$(71) \qquad \delta(u) \leqq y'(u) \quad \text{for almost all} \quad u \geqq 0 .$$

If $\delta(u) = 0$ and $y'(u)$ exists, then (71) is evidently true. Now we shall prove that if $\delta(u) = 1$, if $y'(u)$ exists, if $x'(u) = 0$ and u is an accumulation point of the set $D = \{u : \delta(u) = 1 , 0 \leqq u < \infty\}$, then $y'(u) = 1$. Suppose that $u \in D$ and $u = \lim_{n \to \infty} u_n$ where $u_n \in D$ and $u_n \neq u$. Then $y(u) = u-x(u)$ and $y(u_n) = u_n - x(u_n)$. Accordingly, if $y'(u)$ exists and if $x'(u) = 0$, we have

$$(72) \qquad y'(u) = \lim_{n \to \infty} \frac{y(u)-y(u_n)}{u-u_n} = 1 - \lim_{n \to \infty} \frac{x(u)-x(u_n)}{u-u_n} = 1 - x'(u) = 1 .$$

Since the isolated points of D form a countable (possibly empty) set, therefore (71) follows.

By (70) and (71) we obtain that $y'(u) = \delta(u)$ holds for almost all $u \geq 0$. Thus by (69) we get (67) for $x(t) \leq t$. This completes the proof of the lemma.

By using Lemma 4 we can prove the following result.

Theorem 8. *Let $\{\xi(u), 0 \leq u < \infty\}$ be a separable, homogeneous, real stochastic process with independent increments. If $\mathbf{P}\{\xi(0) = 0\} = 1$, and if the sample functions of the process are nondecreasing step functions with probability 1, then*

$$(73) \qquad \mathbf{P}\{\xi(u) \leq u \text{ for } 0 \leq u \leq t \mid \xi(t) = y\} = \begin{cases} (t-y)/t & \text{for } 0 \leq y \leq t, \\ 0 & \text{otherwise,} \end{cases}$$

where the conditional probability is defined up to an equivalence.

Proof. Define $\xi^*(u)$ for $0 \leq u < \infty$ by assuming that $\xi^*(u) = \xi(u)$ for $0 \leq u \leq t$ and $\xi^*(u+t) = \xi^*(u) + \xi^*(t)$ for $u \geq 0$. Let

$$(74) \qquad \delta(u) = \begin{cases} 1 & \text{if } \xi^*(v) - \xi^*(u) \leq v-u \text{ for } v \geq u, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\delta(u)$ is a random variable which has the same distribution for all $u \geq 0$. Evidently $\delta(0)$ is the indicator variable of the event $\{\xi(u) \leq u$ for $0 \leq u \leq t\}$. Thus we have

$$\mathbf{P}\{\xi(u) \leq u \quad \text{for} \quad 0 \leq u \leq t \,|\, \xi(t)\} = \mathbf{E}\{\delta(0) \,|\, \xi(t)\} =$$

(75)

$$= \frac{1}{t}\int_0^t \mathbf{E}\{\delta(u) \,|\, \xi(t)\}du = \mathbf{E}\{\, \frac{1}{t}\int_0^t \delta(u)du \,|\, \xi(t)\} =$$

$$= \begin{cases} 1 - \dfrac{\xi(t)}{t} & \text{for} \quad 0 \leq \xi(t) \leq t \ , \\ 0 & \text{otherwise,} \end{cases}$$

with probability 1 because by Lemma 4

$$\int_0^t \delta(u)du = \begin{cases} t-\xi(t) & \text{if} \quad 0 \leq \xi(t) \leq t \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{76}$$

holds for almost all sample functions of the process. This completes the proof of the theorem.

We note that Theorem 8 remains also valid if we replace the left-hand side of (73) by $\mathbf{P}\{\xi(u) < u \quad \text{for} \quad 0 < u \leq t \,|\, \xi(t) = y\}$.

From (73) it follows that

$$\mathbf{P}\{\xi(u) \leq u \quad \text{for} \quad 0 \leq u \leq t\} = \mathbf{E}\{[1 - \frac{\xi(t)}{t}]^+\} \tag{77}$$

for $t > 0$ where $[x]^+$ denotes the positive part of x .

If the process $\{\xi(u) \ , \ 0 \leq u < \infty\}$ satisfies the conditions of Theorem 8, then (63) holds with $\Psi(s)$ defined by (64). If in addition

$$\rho = \int_{+0}^{\infty} x dN(x) \tag{78}$$

is a finite nonnegative number, then Theorem 2 is applicable, and we have

(79) $$\mathbf{P}\{\lim_{t \to \infty} \frac{\xi(t)}{t} = \rho\} = 1 \ .$$

In this case by (77) it follows that

(80) $$\mathbf{P}\{\xi(u) \leq u \quad \text{for} \quad 0 \leq u < \infty\} = \begin{cases} 1 - \rho & \text{if} \quad \rho < 1 \ , \\ 0 & \text{if} \quad \rho \geq 1 \ . \end{cases}$$

For by the continuity theorem for probabilities and by (77) we have

(81) $$\begin{aligned} &\mathbf{P}\{\xi(u) \leq u \quad \text{for} \quad 0 \leq u < \infty\} = \lim_{t \to \infty} \mathbf{P}\{\xi(u) \leq u \quad \text{for} \quad 0 \leq u \leq t\} = \\ &= \lim_{t \to \infty} \mathbf{E}\{[1 - \frac{\xi(t)}{t}]^{+}\} = [1 - \rho]^{+} \ . \end{aligned}$$

In the last equality we used that $\xi(t)/t \Rightarrow \rho$ as $t \to \infty$ and that $0 \leq [1 - \frac{\xi(t)}{t}]^{+} \leq 1$ for all $t > 0$.

Examples. We shall mention a few examples for the applications of Theorem 8.

Compound Poisson Processes. Let us suppose that

(82) $$N(x) = -\lambda[1-H(x)]$$

for $x > 0$ where λ is a positive constant and $H(x)$ is the distribution function of a nonnegative random variable. In this case $\{\xi(u) \ , \ 0 \leq u < \infty\}$ is a compound Poisson process and Theorem 8 is applicable. In this particular case we already proved Theorem 8. (See Theorem 48.13). In this case

$$\rho = \lambda \int_0^\infty x dH(x) \tag{83}$$

and (80) also holds if $\rho < \infty$.

Stable Processes. Let us suppose that

$$N(x) = -\frac{1}{\Gamma(1-\alpha)x^\alpha} \tag{84}$$

for $x > 0$ where $0 < \alpha < 1$. In this case

$$\Psi(s) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (e^{-sx}-1) \frac{dx}{x^{\alpha+1}} = -s^\alpha \tag{85}$$

for $\mathrm{Re}(s) \geqq 0$ and $\{\xi(u) , 0 \leqq u < \infty\}$ is a stable process of type $S(\alpha,1,1,0)$. Now Theorem 8 is applicable. However, in this case $\rho = \infty$.

Gamma Processes. Let us suppose that

$$N(x) = -\int_x^\infty \frac{e^{-\mu y}}{y} dy \tag{86}$$

for $x > 0$ where μ is a positive constant. Then

$$\Psi(s) = \int_0^\infty (e^{-sx}-1)e^{-\mu x} \frac{dx}{x} = -\log(1+\frac{s}{\mu}) \tag{87}$$

for $\mathrm{Re}(s) \geqq 0$. In this case we say that $\{\xi(u) , 0 \leqq u < \infty\}$ is a gamma process. Now Theorem 8 holds and (80) also holds with $\rho = 1/\mu$.

52. Weak Convergence of Stochastic Processes.

Let $\{\xi_n(u)\ ,\ 0 \leq u \leq t\}$ $(n = 1,2,\ldots)$ and $\{\xi(u)\ ,\ 0 \leq u \leq t\}$ be real stochastic processes. We say that the finite dimensional distribution functions of the process $\{\xi_n(u)\ ,\ 0 \leq u \leq t\}$ converge to the finite dimensional distribution functions of the process $\{\xi(u)\ ,\ 0 \leq u \leq t\}$ if for any $k = 1,2,\ldots$ and $0 \leq t_1 < t_2 < \ldots < t_k \leq t$ we have

(1) $$\lim_{n \to \infty} \mathbf{P}\{\xi_n(t_1) \leq x_1,\ \xi_n(t_2) \leq x_2, \ldots,\ \xi_n(t_k) \leq x_k\} =$$

$$= \mathbf{P}\{\xi(t_1) \leq x_1,\ \xi(t_2) \leq x_2, \ldots,\ \xi(t_k) \leq x_k\}$$

in every continuity point $(x_1, x_2, \ldots, x_k)$ of the right-hand side of (1).

Let Q be some real functional defined for $\boldsymbol{\xi}_n = \{\xi_n(u)\ ,\ 0 \leq u \leq t\}$ and $\boldsymbol{\xi} = \{\xi(u)\ ,\ 0 \leq u \leq t\}$. The problem arises what conditions should we impose on Q in order that

(2) $$\lim_{n \to \infty} \mathbf{P}\{Q(\boldsymbol{\xi}_n) \leq x\} = \mathbf{P}\{Q(\boldsymbol{\xi}) \leq x\}$$

be satisfied in every continuity point of $\mathbf{P}\{Q(\boldsymbol{\xi}) \leq x\}$?

The importance of the solution of the above problem is twofold. First, it makes possible to determine the probability $\mathbf{P}\{Q(\boldsymbol{\xi}) \leq x\}$ for a process $\boldsymbol{\xi} = \{\xi(u)\ ,\ 0 \leq u \leq t\}$ if we can determine the probabilities $\mathbf{P}\{Q(\boldsymbol{\xi}_n) \leq x\}$ for a sequence of suitable chosen processes $\boldsymbol{\xi}_n = \{\xi_n(u)\ ,\ 0 \leq u \leq t\}$ $(n = 1,2,\ldots)$. Second, it makes possible to determine the limiting distributions of some functionals defined on a class of stochastic processes.

In what follows we assume that the sample functions of the processes

$\{\xi_n(u), 0 \leq u \leq t\}$ and $\{\xi(u), 0 \leq u \leq t\}$ belong to some metric space Ω with probability 1. For $x \in \Omega$, $y \in \Omega$ denote by $\rho(x, y)$ the distance between x and y. Denote by $\mathcal{B}$ the smallest σ-algebra which contains all the open sets (closed sets) in Ω. If Ω is a separable metric space, then $\mathcal{B}$ coincides with the smallest σ-algebra which contains all the open spheres (closed spheres) in Ω. In what follows we shall consider only such spaces $\Omega = \{x(u), 0 \leq u \leq t\}$ for which $\mathcal{A}$, the minimal σ-algebra containing the sets $\{x(u) \leq a\}$ for $u \in [0,t]$ and $a \in (-\infty, \infty)$, contains all spheres in Ω.

For any $A \in \mathcal{B}$ let us define

(3) $$\mu_n(A) = \underset{\sim}{P}\{\underset{\sim}{\xi}_n \in A\},$$

that is, $\mu_n(A)$ is the probability that $\underset{\sim}{\xi}_n = \{\xi_n(u), 0 \leq u \leq t\}$ belongs to A, and

(4) $$\mu(A) = \underset{\sim}{P}\{\underset{\sim}{\xi} \in A\},$$

that is, $\mu(A)$ is the probability that $\underset{\sim}{\xi} = \{\xi(u), 0 \leq u \leq t\}$ belongs to A, provided that the probabilities (3) and (4) are uniquely determined by the finite dimensional distribution functions of the processes $\{\xi_n(u), 0 \leq u \leq t\}$ $(n = 1,2,\ldots)$ and $\{\xi(u), 0 \leq u \leq t\}$.

We say that μ_n converges weakly to μ, that is, $\mu_n \Rightarrow \mu$ as $n \to \infty$, if and only if

(5) $$\lim_{n \to \infty} \int_{\Omega} h(x) d\mu_n = \int_{\Omega} h(x) d\mu$$

for every continuous and bounded real functional $h(x)$ on Ω. The functional $h(x)$ is continuous on Ω if for every $x \in \Omega$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|h(x) - h(y)| < \varepsilon$ whenever $y \in \Omega$ and $\rho(x, y) < \delta$.

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the class of Borel subsets of Ω, that is, $\mathcal{B}$ is

If we suppose that the space Ω is a separable metric space and Q is a continuous functional on Ω, then $Q(\underset{\sim}{\xi}_n)$ $(n = 1,2,\ldots)$ and $Q(\underset{\sim}{\xi})$ will be random variables and

$$\lim_{n\to\infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n) \leqq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leqq x\} \tag{6}$$

holds in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leqq x\}$ if and only if μ_n converges weakly to μ. For (6) holds if and only if

$$\lim_{n\to\infty} \int_\Omega e^{i\omega Q(x)} d\mu_n = \int_\Omega e^{i\omega Q(x)} d\mu \tag{7}$$

for every real ω. Since $\cos[\omega Q(x)]$ and $\sin[\omega Q(x)]$ are continuous and bounded functionals on Ω, the statement is obvious.

Accordingly, if we restrict ourself to separable metric spaces Ω and continuous functionals Q, then (2) holds if and only if $\mu_n \Rightarrow \mu$ as $n \to \infty$. Thus the problem is reduced to find sufficient conditions for $\mu_n \Rightarrow \mu$. The following definition will be useful in solving this problem.

We say that the sequence $\{\mu_n\}$ is weakly compact if every subsequence of $\{\mu_n\}$ contains a subsequence which is weakly convergent.

<u>Yu. V. Prokhorov</u> [523] proved that if Ω is a metric space and if for every $\varepsilon > 0$ there exists a compact set K_ε in Ω such that

$$\sup_{1\leqq n<\infty} \mu_n(\Omega - K_\varepsilon) < \varepsilon , \tag{8}$$

then $\{\mu_n\}$ is weakly compact. (See Theorem 3.2 in the Appendix.)

If we suppose that Ω is a separable metric space, if $\{\mu_n\}$ is weakly compact and if (1) is satisfied, then we can prove that $\mu_n \Rightarrow \mu$ as $n \to \infty$. The proof is exactly the same as the proof of the fourth statement in Theorem 46.7. (Formulas (46.143) to (46.157). The only difference is that in (46.151) $f \in \Omega$.)

Thus we can conclude that if Ω is a separable metric space and if for every $\varepsilon > 0$ there exists a compact set K_ε in Ω such that (8) is satisfied, then (2) holds for every continuous functional Q on Ω. Actually, (2) also holds if we assume only that Q is measurable with respect to $\mathcal{B}$ and almost everywhere continous with respect to μ. The proof of this last statement is exactly the same as the proof of the last statement in Theorem 46.7. (Formulas (46.160) to (46.165).)

We can summarize the above results in the following theorem.

Theorem 1. Let $\underset{\sim}{\xi}_n = \{\xi_n(u),\ 0 \leqq u \leqq t\}$ $(n = 1,2,\ldots)$ and $\underset{\sim}{\xi} = \{\xi(u),\ 0 \leqq u \leqq t\}$ be real stochastic processes whose sample functions belong to some separable metric space Ω with probability 1. Denote by $\mathcal{B}$ the class of Borel subsets of Ω and let us define $\mu_n(A)$ for $A \in \mathcal{B}$ by (3) and $\mu(A)$ for $A \in \mathcal{B}$ by (4). If (1) is satisfied and if for every $\varepsilon > 0$ there exists a compact set K_ε in Ω for which (8) is satisfied, and if Q is a functional on Ω which is measurable with respect to $\mathcal{B}$ and almost everywhere continuous with respect to μ, then

$$\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n) \leqq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leqq x\} \tag{9}$$

in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leqq x\}$.

Theorem 1 has many useful applications in the theory of stochastic processes.

First, let us consider the case when the sample functions of the processes $\{\xi_n(u), 0 \leq u \leq t\}$ $(n = 1,2,\ldots)$ and $\{\xi(u), 0 \leq u \leq t\}$ are continuous with probability 1. Then Ω can be chosen as the space $C[0, t]$ of continuous functions defined on the interval $[0, t]$. If we introduce the metric $\rho(x, y) = \sup_{0\leq u\leq t} |x(u) - y(u)|$ whenever $x = \{x(u), 0 \leq u \leq t\} \in C[0, t]$ and $y = \{y(u), 0 \leq u \leq t\} \in C[0, t]$, then $C[0, t]$ becomes a complete separable metric space.

The following theorem is due to <u>Yu. V. Prokhorov</u> [522],[523]. See also <u>I. I. Gikhman</u> and <u>A. V. Skorokhod</u> [44].

Theorem 2. *Let us suppose that the sample functions of the processes* $\underset{\sim}{\xi}_n = \{\xi_n(u), 0 \leq u \leq t\}$ $(n = 1,2,\ldots)$ *and* $\underset{\sim}{\xi} = \{\xi(u), 0 \leq u \leq t\}$ *are continuous with probability 1, and the finite dimensional distribution functions of the process* $\{\xi_n(u), 0 \leq u \leq t\}$ *converge to the finite dimensional distribution functions of the process* $\{\xi(u), 0 \leq u \leq t\}$ *as* $n \to \infty$. *If for any* $\varepsilon > 0$

$$\lim_{h \to 0} \limsup_{n \to \infty} \underset{\sim}{P}\{\sup_{|u-v|\leq h} |\xi_n(u) - \xi_n(v)| > \varepsilon\} = 0 \tag{10}$$

and if Q *is a real continuous functional on* $C[0, t]$, *then*

$$\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n) \leq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\} \tag{11}$$

in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$.

Proof. By Theorem 1 it is sufficient to prove that for every $\varepsilon > 0$ there exists a compact set K_ε in $C[0, t]$ such that $\mu_n(K_\varepsilon) \geqq 1-\varepsilon$ for $n = 1,2,\ldots$. We can construct a compact set K_ε in the same way as in the proof of the second statement of Theorem 46.7 . (Formulas (46.133) to (46.142).) Only the set F_0 should be chosen differently. Since $\underset{\sim}{P}\{\xi_n(0) \leqq x\} \Rightarrow \underset{\sim}{P}\{\xi(0) \leqq x\}$ as $n \to \infty$, therefore for any $\varepsilon > 0$ we can find an m_0 such that $\underset{\sim}{P}\{|\xi_n(0)| \leqq m_0\} > 1-\varepsilon$ for $n = 1,2,\ldots$. If we choose $F_0 = \{f: |f(0)| \leqq m_0\}$, then $\mu_n(F_0) > 1-\varepsilon$ and the remaining part of the proof remains unchanged.

Now let us consider a few examples for the application of this theorem.

Let us suppose that $\{\xi(u) , 0 \leqq u \leqq t\}$ is a separable Brownian motion process defined on the interval $[0, t]$. Then with probability 1 the sample functions of the process are continuous functions. (See Theorem 50.1.)

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed real random variables for which $\underset{\sim}{E}\{\xi_n\} = 0$ and $\underset{\sim}{E}\{\xi_n^2\} = 1$. Define $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n \geqq 1$ and $\zeta_0 = 0$. Let

$$\xi_n(u) = \zeta_{[nu]} / \sqrt{n} \tag{12}$$

for $0 \leqq u \leqq t$. Then the finite dimensional distribution functions of the stochastic process $\underset{\sim}{\xi}_n = \{\xi_n(u) , 0 \leqq u \leqq t\}$ converge to the finite dimensional distribution functions of the Brownian motion process $\underset{\sim}{\xi} = \{\xi(u) , 0 \leqq u \leqq t\}$. This follows from the central limit theorem and from the fact that the process $\{\xi_n(u) , 0 \leqq u \leqq t\}$ has independent increments.

Now we cannot apply Theorem 2 directly because the sample functions of the processes $\{\xi_n(u), 0 \leq u \leq t\}$ are not continuous functions. However, we can easily overcome this difficulty. Let us define

$$(13) \qquad \xi_n^*(u) = \frac{\zeta_{[nu]} + (nu - [nu])\xi_{[nu+1]}}{\sqrt{n}}$$

for $u \geq 0$. Then the stochastic process $\underset{\sim}{\xi}_n^* = \{\xi_n^*(u), 0 \leq u \leq t\}$ has continuous sample functions and the finite dimensional distribution functions of the process $\{\xi_n^*(u), 0 \leq u \leq t\}$ converge to the finite dimensional distribution functions of the Brownian motion process $\underset{\sim}{\xi} = \{\xi(u), 0 \leq u \leq t\}$. Since $\xi_{[nu+1]}/\sqrt{n} \Rightarrow 0$ as $n \to \infty$, this follows immediately from the results mentioned above.

For the process $\{\xi_n^*(u), 0 \leq u \leq t\}$ we can apply Theorem 2 and we can conclude that if Q is any real continuous functional on $\mathbb{C}[0, t]$, then

$$(14) \qquad \lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n^*) \leq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$$

in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$.

For in this case (10) is satisfied which follows from the inequality (46.126). See formulas (46.126) to (46.132). This result is in agreement with Theorem 46.7.

If we suppose, for example, that $\underset{\sim}{P}\{\xi_n = 1\} = \underset{\sim}{P}\{\xi_n = -1\} = 1/2$ for $n = 1,2,\ldots$, then the random variables $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$ describe a symmetric random walk, and for several functionals Q the limit (14) can be determined directly.

On the one hand this result makes it possible to find the probability $P\{Q(\xi) \leq x\}$ for a Brownian motion process $\xi = \{\xi(u), 0 \leq u \leq t\}$ and on the other hand it shows that the limiting distribution (14) does not depend on the particular sequence of random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$, it depends only on the limiting distribution of $\zeta_n/\sqrt{n}$ as $n \to \infty$.

As a next example let us suppose that $\{\xi(u), 0 \leq u \leq t\}$ is a general Brownian motion process for which $E\{\xi(u)\} = \alpha u$ and $Var\{\xi(u)\} = \sigma^2 u$ where σ is a positive constant. Then with probability 1 the sample functions of the process are continuous functions. (See Theorem 50.1.)

For every $n = 1,2,\ldots$ let $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk}, \ldots$ be mutually independent and identically distributed random variables for which

(15) $$P\{\xi_{nk} = 1\} = \frac{1}{2} + \frac{\alpha}{2\sigma\sqrt{n}} \quad \text{and} \quad P\{\xi_{nk} = -1\} = \frac{1}{2} - \frac{\alpha}{2\sigma\sqrt{n}}$$

whenever $n > \alpha^2/\sigma^2$. Let $\zeta_{nk} = \xi_{n1} + \xi_{n2} + \ldots + \xi_{nk}$ for $n \geq 1$ and $k \geq 1$ and $\zeta_{n0} = 0$ for $n \geq 1$. Define

(16) $$\xi_n(u) = \frac{\sigma\zeta_{n,[nu]}}{\sqrt{n}}$$

for $0 \leq u \leq t$ and

(17) $$\xi_n^*(u) = \frac{\sigma[\zeta_{n,[nu]} + (n - [nu])\xi_{n,[nu+1]}]}{\sqrt{n}}$$

for $0 \leq u \leq t$.

The finite dimensional distribution functions of both processes $\underset{\sim}{\xi}_n = \{\xi_n(u) , 0 \leq u \leq t\}$ and $\underset{\sim}{\xi}_n^* = \{\xi_n^*(u) , 0 \leq u \leq t\}$ converge to the finite dimensional distribution functions of the process $\underset{\sim}{\xi} = \{\xi(u) , 0 \leq u \leq t\}$. While the sample functions of the process $\{\xi_n(u) , 0 \leq u \leq t\}$ are step functions, the sample functions of the process $\{\xi_n^*(u) , 0 \leq u \leq t\}$ are continuous functions. Furthermore, we can easily prove that (10) is satisfied for the process $\{\xi_n^*(u) , 0 \leq u \leq t\}$. Thus Theorem 2 is applicable, and we can conclude that if Q is any real continuous functional on $C[0, t]$, then

$$\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n^*) \leq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$$

in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$.

For several functionals the probability $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n^*) \leq x\}$ can be calculated explicitly and by forming its limit as $n \to \infty$ we can obtain $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$ for a general Brownian motion process $\underset{\sim}{\xi} = \{\xi(u) , 0 \leq u \leq t\}$.

A second important case for the application of Theorem 1 is the following. Let us suppose that the sample functions of the processes $\{\xi_n(u) , 0 \leq u \leq 1\}$ $(n = 1,2,\ldots)$ and $\{\xi(u) , 0 \leq u \leq 1\}$ belong to the space $D[0, 1]$ with probability 1. Here $D[0, 1]$ denotes the space of real functions $f(u)$ defined on the interval $[0, 1]$ for which $f(u+0)$ and $f(u-0)$ exist at every point and $f(u+0) = f(u)$, $f(0) = f(+0)$ and $f(1) = f(1-0)$.

Let us introduce a metric in the space $D[0, 1]$ in the following way:

If $f \in D[0, 1]$ and $g \in D[0, 1]$, then let us define the distance between f and g by

$$(18) \qquad d(f,g) = \inf_{\lambda\in\Lambda} \{\sup_{0\le u\le 1} |f(u) - g(\lambda(u))| + \sup_{0\le u\le 1} |u - \lambda(u)|\} ,$$

where Λ is the set of continuous, increasing, real functions $\lambda(u)$ defined on the interval $[0, 1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$. We can easily check that $d(f, g)$ defines a metric on $D[0, 1]$, and the space $D[0, 1]$ with the metric (18) is a separable metric space. For each $f \in D[0, 1]$ let us define

$$(19) \qquad \begin{aligned} \Delta_a(f) = & \sup_{0\le u-a\le t\le u\le v\le v+a\le 1} \{\min(|f(t) - f(u)| , |f(v) - f(u)|)\} + \\ & + \sup_{0\le u\le a} |f(u) - f(0)| + \sup_{1-a\le u\le 1} |f(u) - f(1)| . \end{aligned}$$

The following theorem is due to A. V. Skorokhod [537].

Theorem 3. Let us suppose that the sample functions of the processes $\xi_n = \{\xi_n(u) , 0 \le u \le 1\}$ $(n = 1,2,\ldots)$ and $\xi = \{\xi(u) , 0 \le u \le 1\}$ belong to the space $D[0, 1]$ with probability 1, and the finite dimensional distribution functions of the process $\{\xi_n(u) , 0 \le u \le 1\}$ converge to the finite dimensional distribution functions of the process $\{\xi(u) , 0 \le u \le 1\}$ as $n \to \infty$. If for every $\varepsilon > 0$

$$(20) \qquad \lim_{a\to 0} \lim_{n\to\infty} \sup P\{\Delta_a(\xi_n) > \varepsilon\} = 0 ,$$

and if Q is a real continuous functional on $D[0, 1]$ with the metric (18), then

(21) $$\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n) \leq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$$

in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$.

For the proof of this theorem we refer to I. I. Gikhman and A. V. Skorokhod [44 pp. 469-478] . Here we shall sketch only briefly the proof of Theorem 3. First, (20) implies that

(22) $$\lim_{a \to 0} \sup_n \underset{\sim}{P}\{\Delta_a(\underset{\sim}{\xi}_n) > \varepsilon\} = 0 .$$

Since for any $\varepsilon > 0$ and c we have

(23) $$\underset{\sim}{P}\{\sup_{0 \leq u \leq 1} |\xi_n(u)| > c\} \leq \underset{\sim}{P}\{\max_{0 \leq k \leq m} |\xi_n(\frac{k}{m})| > c - \varepsilon\} + \underset{\sim}{P}\{\Delta_{1/m}(\underset{\sim}{\xi}_n)\} ,$$

and since

(24) $$\lim_{n \to \infty} \underset{\sim}{P}\{\max_{0 \leq k \leq m} |\xi_n(\frac{k}{m})| \leq x\} = \underset{\sim}{P}\{\max_{0 \leq k \leq m} |\xi(\frac{k}{m})| \leq x\}$$

in every continuity point of the right-hand side, therefore by (20) we obtain that

(25) $$\lim_{c \to \infty} \sup_n \underset{\sim}{P}\{\sup_{0 \leq u \leq 1} |\xi_n(u)| > c\} = 0 .$$

Denote by $K(c,\omega)$ the set of functions $\{f(u)\}$ in $D[0, 1]$ which satisfy the inequalities $|f(u)| \leq c$ for $0 \leq u \leq 1$ and $\Delta_a(\underset{\sim}{f}) \leq \omega(a)$

for $a > 0$ where $\omega(a)$ is a nonincreasing continuous function of a for $0 < a$ and $\lim_{a \to 0} \omega(a) = 0$. Then $K(c,\omega)$ is a compact set. If for every $\varepsilon > 0$ we choose $K_\varepsilon = K(c,\omega)$ with a sufficiently large c, then by (25) the inequality (8) is satisfied, and by Theorem 1 we obtain that (21) holds.

In the following we shall give a few examples for the application of Theorem 3.

First, let us suppose that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of mutually independent and identically distributed real random variables. Write $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. Let us assume that

$$\text{(26)} \qquad \lim_{n \to \infty} \underset{\sim}{P}\left\{\frac{\zeta_n}{B_n} \leq x\right\} = R(x)$$

where $R(x)$ is a nondegenerate stable distribution function of type $S(\alpha,\beta,c,0)$ (the case of $\alpha = 1$, $\beta \neq 0$ is excluded), $B_n > 0$ for $n = 1,2,\ldots$ and $\lim_{n \to \infty} B_n = \infty$.

Define

$$\text{(27)} \qquad \xi_n(u) = \frac{\zeta_{[nu]}}{B_n}$$

for $0 \leq u < 1$ and $n = 1,2,\ldots$ and $\xi_n(1) = \zeta_{n-1}/B_n$ for $n = 1,2,\ldots$.

Let $\{\xi(u), 0 \leq u \leq 1\}$ be a stable stochastic process of type $S(\alpha,\beta,c,0)$ where $\beta = 0$ if $\alpha = 1$. (See formulas (51.61) and (51.62).) Then

$$\text{(28)} \qquad \underset{\sim}{P}\{\xi(u) \leq u^{1/\alpha} x\} = R(x)$$

for $0 < u \leq 1$.

Since both $\{\xi_n(u) , 0 \leq u \leq 1\}$ and $\{\xi(u) , 0 \leq u \leq 1\}$ have independent increments, it follows from (25) that the finite dimensional distribution functions of the process $\underset{\sim}{\xi}_n = \{\xi_n(u) , 0 \leq u \leq 1\}$ converge to the finite dimensional distribution functions of the process $\underset{\sim}{\xi} = \{\xi(u) , 0 \leq u \leq 1\}$ as $n \to \infty$.

For the process $\{\xi_n(u) , 0 \leq u \leq 1\}$ the condition (20) is satisfied. This can be proved by using (26) and the inequality

$$\underset{\sim}{P}\{\Delta_a(\underset{\sim}{\xi}_n) > \varepsilon\} \leq 2\underset{\sim}{P}\{\sup_{0\leq u\leq 4a} |\xi_n(u)| > \frac{\varepsilon}{4}\} + \tag{29}$$

$$+ (1+ \frac{1}{a}) [\underset{\sim}{P}\{\sup_{0\leq u\leq 4a} |\xi_n(u)| > \frac{\varepsilon}{4}\}]^2 .$$

For details of this proof see I. I. Gikhman and A. V. Skorokhod [44 pp.480-483].

Thus we can conclude that if Q is a real and continuous functional on $D[0, 1]$ with the metric (19), then (21) holds.

If, in particular,

$$\underset{\sim}{Q}(f) = \sup_{0\leq u\leq 1} \frac{|f(u) + a(u)|}{b(u)} \tag{30}$$

where $a(u)$ and $b(u) > 0$ are continuous functions of u , then Q is continuous in the metric (19), and by Theorem 3 we have

$$\lim_{n\to\infty} \mathbf{P}\{a(\tfrac{k}{n}) - xb(\tfrac{k}{n}) \leq \frac{\zeta_k}{B_n} \leq a(\tfrac{k}{n}) + xb(\tfrac{k}{n}) \quad \text{for } k = 1,2,\ldots,n\} = \tag{31}$$
$$\mathbf{P}\{a(u) - xb(u) \leq \xi(u) \leq a(u) + xb(u) \quad \text{for } 0 \leq u \leq 1\}$$

for $x \geq 0$.

If

$$\mathbf{Q}(f) = \int_0^1 h\,(f(u))du \tag{32}$$

where $h(x)$ is a continuous function defined on the interval $(-\infty, \infty)$, then $\mathbf{Q}(f)$ is a continuous functional in the metric (18), and by Theorem 3 we obtain that

$$\lim_{n\to\infty} \mathbf{P}\{\frac{1}{n}\sum_{k=1}^{n} h(\frac{\zeta_k}{B_n}) \leq x\} = \mathbf{P}\{\int_0^1 h(\xi(u))du \leq x\} \tag{33}$$

in every continuity point of the limiting distribution function.

As a second example, let us suppose that $\{\xi_n(u)\ ,\ 0 \leq u \leq 1\}$ is a compound Poisson process for every $n = 1,2,\ldots$ and that

$$\mathbf{E}\{e^{-s\xi_n(u)}\} = e^{u\Psi_n(s)} \tag{34}$$

for $\mathrm{Re}(s) = 0$. Furthermore, let $\{\xi(u)\ ,\ 0 \leq u \leq 1\}$ be a homogeneous stochastic process with independent increments for which

$$\mathbf{E}\{e^{-s\xi(u)}\} = e^{u\Psi(s)} \tag{35}$$

for $\mathrm{Re}(s) = 0$.

Let us suppose that the finite dimensional distribution functions of the process $\{\xi_n(u) , 0 \leq u \leq 1\}$ converge to the finite dimensional distribution functions of the process $\{\xi(u) , 0 \leq u \leq 1\}$.

We can easily see that the finite dimensional distribution functions of the process $\{\xi_n(u) , 0 \leq u \leq 1\}$ converge to the finite dimensional distribution functions of the process $\{\xi(u) , 0 \leq u \leq 1\}$ if and only if

$$\lim_{n \to \infty} \Psi_n(s) = \Psi(s) \tag{36}$$

for $\operatorname{Re}(s) = 0$.

We note that if $\{\xi(u) , 0 \leq u \leq 1\}$ is any homogeneous stochastic process with independent increments, then we can find a sequence of compound Poisson processes $\{\xi_n(u) , 0 \leq u \leq 1\}$ such that the finite dimensional distribution functions of the process $\{\xi_n(u) , 0 \leq u \leq 1\}$ converge to the finite dimensional distribution functions of the process $\{\xi(u) , 0 \leq u \leq 1\}$.

Let us suppose that the processes $\{\xi_n(u) , 0 \leq u \leq 1\}$ and $\{\xi(u) , 0 \leq u \leq 1\}$ are separable. By Theorem 5 we can always choose such versions of these processes for which the sample functions belong to $D[0, 1]$ with probability 1.

Now in a similar way as in the previous example we can prove that (29) holds and that (36) implies (20). Thus Theorem 3 is applicable and (21) holds for any real and continuous functional on $D[0, 1]$ with the metric (18).

As a third example, let us suppose that for each n we distribute n points at random on the interval $(0, 1)$ in such a way that each point has a uniform distribution over $(0, 1)$. For each $n = 1,2,\ldots$ denote by $\nu_n(u)$ the number of random points in the interval $(0, u]$ where $0 \leq u \leq 1$.

Define

$$\eta_n(u) = \frac{\nu_n(u) - nu}{\sqrt{n}} \tag{37}$$

for $0 \leq u \leq 1$. Then $\eta_n = \{\eta_n(u), 0 \leq u \leq 1\}$ is a stochastic process whose sample functions belong to $D[0, 1]$.

Let $\eta = \{\eta(u), 0 \leq u \leq 1\}$ be a Gaussian stochastic process for which $E\{\eta(u)\} = 0$ if $0 \leq u \leq 1$ and $E\{\eta(u)\eta(v)\} = \min(u,v) - uv$ if $0 \leq u \leq 1$ and $0 \leq v \leq 1$. (See Section 50.)

We can easily prove that the finite dimensional distribution functions of the process $\{\eta_n(u), 0 \leq u \leq 1\}$ converge to the finite dimensional distribution functions of the process $\{\eta(u), 0 \leq u \leq 1\}$ as $n \to \infty$.

For the process $\{\eta(u), 0 \leq u \leq 1\}$ we have $P\{\eta(0) = 0\} = P\{\eta(1) = 0\} = 1$ and we can represent $\eta(u)$ for $0 < u < 1$ in the following way:

$$\eta(u) = (1-u)\xi\left(\frac{u}{1-u}\right) \tag{38}$$

where $\{\xi(u), 0 \leq u < \infty\}$ is a Brownian motion process.

If we suppose that $\{\eta(u), 0 \leq u \leq 1\}$ is a separable process, then by Theorem 50.1 we can conclude that the sample functions of the process $\{\eta(u), 0 \leq u \leq 1\}$ are continuous with probability 1. For, in this case, $\xi(u) = (1+u)\eta(u/(1+u))$ $(0 \leq u < \infty)$, is a separable Brownian motion process, and thus Theorem 50.1 is applicable. Accordingly, if $\{\eta(u), 0 \leq u \leq 1\}$ is a separable process, then the sample functions with probability 1 belong to the space $C[0, 1]$ and consequently to the space $D[0, 1]$ too.

Now we can prove that

$$(39)\qquad \lim_{a\to 0}\ \limsup_{n\to\infty} \mathbf{P}\{\Delta_a(\eta_n) > \varepsilon\} = 0$$

for every $\varepsilon > 0$. This follows from the inequality

$$(40)\qquad \mathbf{P}\{\Delta_a(\eta_n) > \varepsilon\} \leq \mathbf{P}\{\sup_{|u-v|\leq a} |\eta_n(u) - \eta_n(v)| > \varepsilon\}$$

and the limit relation

$$(41)\qquad \lim_{a\to 0}\ \limsup_{n\to\infty} \mathbf{P}\{\sup_{|u-v|\leq a} |\eta_n(u) - \eta_n(v)| > \varepsilon\} = 0\ .$$

By Theorem 3 we obtain that if the process $\{\eta(u)\ ,\ 0 \leq u \leq 1\}$ is separable and if Q is a real and continuous functional on $D[0, 1]$ with the metric (19), then

$$(42)\qquad \lim_{n\to\infty} \mathbf{P}\{Q(\eta_n) \leq x\} = \mathbf{P}\{Q(\eta) \leq x\}$$

in every continuity point of $\mathbf{P}\{Q(\eta) \leq x\}$.

In Section 39 we have already mentioned some particular cases of (42). In particular, we considered the functionals $Q(f) = \max_{0\leq u\leq 1} f(u)$, $Q(f) = \max_{0\leq u\leq 1} |f(u)|$, and

$$(43)\qquad Q(f) = \int_0^1 [f(u)]^2\, du$$

which are continuous in the metric (18).

Finally, let us consider the following example. Let us suppose that for each $n = 1,2,\ldots$ we have a box which contains $2n$ cards of which n are marked $+1$ and n are marked -1. We draw each of the $2n$ cards from the box without replacement. Let us suppose that every outcome of this random trial has the same probability. Denote by $\sigma_n(k)$ thé sum of the first k numbers drawn $(k = 1,2,\ldots, 2n)$ and let $\sigma_n(0) = 0$. Define

$$\eta_n(u) = \frac{\sigma_n(2nu)}{\sqrt{2n}} \tag{44}$$

for $0 \leq u < 1$ and let $\eta_n(2n) = \sigma_n(2n-1)/\sqrt{2n}$. Then $\underset{\sim}{\eta}_n = \{\eta_n(u)\ ,\ 0 \leq u \leq 1\}$ is a stochastic proces whose sample functions belong to $D[0, 1]$.

Let $\underset{\sim}{\eta} = \{\eta(u)\ ,\ 0 \leq u \leq 1\}$ be a Gaussian stochastic process for which $\underset{\sim}{E}\{\eta(u)\} = 0$ if $0 \leq u \leq 1$ and $\underset{\sim}{E}\{\eta(u)\eta(v)\} = \min(u,v) - uv$ if $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

We can easily see that the finite dimensional distribution functions of the process $\{\eta_n(u)\ ,\ 0 \leq u \leq 1\}$ converge to the finite dimensional distribution functions of the process $\{\eta(u)\ ,\ 0 \leq u \leq 1\}$ as $n \to \infty$.

As we mentioned earlier, if we suppose that the process $\{\eta(u)\ ,\ 0 \leq u \leq 1\}$ is separable then the sample functions are continuous with probability 1.

By using the inequality (40) we can prove that (39) is satisfied in this case too. Thus by Theorem 3 we can conclude that if the process $\{\eta(u)\ ,\ 0 \leq u \leq 1\}$ is separable and if Q is a real and continuous functional on $D[0, 1]$ with the metric (18), then

$$\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\eta}_n) \leq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\eta}) \leq x\} . \tag{45}$$

For example, if $Q(f) = \max_{0 \leq u \leq 1} f(u)$, then Q is continuous in the metric (18) and by (45) we can conclude that

$$(46) \qquad \lim_{n \to \infty} \underset{\sim}{P}\{\max_{0 \leq k \leq n} \sigma_n(k) \leq \sqrt{2n}\, x\} = \underset{\sim}{P}\{\sup_{0 \leq u \leq 1} \eta(u) \leq x\} .$$

We already saw that

$$(47) \qquad \underset{\sim}{P}\{\max_{0 \leq k \leq n} \sigma_n(k) \leq c\} = 1 - \frac{\binom{2n}{n+1+c}}{\binom{2n}{n}}$$

for $c = 0,1,\ldots, n$. (See formulas (39.71) and (39.172).) If we put $c = [\sqrt{2n}\, x]$ in (47) where $x \geq 0$ and let $n \to \infty$, then we obtain that

$$(48) \qquad \underset{\sim}{P}\{\sup_{0 \leq u \leq 1} \eta(u) \leq x\} = 1 - e^{-2x^2}$$

for $x \geq 0$ whenever $\{\eta(u)\, , 0 \leq u \leq 1\}$ is a separable Gaussian process for which $\underset{\sim}{E}\{\eta(u)\} = 0$ and $\underset{\sim}{E}\{\eta(u)\eta(v)\} = \min(u, v) - uv$ $(0 \leq u \leq 1\, ,$ $0 \leq v \leq 1)$.

We note that if in the last example we define

$$(49) \qquad \eta_n^*(u) = \frac{\sigma_n(2nu) + (2nu - [2nu])[\sigma_n(2nu+1) - \sigma_n(2nu)]}{\sqrt{2n}}$$

for $0 \leq u \leq 1$, then $\{\eta_n^*(u)\, , 0 \leq u \leq 1\}$ has continuous sample functions, and the finite dimensional distribution functions of the process $\{\eta_n^*(u)\, ,$ $0 \leq u \leq 1\}$ converge to the finite dimensional distribution functions of the process $\{\eta(u)\, , 0 \leq u \leq 1\}$.

We can prove that (41) is satisfied for the process $\eta_n^* = \{\eta_n^*(u), 0 \leq u \leq 1\}$. Thus by Theorem 2 we can conclude that if $\eta = \{\eta(u), 0 \leq u \leq 1\}$ is a separable Gaussian process for which $E\{\eta(u)\} = 0$ and $E\{\eta(u)\eta(v)\} = \min(u, v) - uv$ $(0 \leq u \leq 1, 0 \leq v \leq 1)$ and if Q is a real continuous functional on $C[0, 1]$ with the metric ρ, then

$$(50) \qquad \lim_{n \to \infty} P\{Q(\eta_n^*) \leq x\} = P\{Q(\eta) \leq x\}$$

in every continuity point of $P\{Q(\eta) \leq x\}$.

If, in particular, $Q(f) = \max_{0 \leq u \leq 1} f(u)$, then Q is continuous on $C[0, 1]$ and (50) reduces to (46).

We shall close this section by giving a brief account of the historical development of the subject of weak convergence of stochastic processes.

The problem of weak convergence of stochastic processes goes back to 1900 when L. Bachelier [481] approximated a Brownian motion process $\{\xi(u), 0 \leq u \leq \infty\}$ by a sequence of random walk processes and found the probability $P\{\sup_{0 \leq u \leq t} \xi(u) \leq x\}$.

The general problem of finding conditions for the validity of (2) has received considerable attention.

In the case where the process $\xi_n = \{\xi_n(u), 0 \leq u \leq t\}$ is defined as suitably normalized sums of mutually independent random variables and $\xi = \{\xi(u), 0 \leq u \leq t\}$ is a Brownian motion process, the limit theorem (2) was proved for various functionals Q in 1931 by A. N. Kolmogorov [511], [512] and in 1946 by P. Erdős and M. Kac [502],[503]. Their results were extended in 1951 by M. D. Donsker [494]. Several results are mentioned in Section 45 for the applications of Donsker's theorem. Theorem 2 was found in 1953 by Yu. V. Prochorov [522], [523]. See also A.N. Kolmogorov and Yu. V. Prochorov [514].

In 1949 J. L. Doob [328] considered the case where $\underset{\sim}{\xi}_n$ is defined by (37) and $\underset{\sim}{\xi}$ is defined by (38) and $Q = \sup_{0 \le u \le 1} |f(u)|$. Doob's heuristic results were justified in 1952 by M. D. Donsker [495].

In 1955 A. V. Skorokhod [535] proved Theorem 3 for stochastic processes with independent increments and in 1956 A. V. Skorokhod [537] proved Theorem 3 in the general case.

Further extensions of the results given in this section can be found in the references at the end of this chapter.

53. Problems

53.1. Let $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ be a recurrent process with mean recurrence time a where a is a finite positive number. Prove that

$$\underset{\sim}{P}\{\lim_{t \to \infty} \frac{\nu(t)}{t} = \frac{1}{a}\} = 1\ .$$

(See J. L. Doob [199].)

53.2. Let ξ_1 and ξ_2 be independent random variables for which

$$\underset{\sim}{P}\{\xi_1 + \xi_2 = k\} = e^{-a} \frac{a^k}{k!} \quad (k = 0,1,\ldots)\ .$$

Prove that there exists a constant c such that $\xi_1 + c$ and $\xi_2 - c$ both have a Poisson distribution. (See D. A. Raikov [157].)

53.3. Let $\{\nu(u)\ ,\ 0 \leq u < \infty\}$ be a Poisson process of density λ. Prove that

$$\underset{\sim}{P}\{\nu(i) = i \text{ for } k \text{ values } i = 1,2,\ldots,n \mid \nu(n) = n\} = \frac{n!k}{(n-k)!\ n^{k+1}}$$

for $k = 1,2,\ldots, n$.

53.4. Let $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ be a recurrent process where the recurrence times θ_k $(k = 1,2,\ldots)$ have the distribution function

$$F(x) = \begin{cases} 1 - \dfrac{1}{x(\log x)^2} & \text{for } x \geq e\ , \\ 0 & \text{for } x < e\ . \end{cases}$$

Determine the asymptotic distribution of $\nu(t)$ as $t \to \infty$.

53.5. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables having the same stable distribution function of type $S(\alpha,\beta,c,0)$ where $\alpha \neq 1$, $c > 0$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. Denote by $\tau_1, \tau_2, \ldots, \tau_k, \ldots$ the successive ladder indices in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$, that is, τ_1 is the smallest $n = 1,2,\ldots$ for which $\zeta_n > \zeta_0$, τ_2 is the smallest $n = 2,3,\ldots$ for which $\zeta_n > \zeta_{\tau_1}$ and so on. Then $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots$ is a sequence of mutually independent and identically distributed random variables taking on positive integers only. Define $\nu(t)$ for $t \geqq 0$ as a discrete random variable taking on nonnegative integers only and satisfying the relation $\{\nu(t) \geqq k\} \equiv \{\tau_k \leqq t\}$ for all $t \geqq 0$ and $k = 0,1,2,\ldots$. Then $\{\nu(t), 0 \leqq t < \infty\}$ is a recurrent process. Determine the asymptotic distribution of $\nu(t)$ as $t \to \infty$.

53.6. Find the asymptotic distribution of ζ_{τ_n} as $n \to \infty$ in Problem 53.5.

REFERENCES

General Theory

[1] Ambrose, W., "On measurable stochastic processes," Transactions of the American Mathematical Society 47 (1940) 66-79.

[2] Andersen, E. S., "Indhold og maal i produktmaengder," Matematisk Tidsskrift B (1944) 19-23.

[3] Andersen, E. S., and B. Jessen, "Some limit theorems on integrals in an abstract set," D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. 22 No. 14 (1946) 29 pp.

[4] Andersen, E. S., and B. Jessen, "On the introduction of measures in infinite product sets," D. Kgl. Danske Vidensk. Selskab. Mat.-fys. Medd. 25 No. 4 (1948) 8 pp.

[5] Andersen, E. S., and B. Jessen, "Some limit theorems on set-functions," D. Kgl. Danske Vidensk. Selskab, Mat.-Fys. Medd. 25 No. 5 (1948) 8 pp.

[6] Billingsley, P., "Hausdorff dimension in probability theory I, II," Illinois Journal of Mathematics 4 (1960) 187-209 and 5 (1961) 291-298.

[7] Blackwell, D., "On a class of probability spaces," Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability. Vol. II. University of California Press, 1956, pp. 1-6.

[8] Blumenthal, R. M., "An extended Markov property," Transactions of the American Mathematical Society 85 (1957) 52-72.

[9] Borel, É., Probabilities and Life. Dover, New York, 1962. [English translation of Émile Borel: Les Probabilités et la Vie. Presses Universitaires de France, 1943.]

[10] Borges, R., "Zur Existenz von separablen stochastischen Prozessen," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 6 (1966) 125-128.

[11] Bravais, A., "Analyse mathématique sur les probabilités des erreurs de situation d'un point," Mémoires présentes par divers savants à l'Académie Royale des Sciences de l'Institut de France. Sci. Math. et Phys. 9 (1846) 255-332.

[12] Breiman, L., Probability. Addison-Wesley, Reading, Massachusetts, 1968.

[13] Brooks, R. A., "Conditional expectations associated with stochastic processes," Pacific Journal of Mathematics 41 (1972) 33-42.

[14] Brunk, H. D., "Note on a theorem of Kakutani," Proceedings of the American Mathematical Society 1 (1950) 409-414.

[15] Buch, K. R., "Some investigations of the set of values of measures in abstract space," D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. 21 (1945) 70 pp.

[16] Cauchy, A. L., Course d'Analyse de l'École Royale Polytechnique. I^{re} Partie. Analyse Algébrique. Paris 1821. [Oeuvres complètes d'Augustin

[17] Cohn, D. L., "Measurable choice of limit points and the existence of separable and measurable processes," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 22 (1972) 161-165.

[18] Cramér, H., "A contribution to the theory of stochastic processes," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (1950), University of California Press, 1951, pp. 329-340.

[19] Cramér, H., "On stochastic processes whose trajectories have no discontinuities of the second kind," Annali di Mathematica Pura ed Applicate (Bologna) (4) 71 (1966) 85-92.

[20] Daley, D. J., "Weakly stationary point processes and random measures," Journal of the Royal Statistical Society. Ser. B. 33 (1971) 406-428.

[21] Daniell, P. J., "A general form of integral," Annals of Mathematics 19 (1917-1918) 279-294.

[22] Daniell, P. J., "Integrals in an infinite number of dimensions," Annals of Mathematics 20 (1918-1919) 281-288.

[23] Daniell, P. J., "Functions of limited variation in an infinite number of dimensions," Annals of Mathematics 21 (1919) 30-38.

[24] Dobrushin, R. L., "Prescribing a system of random variables by conditional distributions," Theory of Probability and its Applications 15 (1970) 458-486.

[25] Doob, J. L., "Stochastic processes depending on a continuous parameter," Transactions of the American Mathematical Society 42 (1937) 107-140.

[26] Doob, J. L., "Stochastic processes with an integral-valued parameter," Transactions of the American Mathematical Society 44 (1938) 87-150.

[27] Doob, J. L., "The law of large numbers for continuous stochastic processes," Duke Mathematical Journal 6 (1940) 290-306.

Cauchy. II^{e} Série. Tome III. Gauthier-Villars, Paris, 1897, pp. 1-332.]

[28] Doob, J. L., "Regularity properties of certain families of chance variables," Transactions of the American Mathematical Society 47 (1940) 455-486.

[29] Doob, J. L., "Probability in function space," Bulletin of the American Mathematical Society 53 (1947) 15-30.

[30] Doob, J. L., Stochastic Processes. John Wiley and Sons, New York, 1953.

[31] Doob, J. L., "An application of stochastic process separability," L'Enseignement Mathématique 15 (1969) 101-105.

[32] Doob, J. L., "Separability and measurable processes," Journal of the Faculty of Science, University of Tokyo, Sec. I., 17 (1970) 297-304.

[33] Doob, J. L., and W. Ambrose, "On two formulations of the theory of stochastic processes depending upon a continuous parameter," Annals of Mathematics 41 (1940) 737-745.

[34] Dudley, R., "A counterexample on measurable processes," Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. Vol. II. Probability Theory. University of California Press, 1972, pp. 57-66. [Correction: The Annals of Probability 1(1973) 191-192.]

[35] Dugue, D., "Characteristic functions and Bernoulli numbers," Journal of Multivariate Analysis 2 (1972) 230-235.

[36] Elliot, E. O., "An extension theorem for obtaining measures on uncountable product spaces," Proceedings of the American Mathematical Society 19 (1968) 1089-1093.

[37] Elliot, E. O., "Extension of measures and abstract stochastic processes," Fundamenta Mathematicae 73 (1972) 105-111.

[38] Elliott, E. O., "A generalization of separable stochastic processes," The Annals of Mathematical Statistics 43 (1972) 320-325.

[39] Elliot, E. O., and A. P. Morse, "General product measures," Transactions of the American Mathematical Society 110 (1964) 245-283.

[40] Erdös, P., "On the distribution function of additive functions," Annals of Mathematics 47 (1946) 1-20.

[41] Feller, W., "Chance processes and fluctuations," Modern Mathematics for the Engineer. Second Series. Ed. E. F. Beckenbach. McGraw-Hill, New York, 1961, pp. 167-181.

[42] Fisz, M., "Characterization of sample functions of stochastic processes by some absolute probabilities," Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (1960) Vol. II: Probability Theory, University of California Press, 1961, pp. 143-151.

[43] Fréchet, M., "Sur l'intégrale d'une fonctionelle étendue à un ensemble abstrait," Bulletin de la Société Mathématique de France 43 (1915) 249-267.

[44] Gikhman, I. I., and A. V. Skorokhod, Introduction to the Theory of Random Processes. W. B. Saunders, Philadelphia, 1969. [English translation of the Russian original published by Izd-vo "Nauka", Moscow, 1965.]

[45] Hamel, G., "Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x+y) = f(x)+f(y)$," Mathematische Annalen 60 (1905) 459-462.

[46] Jensen, J. L. W. V., "Sur les fonctions convexes et les inégalités entre les valeurs moyennes," Acta Mathematica 30 (1906) 175-193.

[47] Jessen, B., "The theory of integration in a space of an infinite number of dimensions," Acta Mathematica 63 (1934) 249-323.

[48] Jessen, B., "Abstrakt maal-og integralteori. 4," Matematisk Tidsskrift B (1939) 7-21.

[49] Jordan, Ch., Calculus of Finite Differences. Budapest, 1939. [Reprinted by Chelsea, New York, 1947.]

[50] Kakutani, S., "Notes on infinite product measure spaces, I," Proc. Imper. Acad. Tokyo 19 (1943) 148-151.

[51] Kakutani, S., "On equivalence of infinite product measures," Annals of Mathematics 49 (1948) 214-224.

[52] Karhunen, K., "Über die Struktur stationärer zufälliger Funktionen," Arkiv för Matematik 1 (1949-1950) 141-160.

[53] Karlin, S., and J. McGregor, "Coincidence probabilities," Pacific Journal of Mathematics 9 (1959) 1141-1164.

[54] Kolmogoroff, A., "Untersuchungen über den Integralbegriff," Mathematische Annalen 103 (1930) 654-696.

[55] Kolmogorov, A. N., Foundations of the Theory of Probability. Chelsea, New York, 1950. [German translation of A. Kolmogoroff: Grundbegriffe der Wahrscheinlichkeitsrechnung. Ergebnisse der Mathematik, Springer, Berlin, 1933.]

[56] Kolmogorov, A. N., and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis. Vol. 1 Metric and Normed Spaces. Graylock Press, Rochester, N. Y. 1957. [English translation of the Russian original published in 1954.]

[57] Kostka, D. G., "Deviations in the Skorokhod-Strassen approximation scheme," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 24 (1972(139-153.

[58] Kuo, H. H., "Integration theory on infinite-dimensional manifolds," Transactions of the American Mathematical Society 159 (1971) 57-78.

[59] Łomnicki, Z., and S. Ulam, "Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités I. Variables indépendantes." Fundamenta Mathematicae 23 (1934) 237-278.

[60] McShane, E. J., "A weak topology for stochastic processes," Boletim da Sociedade de Matemática de São Paulo 14 (1959) 83-101.

[61] Meyer, P. A., "Séparabilité d'un processus stochastiques," Comptes Rendus Acad. Sci. Paris 248 (1959) 3106-3107.

[62] Meyer, P. A., Probability and Potentials. Blaisdell, Waltham, Mass., 1966.

[63] Milkman, J., "Note on the functional equations $f(xy) = f(x) + f(y)$, $f(x^n) = nf(x)$," Proceedings of the American Mathematical Society 1 (1950) 505-508.

[64] Nelson, E., "Regular probability measures on function space," Annals of Mathematics 69 (1959) 630-643,

[65] Neumann, J. v., Functional Operators. Vol. I. Measures and Integrals. Annals of Mathematics Studies. No. 21. Princeton University Press, 1950.

[66] Neuts, M. F., and S. I. Resnick, "On the times of births in a linear birthprocess," The Journal of the Australian Mathematical Society 12 (1971) 473-475.

[67] Paley, R. E. A. C., and N. Wiener, "Notes on the theory and application of Fourier transforms. I-VII.," Transactions of the American Mathematical Society 35 (1933) 348-355 and 761-791.

[68] Paley, R. E. A. C., and N. Wiener, "Notes on random functions," Mathematische Zeitschrift 37 (1933) 647-668.

[69] Paley, R. E. A. C., and N. Wiener, Fourier Transforms in the Complex Domain. American Mathematical Society, 1934.

[70] Parthasarathy, K. R., Probability Measures on Metric Spaces, Academic Press, New York, 1967.

[71] Rogers, C. A., Hausdorff Measures. Cambridge University Press, 1970.

[72] Sawyer, S., "A remark on the Skorokhod representation," Zeitschrift für Wahrscheinlichketistheorie und verw. Gebiete 23 (1972) 67-74.

[73] Segal, I. E., "Abstract probability spaces and a theorem of Kolmogorov," American Journal of Mathematics 76 (1954) 721-732.

[74] Silvestrov, D. S., "Limiting distributions for compositions of random functions," Doklady Akad. Nauk SSSR 199 (1971) 1251-1252. [English translation: Soviet Mathematics-Doklady 12 (1971) 1282-1284.]

[75] Skorokhod, A. V., Studies in the Theory of Random Processes. Addison-Wesley, Reading, Massachusetts, 1965. [English translation of the Russian original published by Kiev University Press, 1961.]

[76] Skorokhod, A. V., "Homogeneous Markov processes without discontinuities of the second kind," Theory of Probability and its Applications 12 (1967) 222-240.

[77] Slutsky, E., "Sur les fonctions éventuelles continues, intégrables et dérivables dans le sens stochastique," Comptes Rendus Acad. Sci. Paris 187 (1928) 878-880.

[78] Slutsky, E., "Qualche proposizione relative alla teoria delle funzioni aleatorie," Giornale dell'Istituto Italiano degli Attuari 8 (1937) 183-199.

[79] Smith, Wilbur L., "A generalized infinite product measure," Duke Mathematical Journal 38 (1971) 765-769.

[80] Stone, C., "Limit theorems for random walks, birth and death processes, and diffusion processes," Illinois Jour. Math. 7 (1963) 638-660.

[81] Strassen, V., "The existence of probability measures with given marginals," The Annals of Mathematical Statistics 36 (1965) 423-439.

[82] Szász, D., "Limit theorems for stochastic processes stopped at random," Theory of Probability and its Applications 16 (1971) 554-555.

[83] Takács, L., "The time dependence of a single-server queue with Poisson input and general service times," The Annals of Mathematical Statistics 33 (1962) 1340-1348.

[84] Takács, L., "On the method of inclusion and exclusion," Journal of the American Statistical Association 62 (1967) 102-113.

[85] Takács, L., "A combinatorial theorem for stochastic processes," Bulletin of the American Mathematical Society 71 (1965) 649-650.

[86] Takács, L., "On combinatorial methods in the theory of stochastic Processes," Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability (1965). University of California Press. Vol. 2. Part 1 (1967) 431-447.

[87] Vere-Jones, D., "Some applications of probability generating functionals to the study of input-output streams," Jour. Roy. Statist. Soc. Ser. B. 30 (1968) 321-333.

[88] Wang, Chia-kang, "Properties and limit theorems of sample functions of the Ito stochastic integral," (Chinese) Acta Mathematica Sinica 14 (1964) 517-531. [English translation: Chinese Mathematics 5 (1964) 556-570.]

[89] Whittle, P., "Stochastic processes in several dimensions," Bulletin of the International Statistical Institute 40 (1963) 974-994.

[90] Wiener, N., "The average of an analytical functional," Proceedings of the National Academy of Sciences U.S.A. 7 (1921) 253-260.

[91] Wiener, N., "The average of an analytical functional and the Brownian movement," Proceedings of the National Academy of Sciences U. S. A. 7 (1921) 294-298.

[92] Wiener, N., "The average value of a functional," Proceedings of the London Mathematical Society 22 (1924) 454-467.

[93] Wiener, N., "Un problème de probabilités dénombrables," Bulletin de la Société Mathématique de France 52 (1924) 569-578.

[94] Wiener, N., "The homogeneous chaos," American Journal of Mathematics 60 (1938) 897-936. [Selected Papers of Norbert Wiener. The M.I.T. Press, Cambridge, Mass., 1964, pp. 372-411.]

Poisson Processes

[95] Aczél, J., "On composed Poisson distributions, III." Acta Mathematica Acad. Sci. Hungaricae 3 (1952) 219-224.

[96] Andersson,Th., "Ladislaus von Bortkiewicz 1868-1931," Nordic Statistical Journal 3 (1931) 9-26.

[97] Bateman, H., "On the probability distribution of α particles," Philosophical Magazine (6) 20 (1910) 704-707.

[98] Bateman, H., "Some problems in the theory of probability," Philosophical Magazine (6) 21 (1911) 745-752.

[99] Benczur, A., "On sequences of equivalent events and the compound Poisson process," Studia Scientiarum Mathematicarum Hungarica 3 (1968) 451-458.

[100] Blanc-Lapierre, A., et R. Fortet, "Sur les répartitions de Poisson," Comptes Rendus Acad. Sci. 240 (1955) 1045-1046.

[101] Bolshev, L. N., "On characterization of the Poisson distribution and its statistical applications," Theory of Probability and its Applications 10 (1965) 446-456.

[102] Bortkewitsch, L. v., Das Gesetz der kleinen Zahlen. Teubner, Leipzig, 1898.

[103] Bortkiewicz, L. v., Die Iterationen. Ein Beitrag zur Wahrscheinlichkeitstheorie. Springer, Berlin, 1917.

[104] Brown, M., "An invariance property of Poisson processes," Journal of Applied Probability 6 (1969) 453-458.

[105] Brown, M., "A property of Poisson processes and its application to macroscopic equilibrium of particle systems," The Annals of Mathematical Statistics 41 (1970) 1935-1941.

[106] Brown, M., "Discrimination of Poisson processes," The Annals of Mathematical Statistics 42 (1971) 773-776.

[107] Chatterji, S. D., "Some elementary characterizations of the Poisson distribution," American Mathematical Monthly 70 (1963) 958-964.

[108] Cuppens, R., "Sur les produits finis de lois de Poisson," Comptes Rendus Acad. Sci. Paris. Sér. A 266 (1968) 726-728. [P. Lévy: Observations sur la Note précédente. Ibid pp. 728-729.]

[109] De Moivre, A., The Doctrine of Chances: or, A Method of Calculating the Probabilities of Events in Play. Third edition. A. Millar, London, 1756. [Reprinted by Chelsea, New York, 1967.]

[110] Dobrushin, R. L., "On Poisson's law for distribution of particles in space," (Russian) Ukrainskii Matemat. Zhur. 8 (1956) 127-134.

[111] Erlang, A. K., "The theory of probabilities and telephone conversations," (Danish) Nyt Tidsskrift for Matematik B 20 (1909) 33-39. [French translation: Revue Général de l'Électricité 18 (1925) 305-309. English translation: Transactions of the Danish Academy of Technical Sciences No. 2 (1948) 131-137.]

[112] Euler, L., "Calcul de la probabilité dans le jeu de recontre," Memoires de l'Académie des Sciences de Berlin (1751) 7 (1753) 255-270. [Leonhardi Euleri Opera Omnia I 7 (1923) 11-25.]

[113] Feller, W., "On a general class of 'contagious' distributions," The Annals of Mathematical Statistics 14 (1943) 389-400.

[114] Fisz, M., "The limiting distribution of the difference of two Poisson random variables," Zastosowania Matematyki 1 (1953) 41-45.

[115] Fisz, M., "Refinement of a probability limit theorem and its application to Bessel functions," Acta Mathematica Acad. Sci. Hungaricae 6 (1955) 199-202.

[116] Fisz, M., and K. Urbanik, "Analytical characterization of a composed, non-homogeneous Poisson process," Studia Mathematica 15 (1956) 328-336.

[117] Florek, K., E. Marczewski, and C. Ryll-Nardzweski, "Remarks on the Poisson stochastic process. I," Studia Math. 13 (1953) 122-129.

[118] Fortet, R., "Random functions from a Poisson process," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (1950). University of California Press, Berkeley, 1951, pp. 373-385.

[119] Fortet, R., "Sur les répartitions ponctuelles aléatoires, en particulier de Poisson," Ann. Inst. Henri Poincaré, Sect. B. 4 (1968) 99-142.

[120] Freedman, D. A., "Poisson processes with random arrival rate," The Annals of Mathematical Statistics 33 (1962) 924-929.

[121] Fujiwara, M., "Ein Problem in der Wahrscheinlichkeitsrechnung," The Tôhoku Mathematical Journal 19 (1921) 48-50.

[122] Geiger, H., and E. Rutherford, "The number of α particles emitted by uranium and thorium and by uranium minerals," Philosophical Magazine (6) 20 (1910) 691-698.

[123] Geiringer, H. Pollaczek, "Über die Poisonsche Verteilung und die Entwicklung willkürlicher Verteilungen," Zeitschrift für angewandte Mathematik und Mechanik 8 (1928) 292-309.

[124] Goldman, J. R., "Stochastic point processes: Limit theorems," The Annals of Mathematical Statistics 38 (1967) 771-779.

[125] Haight, F. A., Handbook of the Poisson Distribution. John Wiley and Sons, New York, 1967.

[126] Jánossy, L., A. Rényi, and J. Aczél, "On composed Poisson distributions, I." Acta Mathematica Acad. Sci. Hungaricae 1 (1950) 209-224.

[127] Kerstan, J., "Teilprozesse Poissonscher Prozesse," Transactions of the Third Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, 1962. Czechoslovak Academy of Sciences, Prague, 1964, pp. 377-403.

[128] Khintchine, A., Asymptotische Gesetze der Wahrscheinlichkeitsrechnung. Springer, Berlin, 1933. [Reprinted by Chelsea, New York, 1948.]

[129] Khinchin, A. Ya., "On Poisson sequences of chance events," Theory of Probability and its Applications 1 (1956) 291-297.

[130] Lee, P. M., "Some examples of infinitely divisible point processes," Studia Scientiarum Mathematicarum Hungarica 3 (1968) 219-224.

[131] Leslie, R. T., "Recurrence times of clusters of Poisson points," Journal of Applied Probability 6 (1969) 372-388.

[132] Letac, G., "Une propriété de fluctuation des processus de Poisson composés croissants," Comptes Rendus Acad. Sci. Paris 258 (1964) 1700-1703.

[133] Lewis, P. A. W., "A branching Poisson process model for the analysis of computer failure patterns," Journal of the Royal Statistical Society. Ser. B 26 (1964) 398-456.

[134] Lundberg, F., I. Approximerad Framställning af Sannolikhetsfunktionen. II. Återförsäkring af Kollektivrisker. Akademisk Afhandling, Uppsala, 1903.

[135] Maceda, E. C., "On the compound and generalized Poisson distributions," The Annals of Mathematical Statistics 19 (1948) 414-416.

[136] Marczewski, E., "Remarks on the Poisson stochastic process. II," Studia Math. 13 (1953) 130-136.

[137] Maruyama, G., "On the Poisson distribution derived from independent random walks," Nat. Sci. Reports of Ochanomizu Univ. 6 (1955) 1-6.

[138] McFadden, J. A., "The mixed Poisson process," Sankhyā: The Indian Journal of Statistics. Ser. A. 27 (1965) 83-92.

[139] Miles, R. E., "Poisson flats in Euclidean spaces. Part I: Finite number of random uniform flats," Advances in Applied Probability 1 (1969) 211-237.

[140] Miles, R. E., "Poisson flats in Euclidean spaces. Part II: Homogeneous Poisson flats and the complementary theorem," Advances in Applied Probability 3 (1971) 1-43.

[141] Milne, R. K., "Identifiability for random translations of Poisson processes," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 15 (1970) 195-201.

[142] Mises, R. v., "Über die Wahrscheinlichkeit seltener Ereignisse," Zeitschrift für angewandte Mathematik und Mechanik 1 (1921) 121-124. [Reprinted in Selected Papers of Richard von Mises. Vol. II. Probability and Statistics, General. American Mathematical Society, Providence, R. I., 1964, pp. 107-112.]

[143] Montmort, P. R., Essay d'Analyse sur les Jeux de Hazard. Second edition, Paris, 1713. [Reprinted by Chelsea, New York, 1973.]

[144] Moran, P. A. P., "A characteristic property of the Poisson distribution," Proceedings of the Cambridge Philosophical Society 48 (1952) 206-207.

[145] Moran, P. A. P., "A non-Markovian quasi-Poisson process," Studia Sci. Math. Hungar. 2 (1967) 425-429.

[146] Mönch, G., "Verallgemeinerung eines Satzes von A. Rényi," Studia Scientiarum Mathematicarum Hungarica 6 (1971) 81-90.

[147] Mycielski, J., "On the distances between signals in the non--homogeneous Poisson stochastic process," Studia Mathematica 15 (1956) 300-313.

[148] Neyman, J., "On a new class of 'contagious' distributions, applicable in entomology and bacteriology," The Annals of Mathematical Statistics 10 (1939) 35-57.

[149] Poisson, S. D., Recherches sur la probabilité des jugements en matière criminelle et en matière civile, précédées des règles générales du calcul des probabilités. Paris, 1837.

[150] Poisson, S. D., Lehrbuch der Wahrscheinlichkeitsrechnung und deren wichtigsten Anwendungen. (German translation of the French original published in 1837, Peris) G. C. E. Meyer, Braunschweig, 1841.

[151] Prékopa, A., "On composed Poisson distributions, IV." Acta Mathematica Acad. Sci. Hungaricae 3 (1952) 317-325.

[152] Prékopa, A., "On the compound Poisson distribution," Acta Scientiarum Mathematicarum (Szeged) 18 (1957) 23-28.

[153] Prékopa, A., "On Poisson and composed Poisson stochastic set functions," Studia Mathematica 16 (1957) 142-155.

[154] Prékopa, A., "On secondary processes generated by a random point distribution of Poisson type," Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös nominatae. Sect. Mathematica 1 (1958) 153-170.

[155] Prékopa, A., "On secondary processes generated by random point distributions," Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös nominatae. Sect. Mathematica. 2 (1959) 139-146.

[156] Prékopa, A., "On the spreading process," Transactions of the Second Prague Conference on Information Theory, Statistical Decision Functions, Random Processes (1959), Czechoslovak Academy of Sciences, Prague, 1960, pp. 521-529.

[157] Raikov, D., "On the decomposition of Poisson laws," Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS (N.S.) 14 (1937) 9-11.

[158] Rao, C. R., and H. Rubin "On a characterization of the Poisson distribution," Sankhya: The Indian Journal of Statistics. Ser. A. 26 (1964) 295-298.

[159] Redheffer, R. M., "A note on the Poisson law," Mathematics Magazine 26 (1953) 185-188.

[160] Rényi, A., "On some problems concerning Poisson processes," Publicationes Mathematicae 2 (1951) 66-73.

[161] Rényi, A., "On composed Poisson distributions, II." Acta Mathematica Acad. Sci. Hungaricae 2 (1951) 83-98.

[162] Rényi, A., "A characterization of Poisson processes," (Hungarian) Publications of the Mathematical Institute of the Hungarian Academy of Sciences 1 (1956) 519-527.

[163] Rényi, A., "On an extremal property of the Poisson process," Annals of the Institute of Statistical Mathematics 16 (1964) 129-133.

[164] Rényi, A., "Remarks on the Poisson process," Studia Math. Acad. Sci. Hungar. 2 (1967) 119-123.

[165] Riordan, J., "Moment recurrence relations for binomial, Poisson and hypergeometric frequency distributions," The Annals of Mathematical Statistics 8 (1937) 103-111.

[166] Rutherford, E., and F. Soddy, "The cause and nature of radioactivity.- Part I." Philosophical Magazine (6) 4 (1902) 370-396.

[167] Rutherford, E., and H. Geiger, "The probability variations in the distribution of α particles," Philosophical Magazine (6) (1910) 698-704.

[168] Ryll-Nardzewski, C., "On the non-homogeneous Poisson process (I)," Studia Mathematica 14 (1954) 124-128.

[169] Ryll-Nardzewski, C., "Remarks on the Poisson stochastic process. III" Studia Math. 14 (1954) 314-318.

[170] Ryll-Nardzewski, C., "On the non-homogeneous Poisson processes," Colloquium Math. 3 (1955) 192-195.

[171] Satterthwaite, F. E., "Generalized Poisson distribution," The Annals of Mathematical Statistics 13 (1942) 410-417.

[172] Schottky, W., "Über spontane Stromschwankungen in verschiedenen Elektrizitätsleitern," Annalen der Physik 57 (1918) 541-567.

[173] Shanbhag, D. N., "Another characteristic property of the Poisson distribution," Proceedings of the Cambridge Philosophical Society 68 (1970) 167-169.

[174] Srivastava, R. C., and A. B. I. Srivastava, "On a characterization of Poisson distribution," Journal of Applied Probability 7 (1970) 497-501.

[175] Steinhaus, H., and K. Urbanik, "Poissonsche Folgen," Mathematische Zeitschrift 72 (1959) 127-145.

[176] Szász, D., "Once more on the Poisson process," Studia Sci. Math. Hungarica 5 (1970) 441-444.

[177] Szász, D., and W. A. Woyczyński, "Poissonian random measures and linear processes with independent pieces." Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. 18 (1970) 475-482.

[178] Takács, L., "On secondary processes generated by a Poisson process and their applications in physics," Acta Mathematica Acad. Sci. Hungaricae 5 (1954) 203-236.

[179] Takács, L., "On secondary stochastic processes generated by a multidimensional Poisson process," Magyar Tud. Akad. Mat. Kutató Int. Közl. 2 (1957) 71-80.

[180] Teicher, H., "On the multivariate Poisson distribution," Skandinavisk Aktuarietidskrift 37 (1954) 1-9.

[181] Tulya-Muhika, S., "A characterization of E-processes and Poisson processes in R^n," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 20 (1971) 199-216.

[182] Watanabe, H., "On the Poisson distribution," Journal of the Mathematical Society of Japan 8 (1956) 127-134.

[183] Whittaker, J. M., "The shot effect for showers," Proceedings of the Cambridge Philosophical Society 33 (1937) 451-458.

Recurrent Processes

[184] Belyaev, Yu. K., and V. M. Maksimov, "Analytic properties of a generating function for the number of renewals," Theory of Probability and its Applications 8 (1963) 102-105.

[185] Bingham, N. H., "Limit theorems for regenerative phenomena, recurrent events and renewal theory," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 21 (1972) 20-44.

[186] Bickel, P. J., and J. A. Yahav, "Renewal theory in the plane," The Annals of Mathematical Statistics 36 (1965) 946-955.

[187] Blackwell, D., "A renewal theorem," Duke Mathematical Journal 15 (1948) 145-150.

[188] Blackwell, D., "Extension of a renewal theorem," Pacific Journal of Mathematics 3 (1953) 315-320.

[189] Borovkov, A. A., "Remarks on Wiener's and Balckwell's theorems," Theory of Probability and its Applications 9 (1964) 303-312.

[190] Chow, Y. S., and H. Robbins, "A renewal theorem for random variables which are dependent or non-identically distributed," The Annals of Mathematical Statistics 34 (1963) 390-395.

[191] Chung, K. L., and P. Erdös, "Probability limit theorems assuming only the first moment, I." Four Papers on Probability. Memoirs of the American Mathematical Society. No. 6 (1951) 19 pp.

[192] Chung, K. L., and H. Pollard, "An extension of renewal theory," Proceedings of the American Mathematical Society 3 (1952) 303-309.

[193] Chung, K. L., and J. Wolfowitz, "On a limit theorem in renewal theory," Annals of Mathematics 55 (1952) 1-6.

[194] Choquet, G. et J. Deny, "Sur l'équation de convolution $\mu = \mu * \sigma$," Comptes Rendus Acad. Sci. Paris 250 (1960) 799-801.

[195] Cox, D. R., Renewal Theory. Methuen, London, 1962.

[196] Cox, D. R., and W. L. Smith, "A direct proof of a fundamental theorem of renewal theory," Skandinavisk Aktuarietidskrift 36 (1953) 139-150.

[197] Crump, K. S., "On systems of renewal equations," Jour. Math. Anal. and Appl. 30 (1970) 425-434.

[198] De Bruijn, N. G., and P. Erdös, "On a recursion formula and on some Tauberian theorems," Journal of Research of the National Bureau of Standards 50 (1953) 161-164.

[199] Doob, J. L., "Renewal theory from the point of view of the theory of probability," Transactions of the American Mathematical Society 63 (1948) 422-438.

[200] Dynkin, E. B., "Some limit theorems for sums of independent random variables with infinite mathematical expectations," (Russian) Izvestija Akad. Nauk SSSR. Ser. Mat. 19 (1955) 247-266. [English translation: Selected Translations in Mathematical Statistics and Probability, IMS and AMS, 1 (1961) 171-189.]

[201] Ežov, Ī. Ī. "A generalization of renewal processes," (Russian) Vīsnik Kiīv, Univ. Ser. Mat. Meh. No. 10 (1968) 55-59. [English translation: Selected Translations in Mathematical Statistics and Probability, IMS and AMS, 10 (1972) 246-250.]

[202] Erdös, P., W. Feller and H. Pollard, "A property of power series with positive coefficients," Bull. Amer. Math. Soc. 55 (1949) 201-204.

[203] Erickson, K. B., "Strong renewal theorems with infinite mean," Trans. Amer. Math. Soc. 151 (1970) 263-291.

[204] Erickson, K. B., "A renewal theorem for distributions on R^1 without expectation," Bulletin of the American Mathematical Society 77 (1971) 406-410.

[205] Feller, W., "On the integral equation of renewal theory," The Annals of Mathematical Statistics 12 (1941) 243-267.

[206] Feller, W., "Fluctuation theory of recurrent events," Trans. Amer. Math. Soc. 67 (1949) 98-119.

[207] Feller, W., "A simple proof for renewal theorems," Comm. Pure and Appl. Math. 14 (1961) 285-293.

[208] Feller, W., and S. Orey "A renewal theorem," Journal of Mathematics and Mechanics 10 (1961) 619-624.

[209] Garsia, A., and J. Lamperti, "A discrete renewal theorem with infinite mean," Commentarii Mathematici Helvetici 37 (1962-63) 221-234.

[210] Gelfond, A. O., "An estimate for the remainder term in a limit theorem for recurrent events," Theory of Probability and its Applications 9 (1964) 299-303.

[211] Heyde, C. C., "Some renewal theorems with applications to a first passage problem," The Annals of Mathematical Statistics 37 (1966) 699-710.

[212] Heyde, C. C., "Asymptotic renewal results for a natural generalization of classical renewal theory," Journal of the Royal Statistical Society. Ser. B 29 (1967) 141-150.

[213] Heyde, C. C., "Variations on a renewal theorem of Smith," The Annals of Mathematical Statistics 39 (1968) 155-158.

[214] Hadwiger, H., "Zur Berechnung der Erneuerungsfunktion nach einer Formel von V. A. Kostitzin," Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker. Heft 34 (1937) 37-43.

[215] Hadwiger, H., "Untersuchungen über das asymptotische Verhalten rekurrenter Zahlenreihen," Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker. Heft 35 (1938) 93-109.

[216] Hinderer, K., and H. Walk, "Anwendung von Erneuerungstheoremen und Taubersätzen für eine Verallgemeinerung der Erneuerungsprozesse," Mathematische Zeitschrift 126 (1972) 95-115.

[217] Iosifescu, M., "An extension of the renewal equation," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 23 (1972) 148-152.

[218] Karlin, S., "On the renewal equation," Pacific Jour. Math. 5 (1955) 229-257.

[219] Kingman, J. F. C., "An application of the theory of regenerative phenomena," Proceedings of the Cambridge Philosophical Society 68 (1970) 697-701.

[220] Klimov, G. P., and L. S. Frank, The structure of a stationary renewal process," Theory of Prob. and Appl. 12 (1967) 120-124.

[221] Kolmogorov, A. N., "Markov chains with a countable number of possible states," (Russian) Bulletin de l'Université d'État à Moscou, Sect. A 1 No. 3 (1937) 1-16.

[222] Lamperti, J., "Some limit theorems for stochastic processes," Journal of Mathematics and Mechanics 7 (1958) 433-448.

[223] Lamperti, J., "A contribution to renewal theory," Proceedings of the American Mathematical Society 12 (1961) 724-731.

[224] Lamperti, J., "An invariance principle in renewal theory," The Annals of Mathematical Statistics 33 (1962) 685-696.

[225] Lotka, A., "A contribution to the theory of self-renewing aggregates, with special reference to industrial replacement," The Annals of Mathematical Statistics 10 (1939) 1-25.

[226] Miller, D. R., "Existence of limits in regenerative processes," The Annals of Mathematical Statistics 43 (1972) 1275-1282.

[227] Morimura, H., "A note on sums of independent random variables," Kōdai Mathematical Seminar Reports. (Tokyo) 13 (1961) 255-260.

[228] Mogyoródi, J., "On the rarefaction of recurrent processes," (Hungarian) Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 19 (1970) 25-31. [English translation: Selected Translations in Mathematical Statistics and Probability. IMS and AMS. 10 (1972) 224-231.]

[229] Mogyoródi, J., and T. Szántai, "On thinning of point processes," (Hungarian) Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 20 (1971) 85-95.

[230] Nagaev, S. V., "Some renewal theorems," Theory of Probability and its Applications 13 (1968) 547-563.

[231] Ney, P., and S. Wainger, "The renewal theorem for a random walk in two-dimensional time," Studia Mathematica 44 (1972) 71-85.

[232] Orey, S., "Tail events for sums of independent random variables," Journal of Mathematics and Mechanics 15 (1966) 937-951.

[233] Paley, R. E. A. C., and N. Wiener, "Notes on the theory and application of Fourier transforms. VII. On the Volterra equation. Transactions of the American Mathematical Society 35 (1933) 785-791.

[234] Richter, H., "Untersuchungen zum Erneuerungsproblem," Mathematische Annalean 118 (1941-1943) 145-194.

[235] Rogozin, B. A., "A remark on a theorem due to W. Feller," Theory of Probability and its Applications 14 (1969) 529.

[236] Schwarz, H., "Zur 'wahrscheinlichkeitstheoretischen Stabilisierung' beim Erneuerungsproblem," Mathematische Annalen 118 (1941-1943) 771-779.

[237] Sevastyanov, B. A., "Multidimensional renewal equations and moments of branching processes," Theory of Probability and its Applications 16 (1971) 199-214.

[238] Smith, W. L., "Regenerative stochastic processes," Proceedings of the Royal Society. Ser.A 232 (1955) 6-31.

[239] Smith, W. L., "On renewal theory, counter problems, and quasi-Poisson processes," Proceedings of the Cambridge Philosophical Society 53 (1957) 175-193. [Addendum. Ibid. 54 (1958) 305.]

[240] Smith, W. L., "Renewal theory and its ramifications," Journal of the Royal Statistical Society. Ser. B 20 (1958) 243-302.

[241] Smith, W. L., "On the cumulants of renewal processes," Biometrika 46 (1959) 1-29.

[242] Smith, W. L., "Infinitesimal renewal processes," Contributions to Probability and Statistics. Essays in Honor of Harold Hotelling. Editors: I. Olkin et al., Stanford University Press, 1960, pp. 396-413.

[243] Smith, W. L., "A note on the renewal function when the mean renewal lifetime is infinite," Journal of the Royal Statistical Society. Ser. B 23 (1961) 230-237.

[244] Smith, W. L., "On some general renewal theorems for nonidentically distributed variables," Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probaiblity (1960) Vol. II: Probability Theory, University of California Press, 1961, 467-514.

[245] Smith, W. L., "On necessary and sufficient conditions for the convergence of the renewal density," Transactions of the American Mathematical Society 104 (1962) 79-100.

[246] Smith, W. L., "On the elementary renewal theorem for non-identically distributed variables," Pacific Journal of Mathematics 14 (1964) 673-699.

[247] Smith. W. L., "On the weak law of large numbers and the generalized elementary renewal theorem," Pacific Journal of Mathematics 22 (1967) 171-188.

[248] Smith, W. L., "On infinitely divisible laws and a renewal theorem for non-negative random variables," The Annals of Mathematical Statistics 39 (1968) 139-154.

[249] Smith, W. L., "Some results using general moment functions," Journal of the Australian Mathematical Society 10 (1969) 429-441.

[250] Stam, A. J., "Two theorems in r-dimensional renewal theory," Zeitschr. Wahrscheinlichkeitstheorie 10 (1968) 81-86.

[251] Stam, A. J., "Renewal theory in r dimensions, I." Compositio Mathematica 21 (1969) 383-399.

[252] Stam, A. J., "Renewal theory in r dimensions, II." Compositio Mathematica 23 (1971) 1-13.

[253] Stein, Ch., "A note on cumulative sums," The Annals of Mathematical Statistics 17 (1946) 498-499.

[254] Stone, Ch., "On characteristic functions and renewal theory," Transactions of the American Mathematical Society 120 (1965) 327-342.

[255] Stone, Ch., "On absolutely continuous components and renewal theory," The Annals of Mathematical Statistics 37 (1966) 271-275.

[256] Störmer, H., Semi-Markoff-Prozesse mit endlich vielen Zuständen. Theorie und Anwendungen. Lecture Notes in Operations Research and Mathematical Systems. Vol. 34. Springer, Berlin, 1970.

[257] Szántai, T., "On limiting distributions for the sums of random number of random variables concerning the rarefaction of recurrent process," Studia Scientiarum Mathematicarum Hungarica 6 (1971) 443-452.

[258] Szántai, T., "On an invariance problem related to different rarefactions of recurrent processes," Studia Scientiarum Mathematicarum Hungarica 6 (1971) 453-456.

[259] Taga, Y., "On high order moments of the number of renewals," Annals of the Institute of Statistical Mathematics 15 (1963) 187-196.

[260] Takács, L., "Occurrence and coincidence phenomena in case of happenings with arbitrary distribution law of duration," Acta Mathematica Acad. Sci. Hungaricae 2 (1951) 275-298.

[261] Takács, L., "A new method for discussing recurrent stochastic processes," (Hungarian) Magyar Tudományos Akadémia Alkalmazott Matematikai Intézetének Közleményei 2 (1953) 135-151.

[262] Takács, L., "Some investigations concerning recurrent stochastic processes of a certain type," (Hungarian) Magyar Tudományos Akadémia Alkalmazott Matematikai Intézetének Közleményei 3 (1954) 115-128.

[263] Takács, L., "On a probability problem arising in the theory of counters," Proceedings of the Cambridge Philosophical Society 52 (1956) 488-498.

[264] Takács, L., "On stochastic processes arising in the theory of particle counters," Magyar Tudományos Akadémia III. (Mat. és Fiz.) Oszt. Közl. 6 (1956) 369-421.

[265] Takács, L., "On the sequence of events, selected by a counter from a recurrent process of events," Theory of Probability and its Applications 1 (1956) 81-91.

[266] Takács, L., "On a generalization of the renewal theory," Publications of the Mathematical Institute of the Hungarian Academy of Sciences 2 (1957) 91-103.

[267] Takács, L., "On some probability problems concerning the theory of counters," Acta Mathematica Acad. Sci. Hungaricae 8 (1957) 127-138.

[268] Takács, L., "On a probability problem in the theory of counters," The Annals of Mathematical Statistics 29 (1958) 1257-1263.

[269] Takács, L., "Comments on W. L. Smith's paper," Journal of the Royal Statistical Society. Ser. B. 20 (1958) 296-298.

[270] Takács, L., "On a coincidence problem concerning particle counters," The Annals of Mathematical Statistics 32 (1961) 739-756.

[271] Täcklind, S., "Elementare Behandlung vom Erneuerungsproblem für den stationären Fall," Skandinavisk Aktuarietidskrift 27 (1944) 1-15.

[272] Täcklind, S., "Fourieranalytische Behandlung vom Erneuerungsproblem," Skandinavisk Aktuarietidskrift 28 (1945) 69-105.

[273] Teugels, J. L., "Renewal theorems when the first or the second moment is infinite," The Annals of Mathematical Statistics 39 (1968) 1210-1219.

[274] Teugels, J. L., "Regular variation of Markov renewal functions," Journal of the London Mathematical Society (2) 2 (1970) 179-190.

[275] Williamson, J. A., "Some renewal theorems for non-negative independent random variables," Transactions of the American Mathematical Society 114 (1965) 417-445.

[276] Williamson, J. A., "A relation between a class of limit laws and a renewal theorem," Illinois Journal of Mathematics 10 (1966) 210-219.

Point Processes.

[277] Bartlett, M. S., "The spectral analysis of point processes," Journal of the Royal Statistical Society. Ser. B 25 (1963) 264-296.

[278] Belyaev, Yu. K., "Limit theorems for dissipative flows," Theory of Probability and its Applications 8 (1963) 165-173.

[279] Belyaev, Yu. K., "Random flows generated by random processes (properties of random flows, level-crossing problems)," (Russian) Matem. Zametki 8 (1970) 393-407. [English translation: Mathematical Notes of the Academy of Sciences of the USSR 8 (1970) 691-698.]

[280] Belyayev, Yu. K., "Point processes and first passage problems," Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. Vol. III. Probability Theory. University of California Press, 1972, pp. 1-17.

[281] Belyaev, Yu. K., and V. I. Piterbarg, "Generalized flows with finite memory," Theory of Probability and its Applications 15 (1970) 204-214.

[282] Beutler, F. J., and O. A. Z. Leneman, "The theory of stationary point processes," Acta Mathematica 116 (1966) 159-197.

[283] Breiman, L., "The Poisson tendency in traffic distribution," The Annals of Mathematical Statistics 34 (1963) 308-311.

[284] Chung, K. L., "Crudely stationary counting processes," The American Mathematical Monthly 79 (1972) 867-877.

[285] Cox, D. R., and P. A. W. Lewis, "Multivariate point processes," Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. Vol. III. Probability Theory. University of California Press, 1972, pp. 401-448.

[286] Cox, D. R., and W. L. Smith, "The superposition of several strictly periodic sequences of events," Biometrika 40 (1953) 1-11.

[287] Cox, D. R., and W. L. Smith, "On the superposition of renewal processes," Biometrika 41 (1954) 91-99.

[288] Cramér, H., M. R. Leadbetter, and R. J. Serfling, "On distribution function-moment relationships in a stationary point process," Zeitschrift für Wahrscheinlichkeitstheorie 18 (1971) 1-8.

[289] Daley, D. J., "Asymptotic properties of stationary point processes with generalized clusters," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 21 (1972) 65-76.

[290] Ebe, M., "On sets of motion of point processes," Jour. Operational Res. Society of Japan 11 (1969) 97-113.

[291] Fürth, R., Schwankungserscheinungen in der Physik. Sammlung Vieweg, Braunschweig, 1920.

[292] Gilles, D. C., and P. A. W. Lewis, "The spectrum of intervals of a geometric branching Poisson process," Journal of Applied Probability 4 (1967) 201-205.

[293] Goldman, J., "Stochastic point processes: Limit theorems." The Annals of Mathematical Statistics 38 (1967) 771-779.

[294] Gnedenko, B. V., and B. Freier, "Einige Bemerkungen zu einer Arbeit von I. Kowalenko," (Russian) Litovsk. Mat. Sbornik 9 (1969) 463-470.

[295] Kerstan, J., and K. Matthes, "Ergodische unbegrenzt teilbare stationäre zufällige Punktfolgen," Transactions of the Fourth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, 1965. Czechoslovak Academy of Sciences, Prague, 1967, pp. 399-415.

[296] Khinchin, A. Ya., "Sequences of chance events without after-effects," Theory of Probability and its Applications 1 (1956) 1-15.

[297] Kingman, J. F. C., "The stochastic theory of regenerative events," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 2 (1964) 180-224.

[298] Kingman, J. F. C., "Some further analytical results in the theory of regenerative events," Journal of Mathematical Analysis and Applications 11 (1965) 422-433.

[299] Kingman, J. F. C., Regenerative Phenomena. John Wiley and Sons, New York, 1972.

[300] Lawrance, A. J., "Arbitrary event initial conditions for branching Poisson processes," Journal of the Royal Statistical Society. Ser. B 34 (1972) 114-123.

[301] Leadbetter, M. R., "On basic results of point process theory," Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. Vol. III. Probability Theory. University of California Press, 1972, pp. 449-462.

[302] Lewis, P. A. W., "Asymptotic properties and equilibrium conditions for branching Poisson processes," Journal of Applied Probability 6 (1969) 355-371.

[303] Lewis, P. A. W., (Editor) Stochastic Point Processes: Statistical Analysis, Theory and Applications. Wiley-Interscience, New York, 1972.

[304] Lundberg, O., On Random Processes and their Application to Sickness and Accident Statistics. Dissertation. Uppsala, 1940.

[305] McFadden, J. A., "On the lengths of intervals in a stationary point process," Journal of the Royal Statistical Society. Ser. B 24 (1962) 364-382.

[306] McFadden, J. A., and W. Weissblum, "Higher order properties of a stationary point process," Journal of the Royal Statistical Society. Ser. B. 25 (1963) 413-431.

[307] Mori, T., "On random translations of point processes," The Yokohama Mathematical Journal 19 (1971) 119-139.

[308] Moyal, J. E., "The general theory of stochastic population processes," Acta Mathematica 108 (1962) 1-31.

[309] Nawrotzki, K., "Eine Monotonieeigenschaft zufälliger Punktfolgen," Mathematische Nachrichten 24 (1962) 193-200.

[310] Nawrotzki, K., "Ein Grenzwertsatz für homogene zufällige Punktfolgen (Verallgemeinerung eines Satzes van A. Rényi), Mathematische Nachrichten 24 (1962) 201-217.

[311] Ososkov, G. A., "A limit theorem for flows of similar events," Theory of Probability and its Applications 1 (1956) 248-255.

[312] Palm, C., "Intensitätschwankungen im Fernsprechverkehr," Ericsson Technics No. 44 (1943) 1-189.

[313] Ryll-Nardzewski, C., "Remarks on processes of calls," Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (1960). University of California Press, Vol. 2 (1961) 455-465.

[314] Slivnyak, I. M., "Some properties of stationary flows of homogeneous random events," Theory of Probability and its Applications 7 (1962) 336-341.

[315] Stone, Ch., "On a theorem by Dobrushin," The Annals of Mathematical Statistics 39 (1968) 1391-1401.

[316] Thedéen, T., "A note on the Poisson tendency in traffic distribution," The Annals of Mathematical Statistics 35 (1964) 1823-1824.

[317] Vere-Jones, D., "Stochastic models for earthquake occurrence," Journal of the Royal Statistical Society. Ser. B 32 (1970) 1-62.

[318] Whitt, W., "Limits for the superposition of m-dimensional point processes," Journal of Applied Probability 9 (1972) 462-465.

Brownian Motion Process

[319] Ambartsumian, R. V., "On an application of the connection between Brownian motion and the Dirichlet problem," Theory of Probability and its Applications 10 (1965) 490-493.

[320] Anderson, T. W., "A modification of the sequential probability ratio test to reduce the sample size," The Annals of Mathematical Statistics 31 (1960) 165-197.

[321] Bachelier, L., "Théorie de la spéculation," Ann. Sci. École Norm. Sup. 17 (1900) 21-86. [English translation: The Random Character of Stock Market Process. Editor: P. H. Cootner. M.I.T. Pres, Cambridge, Massachusetts, 1964, pp. 17-78.]

[322] Clark, J. M. C., "The representation of functionals of Brownian motion by stochastic integrals," The Annals of Mathematical Statistics 41 (1970) 1282-1295.

[323] Cameron, R. H., and W. T. Martin, "Evaluation of various Wiener integrals by use of certain Sturm-Liouville differential equations," Bulletin of the American Mathematical Society 51 (1945) 73-90.

[324] Cherkasov, I. D., "On the transformation of the diffusion process to a Wiener process," Theory of Probability and its Applications 2 (1957) 373-377.

[325] De Moivre, A., "Approximatio ad summam terminorum binomii $(a+b)^n$ in seriem expansi," November 12, 1733 pp. 1-7.

[326] Dinges, H., "Einige Verteilungen, die mit dem Wienerprozess zusammenhängen," Transactions of the Third Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, 1962. Czechoslovak Academy of Sciences, Prague, 1964, pp. 75-83.

[327] Doob, J. L., "The Brownian movement and stochastic equations," Annals of Mathematics 43 (1942) 351-369. (See also in N. Wax [369] pp. 319-337.)

[328] Doob, J. L., "Heuristic approach to the Kolmogorov-Smirnov theorems," The Annals of Mathematical Statistics 20 (1949) 393-403.

[329] Doob, J. L., "A probability approach to the heat equation," Transactions of the American Mathematical Society 80 (1955) 216-280.

[330] Dvoretzky, A., P. Erdös, and S. Kakutani, "Nonincrease everywhere of the Brownian motion process," Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (1960) Vol. II: Probability Theory, University of California Press, 1961, pp. 103-116.

[331] Fortet, R., "Quelques travaux récents sur le mouvement brownien," Annales de l'Institut Henri Poincaré (Paris) 11 (1949) 175-226.

[332] Földes, A., "A construction of Brownian motion process in r-dimension," Studia Scientiarum Mathematicarum Hungarica 6 (1971) 375-380.

[333] Frank, Ph., and R. v. Mises, Die Differential- und Integralgleichungen der Mechanik und Physik. Second edition. Vol. I (1930). Vol. II. (1935). Friedr. Vieweg und Sohn, Braunschweig.

[334] Freedman, D., Brownian Motion and Diffusion. Holden-Day, San Francisco, 1971.

[335] Fürth, R., "Wärmeleitung und Diffusion," Die Differential- und Integralgleichungen der Mechanik und Physik von Ph. Frank und R. v. Mises. Vol. II (1935) 526-626. Friedr. Vieweg und Sohn, Braunschweig.

[336] Gauss, C. F., Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium. Hamburg, 1809. [English translation: Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections. Dover, New York, 1963.]

[337] Haas-Lorentz, G. L. de, Die Brownsche Bewegung und einige verwandte Erscheinungen. Braunschweig, 1913.

[338] Hostinský, B., "Application du calcul des probabilités à la théorie du mouvement Brownien," Annales de l'Institut H. Poincaré 3 (1932) 1-74.

[339] Hunt, G. A., "Some theorems concerning Brownian motion," Transactions of the American Mathematical Society 8 (1956) 294-319.

[340] Ilin, A. M., and R. Z. Khasminskii, "On equations of Brownian motion," Theory of Probability and its Applications 9 (1964) 421-444.

[341] Itô, K., and H. P. McKean, Jr., "Brownian motions on a half line," Illinois Journal of Mathematics 7 (1963) 181-231.

[342] Itô, K., and H. P. McKean, Diffusion Processes and their Sample Paths. Springer-Verlag, Berlin, 1965

[343] Kac, M., "Random walk and the theory of Brownian motion," The American Mathematical Monthly 54 (1947) 369-391. [Reprinted in Selected Papers on Noise and Stochastic Processes. Edited by N. Wax. Dover, New York, 1954, pp. 295-317.]

[344] Kac, M., and A. J. F. Siegert, "On the theory of noise in radio receivers with square law detectors," Journal of Applied Physics 18 (1947) 383-397.

[345] Kaufman, R., "Measures of Hausdorff-type, and Brownian motion," Mathematika 19 (1972) 115-119.

[346] Kiefer, J., "A functional equation technique for obtaining Wiener process probabilities associated with theorems of Kolmogorov-Smirnov type," Proceedings of the Cambridge Philosophical Society 55 (1959) 328-332.

[347] Kinney, J. R., "The convex hull of plane Brownian motion," The Annals of Mathematical Statistics 34 (1963) 327-329.

[348] Knight, F. B., "On the random walk and Brownian motion," Trans. Amer. Math. Soc. 103 (1962) 218-228.

[349] Kuo, H. H., "Stochastic integrals in abstract Wiener Space," Pacific Journal of Mathematics 41 (1972) 469-483.

[350] Kuo, H. H., "Diffusion and Brownian motion on infinite-dimensional manifolds," Transactions of the American Mathematical Society 169 (1972) 439-459.

[351] Laplace, P. S., "Mémoire sur les approximations des formules qui sont fonctions de très grands nombres," Mémoires de l'Académie Royale des Sciences de Paris, année 1782 (1785) 1-88. [Oeuvres complètes de Laplace 10 (1894) 209-291.]

[352] Lévy, P., Processus Stochastiques et Mouvement Brownien. First edition 1948. Second edition 1965. Gauthier-Villars, Paris.

[353] Lévy, P., "Wiener's random function, and other Laplacian random functions," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (1950). University of California Press, Berkeley, 1951, pp. 171-187.

[354] Lévy, P., "Random functions: General theory with special reference to Laplacian random functions, University of California Publication in Statistics 1 No. 12 (1953) 331-390.

[355] Lévy, P., Le Mouvement Brownien. Mémorial des Sciences Mathématiques. Fasc. 126. Gauthier-Villars, Paris, 1954.

[356] Lukács, E., "Some results in the theory of Wiener integrals," Transactions of the Fourth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, 1965. Czechoslovak Academy of Sciences, Prague, 1967, pp. 29-43.

[357] Lüneburg, R., "Das Problem der Irrfahrt ohne Richtungsbeschränkung und die Randwertaufgabe der Potentialtheorie," Mathematische Annalen 104 (1931) 700-738.

[358] Monroe, I., "On embedding right continuous martingales in Brownian motion," The Annals of Mathematical Statistics 43 (1972) 1293-1311.

[359] Novikov, A. A., "On stopping times for a Wiener process," Theory of Probability and its Applications 16 (1971) 449-456.

[360] Park, Ch., "On Fredholm transformations in Yeh-Wiener space," Pacific Journal of Mathematics 40 (1972) 173-195.

[361] Petrowsky, I., "Über das Irrfahrtproblem," Mathematische Annalen 109 (1934) 425-444.

[362] Petrovskii, I. G., "Zur ersten Randwertaufgabe der Wärmeleitungsgleichung," Compositio Mathematica 1 (1935) 383-419.

[363] Silvestrov, D. S., "Limit theorems for functionals of integral type on diffusion processes," (Russian) Doklady Akad. Nauk SSSR 200 (1971) 545-547. [English translation: Soviet Mathematics-Doklady 12 (1971) 1450-1453.]

[364] Skitovich, V. P., "On a characterization of Brownian motion," Theory of Probability and its Applications 1 (1956) 326-328.

[365] Skorokhod, A. V., and N. P. Slobodenjuk, "On the asymptotic behavior of certain functionals of a Brownian motion process," (Russian) Ukrain. Mat. Zur. 18 No. 4 (1966) 60-71. [English translation: Selected Translations in Mathematical Statistics and Probability, IMS and AMS, 9 (1971) 205-217.]

[366] Strait, P. T., "A note on Lévy's Brownian process on the Hilbert sphere," Proceedings of the American Mathematical Society 33 (1972) 207-209.

[367] Stratonovič, R. L., "On the probability functional of diffusion processes," (Russian) Proc. Sixth All-Union Conf. Theory Probability and Math. Statist. (Vilniüs, 1960) Gosudarstv. Izdat. Političesk. i Naučn. Lit. Litovsk. SSR, Vilnius, 1962, pp. 471-482. [English translation: Selected Translations in Mathematical Statistics and Probability, IMS and AMS, 10 (1972) 273-286.]

[368] Sternberg, W., "Über die Gleichung der Wärmeleitung," Mathematische Annalen 101 (1929) 394-398.

[369] Wax, N., (Editor) Selected Papers on Noise and Stochastic Processes. Dover, New York, 1954.

[370] Wiener, N., "Differential-space," Jour. Math. Phys. M.I.T. 2 (1923) 131-174. [Reprinted in Selected Papers of Norbert Wiener, SIAM and M.I.T., 1964, pp. 55-98.]

[371] Williams, D., "Decomposing the Brownian path," Bull. Amer. Math. Soc. 76 (1970) 871-873.

Gaussian Processes.

[372] Baxter, G., "A strong limit theorem for Gaussian processes," Proceedings of the American Mathematical Society 7 (1956) 522-527.

[373] Belayev, Yu. K., "Local properties of the sample functions of stationary Gaussian processes," Theory of Probability and its Applications 5 (1960) 117-120.

[374] Belayev, Yu. K., "Continuity and Hölder's conditions for sample functions of stationary Gaussian processes," Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (1960) Vol. II: Probability Theory, University of California Press, 1961, pp. 23-33.

[375] Beljaev, Ju. K., and V. I. Piterbarg, "Asymptotics of the average number of A-points of overshoot of a Gaussian field beyond a high level," (Russian) Doklady Akademii Nauk SSSR 203 (1972) 9-12. [English translation: Soviet Mathematics-Doklady 13 (1972) 309-313.]

[376] Berman, S. M., "Gaussian processes with stationary increments: Local times and sample function properties," The Annals of Mathematical Statistics 41 (1970) 1260-1272.

[377] Berman, S. M., "A class of limiting distributions of high level excursions of Gaussian processes," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 21 (1972) 121-134.

[378] Berman, S. M., "Gaussian sample functions: uniform dimension and Hölder conditions nowhere," Nagoya Mathematical Journal 46 (1972) 63-86.

[379] Berman, S. M., "Maximum and high level excursion of a Gaussian process with stationary increments," The Annals of Mathematical Statistics 43 (1972) 1247-1266.

[380] Bulinskaya, E. V., "On the mean number of crossings of a level by a stationary Gaussian process," Theory of Probability and its Applications 6 (1961) 435-438.

[381] Cameron, R. H., and W. T. Martin, "The behavior of measure and measurability under change of scale in Wiener space," Bulletin of the American Mathematical Society 53 (1947) 130-137.

[382] Dudley, R. M., "Gaussian processes on several parameters," The Annals of Mathematical Statistics 36 (1965) 771-788.

[383] Doob, J. L., "The elementary Gaussian processes," Annals of Math. Statist. 15 (1944) 229-282.

[384] Garsia, A. M., E. Rodemich and H. Rumsey, Jr., "A real variable lemma and the continuity of paths of some Gaussian processes," Indiana University Mathematics Journal 20 (1970) 565-578.

[385] Geman, D., "On the variance of the number of zeros of a stationary Gaussian process," The Annals of Mathematical Statistics 43 (1972) 977-982.

[386] Gladyshev, E. G., "A new limit theorem for stochastic processes with Gaussian increments," Theory of Probability and its Applications 6 (1961) 52-61.

[387] Itô, K., "The expected number of zeros of continuous stationary Gaussian processes," Journal of Mathematics of Kyoto University 3 (1964) 207-216.

[388] Ivanov, V. A., "On the average number of crossings of a level by sample functions of a stochastic process," Theory of Probability 5 (1960) 319-323.

[389] Kac, M., and D. Slepian, "Large excursions of Gaussian processes," The Annals of Mathematical Statistics 30 (1959) 1215-1228.

[390] Leadbetter, M. R., "On crossings of arbitrary curves by certain Gaussian processes," Proceedings of the American Mathematical Society 16 (1965) 60-68.

[391] Mirošin, R. N., "on the finiteness of the moments of the number of zeros of a differentiable Gaussian stationary process," (Russian) Doklady Akad. Nauk SSSR 200 (1971) 32-34. [English translation: Soviet Mathematics-Doklady 12 (1971) 1321-1324.]

[392] Park, W. J., "On the equivalence of Gaussian processes with factorable covariance functions," Proceedings of the American Mathematical Society 32 (1972) 275-279.

[393] Pitcher, T. S., "On the sample functions of processes which can be added to a Gaussian process," The Annals of Mathematical Statistics 34 (1963) 329-333.

[394] Piterbarg, V. I., "On existence of moments for the number of crossings of a given level by a Gaussian stationary process," Doklady Akad. Nauk SSSR 182 (1968) 46-48. [English translation: Soviet Mathematics-Doklady 9 (1968) 1109-1111.]

[395] Preston, Ch., "Continuity properties of some Gaussian processes," The Annals of Mathematical Statistics 43 (1972) 285-292.

[396] Rice, S. O., "Mathematical analysis of random noise I," Bell System Technical Journal 23 (1944) 282-332; II. Ibid 24 (1945) 46-156. [Reprinted in Selected Papers on Noise and Stochastic Processes. Edited by N. Wax. Dover, New York, 1954, pp. 133-294.]

[397] Rozanov, Yu. A., "On probability measures in functional spaces corresponding to stationary Gaussian processes," Theory of Probability and its Applications 9 (1964) 404-420.

[398] Slepian, D., "Some comments on the detection of Gaussian signals in Gaussian noise," IRE Transactions on Information Theory. Vol. IT 4 (1958) 65-68.

[399] Slepian, D., "The one-sided barrier problem for Gaussian noise," Bell System Technical Journal 41 (1962) 463-501.

[400] Slepian, D., "On the zeros of Gaussian noise," Proceedings of the Symposium on Time Series Analysis. Brown University, June 11-14, 1962. Ed. M. Rosenblatt. John Wiley and Sons, New York, 1963, pp. 104-115.

[401] Skorokhod, A. V., "A note on Gaussian measures in a Banach space," Theory of Probability and its Applications 15 (1970) 508.

[402] Strait, P. T., "Sample function regularity for Gaussian processes with the parameter in a Hilbert space," Pacific Journal of Mathematics 19 (1966) 159-173.

[403] Wong, E., "Some results concerning the zero-crossings of Gaussian noise," SIAM Journal of Applied Mathematics 14 (1966) 1246-1254.

[404] Wong, E., "The distribution of intervals between zeros for a stationary Gaussian process," SIAM Jour. Appl. Math. 18 (1970) 67-73.

[405] Ylvisaker, N. D., "The expected number of zeros of a stationary Gaussian process," The Annals of Mathematical Statistics 36 (1965) 1043-1046.

[406] Ylvisaker, D., "A note on the absence of tangencies in Gaussian sample paths," The Annals of Mathematical Statistics 39 (1968) 261-262.

[407] Zimmerman, G. J., "Some sample function properties of the two-parameter Gaussian process," The Annals of Mathematical Statistics 43 (1972) 1235-1246.

Processes with Independent Increments.

[408] Baxter, G., "On the measure of Hilbert neighborhoods for processes with stationary, independent increments," Proc. Amer. Math. Soc. 10 (1959) 690-695.

[409] Blumenthal, R. M., and R. K. Getoor, "Sample functions of stochastic processes with stationary independent increments," Journal of Mathematics and Mechanics 10 (1961) 493-516.

[410] Bretagnolle, J., and D. Dacunha-Castelle, "Récurrence ponctuelle des processus à accroissements indépendants," Comptes Rendus Acad. Sci. Paris 261 (1965) 4604-4606.

[411] Cogburn, R., and H. G. Tucker, "A limit theorem for a function of the increments of a decomposable process," Transactions of the American Mathematical Society 99 (1961) 278-284.

[412] De Finetti, B., "Sulle funzioni a incremento aleatorio," Atti R, Accad. Naz. Lincei, Rendiconti Cl. Fis. Mat. e Nat. Sixth Ser. 10 (1929) 163-168.

[413] De Finetti, B., "Sulla possibilità di valori eccezionali una legge di incrementi aleatori," Atti R. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 10 (1929) 325-329.

[414] De Finetti, B., "Integrazione delle funzioni a incremento aleatorio," Atti R. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 10 (1929) 548-553.

[415] De Finetti, B., "Le funzioni caratteristiche di legge istantanea," Atti R. Accademia Nazionale dei Lincei. Rendiconti. Classe di Science Fisiche, Matematiche, e Naturali. Sixth Ser. 12 (1930) 278-282.

[416] De Finetti, B., "Le funzioni caratteristiche di legge istantanea dotate di valori eccezionali," Atti R. Accad. Naz. Lincei. Rendiconti. Classe di Science Fisiche, Matematiche e Naturali. Sixth Ser. 14 (1931) 259-265.

[417] Ferguson, Th. S., and M. J. Klass, "A representation of independent increment processes without Gaussian components," The Annals of Mathematical Statistics 43 (1972) 1634-1643.

[418] Fisz, M., "Infinitely divisible distributions: Recent results and applications." The Annals of Mathematical Statistics 33 (1962) 68-84.

[419] Fisz, M., "On the orthogonality of measures induced by L-processes," Transactions of the American Mathematical Society 106 (1963) 185-192.

[420] Fristedt, B. E., "Sample function behavior of increasing processes with stationary, independent increments," Pacific Journal of Mathematics 21 (1967) 21-33.

[421] Fristedt, B., "Variation of symmetric, one-dimensional stochastic processes with stationary, independent increments," Illinois Journal of Mathematics 13 (1969) 717-721.

[422] Fristedt, B., "Upper functions for symmetric processes with stationary, independent increments," Indiana University Mathematics Journal 21 (1971) 177-185.

[423] Gnedenko, B. V., "Sur la croissance des processus stochastiques homogènes à accroissements indépendants," (Russian) Izvestiya Akad. Nauk SSSR Ser. Mat. (Bulletin de l'Académie des Sciences de l'URSS. Sér. Math.) 7 (1943) 89-110.

[424] Goldman, J. R., "Infinitely divisible point processes in R^n," Journal of Mathematical Analysis and Applications 17 (1967) 133-146.

[425] Greenwood, P., and B. Fristedt, "Variations of processes with stationary, independent increments," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 23 (1972) 171-186.

[426] Huff, B. W., "Comments on the continuity of distribution functions obtained by superposition," Proceedings of the American Mathematical Society 27 (1971) 141-146.

[427] Huff, B. W., "Further comments on the continuity of distribution functions obtained by superposition," Proceedings of the American Mathematical Society 35 (1972) 561-564.

[428] Khintchine, A. Ya., "Sur la croissance locale des processus stochastiques homogènes à accroissements indépendants," Izvestiya Akad. Nauk SSSR Ser. Mat. (Bulletin de l'Académie des Sciences de l'URSS. Sér. Math.) 3 (1939) 487-508.

[429] Kesten, H., "A convolution equation and hitting probabilities of single points for processes with stationary increments," Bulletin of the American Mathematical Society 75 (1969) 573-578.

[430] Kesten, H., "Hitting probabilities of single points for processes with stationary independent increments," Memoirs of the American Mathematical Society. No. 93 (1969) 1-129.

[431] Kingman, J. F. C., "Recurrence properties of processes with stationary independent increments," Journal of the Australian Mathematical Society 4 (1964) 223-228.

[432] Kolmogoroff, A., "Sulla forma generale di un processo stocastico omogeneo. (Un problema di Bruno de Finetti)," Atti R. Acad. Naz. Lincei Rend. Cl. Fis. Mat. Nat. Sixth Ser. 15 (1932) 805-808.

[433] Kolmogoroff, A., "Ancore sulla forma generale di un processo stocastico omogeneo," Atti R. Accad. Naz. Lincei Rend. Cl. Fis. Mat. Nat. Sixth Ser. 15 (1932) 866-869.

[434] Kozin, F., "A limit theorem for processes with stationary independent increments," Proceedings of the American Mathematical Society 8 (1957) 960-963,

[435] Lévy, P., "Sur les intégrales dont les éléments sont des variables aléatoires indépendantes," Annali della R. Scuola Normale Superiore di Pisa. Sci. Fis. e Mat. (2) 3 (1934) 337-366 . (See also P. Lévy [436].)

[436] Lévy, P., "Observation sur un précédent mémoire de l'auteur," Annali della R. Scuola Normale Superiore di Pisa. Sci. Fis. e Mat. (2) 4 (1935) 217-218. (See also P. Lévy [435].)

[437] Lévy, P., "Sur certains processus stochastiques homogènes," Compositio Mathematica 7 (1940) 283-339.

[438] Lee, P. M., "Infinitely divisible stochastic processes," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 7 (1967) 147-160.

[439] Lee, P. M., "Some examples of infinitely divisible point processes," Studia Sci. Math. Hungar. 3 (1968) 219-224.

[440] Maksimov, V. M., "Random processes with independent increments with values in an arbitrary finite group," Theory of Probability and its Applications 15 (1970) 215-228.

[441] Monroe, I., "On the γ-variation of processes with stationary independent increments," The Annals of Mathematical Statistics 43 (1972) 1213-1220.

[442] Maruyama, G., "Infinitely divisible processes," Theory of Probability and its Applications 15 (1970) 1-22.

[443] Mercer, A., and C. S. Smith, "A random walk in which the steps occur randomly in time," Biometrika 46 (1959) 32-35.

[444] Rogozin, B. A., "On some classes of processes with independent increments," Theory of Probability and its Applications 10 (1965) 479-483.

[445] Shtatland, E. S., "On local properties of processes with independent increments," Theory of Probability and its Applications 10 (1965) 317-322.

[446] Skorokhod, A. V., Random Processes with Independent Increments. (Russian) Izdat. "Nauka", Moscow, 1964.

[447] Skorokhod, A. V., "Absolute continuity of infinitely divisible distributions under translations," Theory of Probability and its Applications 10 (1965) 465-472.

[448] Stratton, H. H., Jr., "On dimension of support for stochastic processes with independent increments," Transactions of the American Mathematical Society 132 (1968) 1-29.

[449] Wasan, M. T., "On an inverse Gaussian process," Skandinavisk Aktuarietidskrift 51 (1968) 69-96.

[450] Watanabe, T., "Some potential theory of processes with stationary independent increments by means of the Schwartz distribution theory," Journal of the Mathematical Society of Japan 24 (1972) 213-231.

[451] Zolotarev, V. M., "Distribution of the superposition of infinitely divisible processes," Theory of Probability and its Applications 3 (1958) 185-188.

Stable Processes.

[452] Blumenthal, R. M., and R. K. Getoor, "Some theorems on stable processes," Transactions of the American Mathematical Society 95 (1960) 263-273.

[453] Blumenthal, R. M., and R. K., Getoor, "The dimension of the set of zeros and the graph of a symmetric stable process," Illinois Journal of Mathematics 6 (1962) 308-316.

[454] Breiman, L., "A delicate law of the iterated logarithm for non-decreasing stable processes," The Annals of Mathematical Statistics 39 (1968) 1818-1824. [Correction: The Annals of Mathematical Statistics 41 (1970) 1126.]

[455] Elliot, J., "Absorbing barrier processes connected with the symmetric stable densities," Illinois Journal of Mathematics 3 (1959) 200-216.

[456] Getoor, R. K., "The asymptotic distribution of the number of zero free intervals of a stable process," Transactions of the American Mathematical Society 106 (1963) 127-138.

[457] Greenwood, P. E., "The variation of a stable path is stable," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 14 (1969) 140-148.

[458] Hawkes, J., "Polar sets, regular points and recurrent sets for the symmetric and increasing stable processes," Bulletin of the London Mathematical Society 2 (1970) 53-59.

[459] Hawkes, J., "A lower Lipschitz condition for the stable subordinator," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 17 (1971) 23-32.

[460] Hawkes, J., "Some dimension theorems for the sample functions of stable processes," Indiana University Mathematics Journal 20 (1971) 733-738.

[461] Jain, N., and W. E. Pruitt, "The correct measure function for the graph of a transient stable process," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 9 (1968) 131-138.

[462] Jain, N., and W. E. Pruitt, "Collisions of stable processes," Illinois Journal of Mathematics 13 (1969) 241-248.

[463] Kac, M., "Some remarks on stable processes," Publications de l'Institut de Statistique de l'Université de Paris 6 (1957) 303-306.

[464] Kantner, M., "On the spectral representation for symmetric stable random variables," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 23 (1972) 1-6.

[465] Khintchine, A., "Zwei Sätze über stochastische Prozesse mit stabilen Verteilungen," Matematicheskii Sbornik (Recueil Mathématique) N.S. 3 (1938) 577-584.

[466] Lamperti, J., "Semi-stable stochastic processes," Transactions of the American Mathematical Society 104 (1962) 62-78.

[467] Lukács, E., "A characterization of stable processes," Journal of Applied Probability 6 (1969) 409-418.

[468] McKean, H. P., Jr., "Sample functions of stable processes," Annals of Mathematics 61 (1955) 564-579.

[469] Molchanov, S. A., and E. Ostrovskii, "Symmetric stable processes as traces of degenerate diffusion processes," Theory of Probability and its Applications 14 (1969) 128-131.

[470] Port, S. C., "On hitting places for stable processes," The Annals of Mathematical Statistics 38 (1967) 1021-1026.

[471] Pruitt, W. E., and S. J. Taylor, "The potential kernel and hitting probabilities for the general stable process in R^N," Transactions of the American Mathematical Society 146 (1969) 299-321.

[472] Pruitt, W. E., and S. J. Taylor, "Sample path properties of processes with stable components," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 12 (1969) 267-289.

[473] Port, S. C., and C. J. Stone, "Stopping times for recurrent stable processes," Duke Math. Jour. 35 (1968) 663-670.

[474] Stone, Ch., "The set of zeros of a semistable process," Illinois Journal of Mathematics 7 (1963) 631-637.

[475] Takeuchi, J., "A local asymptotic law for the transient stable process," Proceedings of the Japan Academy 40 (1964) 141-144.

[476] Taylor, S. J., "Multiple points for the sample paths of the symmetric stable process," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 5 (1966) 247-267.

[477] Taylor, S. J., "Sample path properties of a transient stable process," Journal of Mathematics and Mechanics 16 (1967) 1229-1246.

[478] Taylor, S. J., and J. G. Wendel, "The exact Hausdorff measure of the zero set of a stable process," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 6 (1966) 170-180.

[479] Widom, H., "Stable processes and integral equations," Transactions of the American Mathematical Society 98 (1961) 430-449.

[480] Zolotarev, V. M., "Analogue of the iterated logarithm law for semi-continuous stable processes," Theory of Probability and its Applications 9 (1964) 512-513.

Convergence of Stochastic Processes.

[481] Bachelier, L., "Théorie de la spéculation," Annales Scientifiques de l'École Normale Supérieure 17 (1900) 21-86. [English translation: The Random Character of Stock Market Process. Editor P. H. Cootner. M.I.T. Press, Cambridge, Massachusetts, 1964, pp. 17-78.]

[482] Bartoszyński, R., "A characterization of the weak convergence of measures," The Annals of Mathematical Statistics 32 (1961) 561-576.

[483] Billingsley, P., "The invariance principle for dependent random variables," Transactions of the American Mathematical Society 83 (1956) 250-268.

[484] Billingsley, P., "Limit theorems for randomly selected partial sums," The Annals of Mathematical Statistics 33 (1962) 85-92.

[485] Billingsley, P., Convergence of Probability Measures. John Wiley and Sons, New York, 1968.

[486] Billingsley, P., Weak Convergence of Measures: Applications in Probability. Regional Conference Series in Applied Mathematics. SIAM, 1971.

[487] Borovkov, A. A. "Convergence of weakly dependent processes to the Wiener process," Theory of Probability and its Applications 12 (1967) 159-186.

[488] Borovkov, A. A., "On the convergence to diffusion processes," Theory of Probability and its Applications 12 (1967) 405-431.

[489] Borovkov, A. A., "On three types of conditions for convergence to diffusion processes," (Russian) Doklady Akademii Nauk SSSR 187 (1969) 974-977. [English translation: Soviet Mathematics-Doklady 10 (1969) 960-963.]

[490] Borovkov, A. A., "Theorems on the convergence to Markov diffusion processes," Zeitschr. Wahrscheinlichkeitstheorie 16 (1970) 47-76.

[491] Brovkov, A. A., "The convergence of distributions of functionals on stochastic processes," (Russian) Uspehi Mat. Nauk 27 (1972) 1-41. [English translation: Russian Mathematical Surveys 27 (1972) 1-42.]

[492] Cheng, Tseng-tung, "Weak convergence of measures in metric spaces," (Chinese) Acta Mathematica Sinica 15 (1965) 153-158. [English translation: Chinese Mathematics 6 (1965) 456-462.]

[493] Chentsov, N. N., "Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the 'heuristic' approach to the Kolmogorov-Smirnov tests," Theory of Probability and its Applications 1 (1956) 140-144.

[494] Donsker, M. D., "An invariance principle for certain probability limit theorems," Four papers on Probability. Memoirs of the American Mathematical Society. No. 6 (1951) 12 pp.

[495] Donsker, M. D., "Justification and extension of Doob's heuristic approach to the Kolmogorov-Smirnov theorems," The Annals of Mathematical Statistics 23 (1952) 277-281.

[496] Doob, J. L., "Heuristic approach to the Kolmogorov-Smirnov theorems," The Annals of Mathematical Statistics 20 (1949) 393-403.

[497] Driml, M., "Convergence of compact measures on metric spaces," Transactions of the Second Prague Conference on Information Theory, Statistical Theory, Statistical Decision Functions, Random Processes (1959), Prague, 1960, pp. 71-92.

[498] Dudley, R. M., "Weak convergence of probabilities on nonseparable metric spaces and empirical measures on Euclidean spaces," Illinois Jour. Math. 10 (1966) 109-126.

[499] Dudley, R. M., "Measures on non-separable metric spaces," Illinois Jour. Math. 11 (1967) 449-453.

[500] Dudley, R. M., "Distances of probability measures and random variables," The Annals of Mathematical Statistics 39 (1968) 1563-1572.

[501] Dudley, R. M., "On measurability over product spaces," Bulleting of the American Mathematical Society 77 (1971) 271-274.

[502] Erdös, P., and M. Kac, "On certain limit theorems of the theory of probability," Bulletin of the American Mathematical Society 52 (1946) 292-302.

[503] Erdős, P., and M. Kac, "On the number of positive sums of independent random variables," Bulletin of the American Mathematical Society 53 (1947) 1011-1020.

[504] Grimvall, A., "A theorem on convergence to a Lévy process," Mathematica Scandinavica 30 (1972) 339-349.

[505] Hitsuda, M., and A. Shimizu, "The central limit theorem for additive functionals of Markov processes and the weak convergence to Wiener measure," Journal of the Mathematical Society of Japan 22 (1970) 551-566.

[506] Jain, N. C., and G. Kallianpur, "A note on uniform convergence of stochastic processes," The Annals of Mathematical Statistics 41 (1970) 1360-1362.

[507] Jiřina, M., "Convergence in distribution of random measures," The Annals of Mathematical Statistics 43 (1972) 1727-1731.

[508] Kiefer, J., "Skorokhod embedding of multivariate RV's, and the sample DF," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 24 (1972) 1-35.

[509] Kimme, E. G., "On the convergence of sequences of stochastic processes," Trans. Amer. Math. Soc. 84 (1957) 208-229.

[510] Kimme, E. G., "Some equivalence conditions for the uniform convergence in distribution of sequences of stochastic processes," Trans. Amer. Math. Soc. 95 (1960) 495-515.

[511] Kolmogorov, A., "Eine Verallgemeinerung des Laplace-Liapounoffschen Satzes," Izvestiya Akad. Nauk SSSR (Bulletin de l'Académie des Sciences URSS Cl. Sci. Math. et Nat.) (1931) 959-962.

[512] Kolmogorov, A., "Über die Grenzwertsätze der Wahrscheinlichkeits-rechnung," Izvestiya Akad. Nauk SSSR (Bulletin de l'Académie des Sciences URSS Cl. Sci. Math. et Nat.) (1933) 363-372.

[513] Kolmogorov, A. N., "On Skorokhod convergence," Theory of Probability and its Applications 1 (1956) 215-222.

[514] Kolmogoroff, A., und J. Prochorow, "Zufällige Funktionen und Grenz-verteilungssätze," Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik, Berlin, 1954, pp. 113-126.

[515] Kuelbs, J., "The invariance principle for a lattice of random variables," The Annals of Mathematical Statistics 39 (1968) 382-389.

[516] LeCam, L., "Convergence in distribution of stochastic processes," University of California Publications in Statistics 2 (1957) 207-236.

[517] Liggett, T. M., "An invariance principle for conditioned sums of independent random variables," Journal of Mathematics and Mechanics 18 (1968) 559-570.

[518] Neuhaus, G., "On weak convergence of stochastic processes with multidimensional time parameter," The Annals of Mathematical Statistics 42 (1971) 1285-1295.

[519] Neveu, J., "Note on the thightness of the metric on the set of complete sub σ-algebras of a probability space," The Annals of Mathematical Statistics 43 (1972) 1369-1371.

[520] Oodaira, H., and K. Yoshihara, "Functional central limit theorems for strictly stationary processes satisfying the strong mixing condition," Kōdai Mathematical Seminar Reports (Tokyo) 24 (1972) 259-269.

[521] Parthasarathy, K. R., Probability Measures on Metric Spaces. Academic Press, New York and London, 1967.

[522] Prochorov, Yu. V., "Probability distributions in functional spaces," (Russian) Uspehi Matem. Nauk 8 No. 3 (1953) 165-167.

[523] Prokhorov, Yu. V., "Convergence of random processes and limit theorems in probability theory," Theory of Probability and its Applications 1 (1956) 157-214.

[524] Prohorov, Yu. V., "The method of characteristic functionals," Proceedings of the Fourth Berkeley Symposium on Mathematixal Statistics and Probability (1960), Vol. II: Probability Theory. University of California Press, 1961, pp. 403-419.

[525] Prohorov, Yu. V., "Random measures on a compactum," (Russian) Doklady Akademii Nauk SSSR 138 (1961) 53-55. [English translation: Soviet Mathematics-Doklady 2 (1961) 539-541.]

[526] Pyke, R., "Applications of almost surely convergent constructions of weakly convergent processes," Probability and Information Theory. Proceedings of the International Symposium at McMaster University, Canada, April 1968. Lecture Notes in Mathematics No. 89, Springer, Berlin, 1969, pp. 187-200.

[527] Rao, R. R., "The law of large numbers for $D[0,1]$-valued random variables," Theory of Probability and its Applications 8 (1963) 70-74.

[528] Rosén, B., "Limit theorems for sampling from finite populations," Arkiv för Matematik 5 (1964) 383-424.

[529] Rosén, B., "On the central limit theorem for sums of dependent random variables," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 7 (1967) 48-82.

[530] Rosenkrantz, W. A., "On rates of convergence for the invariance principle," Transactions of the American Mathematical Society 129 (1967) 542-552.

[531] Sen, P. K., "Finite population sampling and weak convergence to a Brownian bridge," Sankhyā: The Indian Journal of Statistics. Ser. A 34 (1972) 85-90.

[532] Silvestrov, D. S., "On convergence of stochastic processes in the uniform topology," (Russian) Doklady Akad. Nauk SSSR (1971) 43-44. [English translation: Soviet Mathematics - Doklady 12 (1971) 1335-1337.]

[533] Silvestrov, D. S., "Remarks about the limit of composite random functions," (Russian) Teorija Verojatnost. i Primenen. 17 (1972) 707-715.

[534] Silvestrov, D. S., "On convergence of complex random functions in the J-topology," (Russian) Doklady Akad. Nauk SSSR 202 (1972) 539-540. [English translation: Soviet Mathematics-Doklady 13 (1972) 152-154.]

[535] Skorokhod, A. V., "On the limiting transition from a sequence of sums of independent random quantities to a homogeneous random process with independent increments," (Russian) Dokl. Akad. Nauk. SSSR 104 (1955) 364-367.

[536] Skorokhod, A. V., "On a class of limit theorems for Markov chains," (Russian) Dokl. Akad. Nauk. SSSR 106 (1956) 781-784.

[537] Skorokhod, A. V., "Limit theorems for stochastic processes," Theory of Probability and its Applications 1 (1956) 261-290.

[538] Skorokhod, A. V., "Limit theorems for stochastic processes with independent increments," Theory of Probability and its Applications 2(1957) 138-171.

[539] Skorokhod, A. V., "On the differentiability of measures which correspond to stochastic processes. I. Processes with independent increments," Theory of Probability and its Applications 2 (1957) 407-432.

[540] Skorokhod, A. V., "A limit theorem for independent random variables," (Russian) Doklady Akademii Nauk SSSR 133 (1960) 34-35. [English translation: Soviet Mathematics-Doklady 1 (1960) 810-811.]

[541] Skorokhod, A. V., Random Processes with Independent Increments. (Russian) Izd. "Nauka", Moscow, 1964.

[542] Stone, C., "Weak convergence of stochastic processes defined on semi-infinite time intervals," Proc. Amer. Math. Soc. 14 (1963) 694-696.

[543] Straf, M. L., "Weak convergence of stochastic processes with several parameters," Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. Vol. II. Probability Theory. University of California Press, 1972, pp. 187-221.

[544] Topsøe, F., "A criterion for weak convergence of measures with an application to convergence of measures on D[0,1]," Mathematica Scandinavica 25 (1969) 97-104.

[545] Varadarajan, V. S., "Weak convergence of measures on separable metric spaces," Sankhyā: The Indian Journal of Statistics 19 (1958) 15-22.

[546] Varadarajan, V. S., "On the convergence of sample probability distributions," Sankhyā: The Indian Journal of Statistics 19 (1958) 23-26.

[547] Varadarajan, V. S., "A remark on strong measurability," Sankhyā: The Indian Journal of Statistics 20 (1958) 219-220.

[548] Varadarajan, V. S., "Measures on topological spaces," Mat. Sb. (N.S.) 55 (97) (1961) 35-100. [Amer. Math. Soc. Translations. Sec. Ser. 48 (1965) 161-228.]

[549] Varadarajan, V. S., "Convergence of stochastic processes," Bull. Amer. Math. Soc. 67 (1961) 276-280.

[550] Walsh, J., "A note on the uniform convergence of stochastic processes," Proceedings of the American Mathematical Society 18 (1967) 129-132.

[551] Woodroofe, M., "On the weak convergence of stochastic processes without discontinuities of the second kind," Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 11 (1968) 18-25.

Additional References

[552] Sierpiński, W., "Sur les fonctions convexes mesurables," Fundamenta Mathematicae 1 (1920) 125-129.

[553] Smith, W.L., "Asymptotic renewal theorems," Proceedings of the Royal Society of Edinburgh. Sect. A (Mathematics) 64 (1954) 9-48.

[554] Smith, W.L., "Extension of a renewal theorem," Proceedings of the Cambridge Philosophical Society 51 (1955) 629-638.

[555] Whitt, W., "Weak convergence of probability measures on the function space $C[0,\infty)$," The Annals of Mathematical Statistics 41 (1970) 939-944.