CHAPTER VI

LIMIT THEOREMS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

41. Fundamental Theorems. In this section we shall prove some basic theorems which will be needed in studying the limiting behavior of sums of mutually independent and identically distributed real random variables.

Random Trials. If we speak about a random trial, then we suppose that a probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$ is associated with the random trial. In the probability space, Ω is the sample space, that is, the set of all the possible outcomes of the random trial. The elements of Ω are called sample points and are denoted by ω. In the probability space, $\mathcal{B}$ is the class of random events considered in the random trial. Each random event is defined as a subset of Ω. We shall denote random events by capital Latin letters, such as A, B, C. There are two exceptions. We shall denote by Ω the sure event, which always occurs, and by Θ the impossible event, which never occurs. The sure event Ω contains every sample point and the impossible event Θ contains no sample point. If the occurrence of A implies the occurrence of B, then we shall write $A \subset B$. Then every point of A is contained in B. The complementary event of A will be denoted by $\bar{A}$. The simultaneous occurrence of the random events A, B, C,... will be denoted by ABC ... . The event that at least one event occurs among A, B, C,... will be denoted by A+B+C+ ... . We suppose that $\mathcal{B}$ is a σ - algebra of subsets of Ω , that is, $\mathcal{B}$ is a class of subsets of Ω which satisfies the following two requirements:

(i) If $A \varepsilon \mathcal{B}$, then $\bar{A} \varepsilon \mathcal{B}$.

(ii) If $A_n \varepsilon \mathcal{B}$ for $n = 1,2,\ldots$, then $\sum_{n=1}^{\infty} A_n \varepsilon \mathcal{B}$.

With every event $A \varepsilon \mathcal{B}$ we associate a real number $\underset{\sim}{P}\{A\}$, the probability of A . We suppose that the set function $\underset{\sim}{P}\{A\}$ is nonnegative, normed and σ - additive, that is, $\underset{\sim}{P}\{A\}$ defined for $A \varepsilon \mathcal{B}$ satisfies the following three requirements:

(i) $\underset{\sim}{P}\{A\} \geqq 0$ for $A \varepsilon \mathcal{B}$.

(ii) $\underset{\sim}{P}\{\Omega\} = 1$.

(iii) If $A_n \varepsilon \mathcal{B}$ for $n = 1,2,\ldots$ and $A_i A_j = \Theta$ for $i \neq j$, then

$$(1) \qquad \underset{\sim}{P}\{\sum_{n=1}^{\infty} A_n\} = \sum_{n=1}^{\infty} \underset{\sim}{P}\{A_n\} .$$

If $A_1, A_2, \ldots, A_n, \ldots$ is an infinite sequence of events, then

$$(2) \qquad A^* = \limsup_{n\to\infty} A_n = \prod_{n=1}^{\infty} \sum_{r=n}^{\infty} A_r$$

is the event that infinitely many events occur in the sequence $A_1, A_2, \ldots, A_n, \ldots$, and

$$(3) \qquad A_* = \liminf_{n\to\infty} A_n = \sum_{n=1}^{\infty} \prod_{r=n}^{\infty} A_r$$

is the event that all but a finite number of events occur in the sequence $A_1, A_2, \ldots, A_n, \ldots$.

If $A^* = A_*$, then we say that the sequence of events $A_1, A_2, \ldots, A_n, \ldots$ has a limit and it is defined by $\lim_{n\to\infty} A_n = A^* = A_*$.

If $A_n \in \mathcal{B}$ for $n = 1,2,\ldots,$ then obviously $A^* \in \mathcal{B}$ and $A_* \in \mathcal{B}$, and we can easily prove that

(4) $$\underset{\sim}{P}\{A^*\} = \lim_{n\to\infty} \underset{\sim}{P}\{\sum_{r=n}^{\infty} A_r\}$$

and

(5) $$\underset{\sim}{P}\{A_*\} = \lim_{n\to\infty} \underset{\sim}{P}\{\prod_{r=n}^{\infty} A_r\} .$$

If $\lim_{n\to\infty} A_n$ exists, then it follows easily from (4) and (5) that

(6) $$\underset{\sim}{P}\{\lim_{n\to\infty} A_n\} = \lim_{n\to\infty} \underset{\sim}{P}\{A_n\} .$$

If $A_1, A_2, \ldots, A_n, \ldots$ is a monotone sequence of events, that is, either $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ or $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$, then by (2) and (3) we have $A^* = A_*$, that is, $\lim_{n\to\infty} A_n$ exists and (6) holds.

The events $A_1, A_2, \ldots, A_n$ are said to be mutually independent if

(7) $$\underset{\sim}{P}\{A_{i_1} A_{i_2} \ldots A_{i_k}\} = \underset{\sim}{P}\{A_{i_1}\}P\{A_{i_2}\}\ldots P\{A_{i_k}\}$$

for all $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $k = 1,2,\ldots, n$.

We say that $A_1, A_2, \ldots, A_n, \ldots$ is an infinite sequence of mutually independent events if (7) holds for all $n = 1, 2, \ldots$.

The events $A_1, A_2, \ldots, A_n$ are said to be interchangeable if

$$\underset{\sim}{P}\{A_{i_1} A_{i_2} \ldots A_{i_k}\} = \underset{\sim}{P}\{A_1 A_2 \ldots A_k\} \tag{8}$$

for all $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$ and $k = 1, 2, \ldots, n$.

We say that $A_1, A_2, \ldots, A_n, \ldots$ is an infinite sequence of interchangeable events if (8) holds for all $n = 1, 2, \ldots$.

Now we shall prove two theorems for an infinite sequence of random events. These theorems were proved in a particular case in 1909 by <u>É. Borel</u> [16], and in a more general case in 1917 by <u>F. P. Cantelli</u> [18]. (See also <u>P. Lévy</u> [113] pp. 126-128.)

<u>Theorem 1</u>. <u>If</u> $A_1, A_2, \ldots, A_n, \ldots$ <u>are arbitrary events, and</u>

$$\sum_{n=1}^{\infty} \underset{\sim}{P}\{A_n\} < \infty , \tag{9}$$

<u>then</u>

$$\underset{\sim}{P}\{A^*\} = \underset{\sim}{P}\{\prod_{n=1}^{\infty} \sum_{r=n}^{\infty} A_r\} = 0 . \tag{10}$$

<u>Proof</u>. Obviously we have

$$A^* \subset \sum_{r=n}^{\infty} A_r \tag{11}$$

for all $n = 1, 2, \ldots$. Hence by Boole's inequality it follows that

$$0 \leqq P\{A^*\} \leqq \sum_{r=n}^{\infty} P\{A_r\} \tag{12}$$

for $n = 1,2,\ldots$. Now by (9) the extreme right member in (12) tends to 0 as $n \to \infty$. Thus $P\{A^*\} = 0$ which was to be proved.

<u>Theorem 2</u>. <u>If</u> $A_1, A_2, \ldots, A_n, \ldots$ <u>are mutually independent events, and</u>

$$\sum_{n=1}^{\infty} P\{A_n\} = \infty , \tag{13}$$

<u>then</u>

$$P\{A^*\} = P\{\prod_{n=1}^{\infty} \sum_{r=n}^{\infty} A_r\} = 1 . \tag{14}$$

<u>Proof</u>. We shall prove that $P\{\overline{A^*}\} = 0$ which implies (14). By (4) we have

$$P\{\overline{A^*}\} = \lim_{n \to \infty} P\{\prod_{r=n}^{\infty} \overline{A}_r\} . \tag{15}$$

Furthermore, we can write that

$$0 \leqq P\{\prod_{r=n}^{\infty} \overline{A}_r\} \leqq P\{\prod_{r=n}^{m} \overline{A}_r\} = \prod_{r=n}^{m} P\{\overline{A}_r\} \tag{16}$$

for all $m \geqq n$. Here we used the fact that $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n, \ldots$ are also mutually independent events. Since evidently

$$P\{\overline{A}_r\} = 1-P\{A_r\} \leqq e^{-P\{A_r\}} , \tag{17}$$

it follows that

$$0 \leqq P\{\prod_{r=n}^{\infty} \overline{A}_r\} \leqq e^{-\sum_{r=n}^{m} P\{A_r\}} \tag{18}$$

for $m \geqq n$. Now by (13) the extreme right member of (18) tends to 0 as $m \to \infty$. Thus

$$\underset{\sim}{P}\{\prod_{r=n}^{\infty} \bar{A}_r\} = 0 \tag{19}$$

for all $n = 1,2,\ldots$. Finally, by (15) we obtain that $\underset{\sim}{P}\{\overline{A^*}\} = 0$ which was to be proved.

<u>Corollary 1</u>. <u>If</u> $A_1, A_2, \ldots, A_n, \ldots$ <u>are mutually independent events</u>, <u>then</u>

$$\underset{\sim}{P}\{A^*\} = 0 \quad \underline{\text{whenever}} \quad \sum_{n=1}^{\infty} \underset{\sim}{P}\{A_n\} < \infty , \tag{20}$$

<u>and</u>

$$\underset{\sim}{P}\{A^*\} = 1 \quad \underline{\text{whenever}} \quad \sum_{n=1}^{\infty} \underset{\sim}{P}\{A_n\} = \infty . \tag{21}$$

This last statement is the so-called Borel-Cantelli theorem.

<u>Random Variables</u>. Let us consider a random trial and let $(\Omega, \mathcal{B}, \underset{\sim}{P})$ be the associated probability space. If we speak about a real random variable ξ concerning the random trial, then by this we understand a real function $\xi = \xi(\omega)$ defined for $\omega \in \Omega$ and measurable with respect to $\mathcal{B}$, that is, for every real x the event $\{\omega : \xi(\omega) \leqq x\} \in \mathcal{B}$. A random variable $\xi(\omega)$ may be finite or infinite. If it is not specified otherwise, then by a random variable ξ we mean a finite, measurable, real function $\xi(\omega)$ defined on Ω.

If $\xi = \xi(\omega)$ is a finite random variable, then the function

(22) $$F(x) = \underset{\sim}{P}\{\xi \leq x\}$$

defined for $-\infty < x < +\infty$ is called the distribution function of the random variable. We define $F(+\infty) = \lim_{x \to \infty} F(x)$ and $F(-\infty) = \lim_{x \to -\infty} F(x)$.

If we know the distribution function of ξ, then we can determine the probability

(23) $$Q(S) = \underset{\sim}{P}\{\xi \varepsilon S\}$$

for any Borel set S of the real line, that is, for any set S which belongs to the minimal σ- algebra which contains all the intervals of the real line. This follows from the extension theorem of Carathéodory. (See Theorem 1.2 in the Appendix .) The set function $Q(S)$ is nonnegative, normed and σ- additive, that is, a probability measure defined on the σ- algebra of all the Borel subsets of the real line.

We can classify random variables according to their distribution functions. Let ξ be a finite random variable and let $\underset{\sim}{P}\{\xi \leq x\} = F(x)$. The point spectrum S of ξ is defined as the set of discontinuity points of $F(x)$, that is,

(24) $$S = \{x : F(x+0) - F(x-0) > 0\} .$$

The set S is either empty or finite or countably infinite. The continuous spectrum R of ξ is defined as the set of all those continuity points of $F(x)$ in which $F(x)$ is increasing, that is,

(25) $$R = \{x:F(x+\varepsilon)-F(x-\varepsilon) > 0 \text{ for all } \varepsilon > 0 \text{ and } F(x+0)-F(x-0) = 0\} .$$

The set R is perfect (possibly empty), that is, R is closed and contains no isolated points.

If R is the empty set, then we say that ξ is a discrete random variable. In this case each $a \in S$ is called a possible value of the random variable ξ. If ξ is discrete, then

(26) $$\sum_{a \in S} P\{\xi = a\} = 1$$

and

(27) $$F(x) = \sum_{\substack{a \leq x \\ a \in S}} P\{\xi = a\}$$

is a step function.

If ξ is a discrete random variable, if $P\{\xi = 0\} < 1$, and if $S \subset \{n\lambda : n = 0, \pm 1, \pm 2, \ldots\}$ for some positive λ, then we say that ξ is a lattice random variable. The largest λ which satisfies the above requirement is called the step of ξ.

If S is the empty set, then ξ is called a continuous random variable. Then $F(x)$ is a continuous function of x.

Let ξ be a finite real random variable, and $P\{\xi \leq x\} = F(x)$. If

(28) $$\int_{\Omega} |\xi(\omega)| dP = \int_{-\infty}^{\infty} |x| dF(x) < \infty ,$$

then we say that ξ has a finite expectation defined by

(29) $$E\{\xi\} = \int_{\Omega} \xi(\omega) dP = \int_{-\infty}^{\infty} x dF(x) .$$

In this case we say also that $E\{\xi\}$ exists.

If the integrals in (28) diverge, then $E\{\xi\} = +\infty$ or $E\{\xi\} = -\infty$ or $E\{\xi\}$ is indeterminate. Obviously $E\{\xi\}$ exists if and only if $E\{|\xi|\} < \infty$.

If $g(x)$ is a finite Borel measurable function of x, then $\eta = g(\xi)$ is also a random variable and its expectation is given by

$$(30) \qquad E\{\eta\} = \int_{\Omega} g(\xi(\omega))dP = \int_{-\infty}^{\infty} g(x)dF(x)$$

provided that the integrals in (30) are absolutely convergent.

The expectation $E\{\xi^k\}$ $(k = 0,1,2,\ldots)$ is called the k-th moment of ξ (about the origin) provided that it exists.

If the expectation $E\{\xi\}$ exists, then we define $E\{[\xi-E\{\xi\}]^k\}$ $(k = 0,1,2,\ldots)$ as the k-th central moment of ξ provided that it exists. The second central moment is called also the variance of ξ and is denoted by $\text{Var}\{\xi\}$.

The following inequality which was found in 1855 by J. Bienaymé [8] and in 1867 by P. L. Chebyshev [474] for discrete random variables is a very useful one in the theory of probability. (See also A.A. Markov [584] p. 54.)

Theorem 3. If ξ is a nonnegative random variable, then

$$(31) \qquad P\{\xi \geq a\} \leq \frac{E\{\xi\}}{a}$$

for any $a > 0$.

Proof. Let us define a random variable χ in the following way: $\chi = 1$ if $\xi \geq a$ and $\chi = 0$ if $\xi < a$. Then we have

$$a\chi \leqq \xi \ . \tag{32}$$

If $\chi = 0$, then (32) is evident. If $\chi = 1$, then $\xi \geqq a$, and thus (32) holds in this case too. By (32) we obtain that

$$a\mathbf{E}\{\chi\} \leqq \mathbf{E}\{\xi\} \tag{33}$$

and obviously $\mathbf{E}\{\chi\} = \mathbf{P}\{\chi = 1\} = \mathbf{P}\{\xi \geqq a\}$. If $a > 0$, then (31) follows from (33).

Let $\xi_1, \xi_2, \ldots, \xi_n$ be n real random variables. We define

$$F(x_1, x_2, \ldots, x_n) = \mathbf{P}\{\xi_1 \leqq x_1, \xi_2 \leqq x_2, \ldots, \xi_n \leqq x_n\} \tag{34}$$

for all real $x_1, x_2, \ldots, x_n$ as the joint distribution function of $\xi_1, \xi_2, \ldots, \xi_n$.

If we know the joint distribution function of $\xi_1, \xi_2, \ldots, \xi_n$, then we can determine the probability

$$Q(S) = \mathbf{P}\{(\xi_1, \xi_2, \ldots, \xi_n) \in S\} \tag{35}$$

for any Borel set of the n-dimensional Euclidean space, that is, for any set S which belongs to the minimal σ- algebra which contains all the "intervals" of the n-dimensional Euclidean space. This follows from the extension theorem of Carathéodory. (See Theorem 1.2 in the Appendix.) The set function $Q(S)$ is nonnegative, normed and σ- additive, that is, $Q(S)$ is a probability measure defined on the σ- algebra of all the Borel subsets of the n-dimensional Euclidean space.

We can introduce some useful classification for a finite or an infinite sequence of real random variables based on their joint distribution functions. The real random variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be mutually independent if

$$(36) \qquad \mathbf{P}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_n \leq x_n\} = \mathbf{P}\{\xi_1 \leq x_1\}\mathbf{P}\{\xi_2 \leq x_2\} \ldots \mathbf{P}\{\xi_n \leq x_n\}$$

holds for all real $x_1, x_2, \ldots, x_n$.

We say that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is an infinite sequence of mutually independent real random variables if (36) holds for all $n = 1,2,\ldots$.

The real random variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be interchangeable if

$$(37) \qquad \mathbf{P}\{\xi_{i_1} \leq x_1, \xi_{i_2} \leq x_2, \ldots, \xi_{i_n} \leq x_n\} = \mathbf{P}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_n \leq x_n\}$$

holds for every permutation $(i_1, i_2, \ldots, i_n)$ of $(1,2,\ldots, n)$ and for all real $x_1, x_2, \ldots, x_n$.

We say that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is an infinite sequence of interchangeable real random variables if (37) holds for all $n = 1,2,\ldots$.

The Borel-Cantelli theorem (Corollary 1) implies the following result: Let $A_1, A_2, \ldots, A_n, \ldots$ be a sequence of mutually independent events. Define $\xi_n = 1$ if A_n occurs, and $\xi_n = 0$ if A_n does not occur. Then

$$(38) \qquad \mathbf{P}\{\sum_{n=1}^{\infty} \xi_n \text{ converges}\}$$

is either 0 or 1 . This last statement is true for any sequence of mutually independent random variables. This follows from a more general theorem, the so-called "zero-or-one law", which was discovered in several cases in 1931 by P. Lévy [428] and was formulated in a general form in 1933 by A. N. Kolmogorov [100]. (See also P. Lévy [112], and [113] pp. 128-130.) Before proving this theorem let us make some preliminary statements.

A class of events $\mathcal{A}$ is called an algebra of events if it satisfies the following two requirements:

(i) If $A \in \mathcal{A}$, then $\bar{A} \in \mathcal{A}$

and

(ii) If $A_k \in \mathcal{A}$ for $k = 1,2,\dots, n$, then $\sum_{k=1}^{n} A_k \in \mathcal{A}$ for any finite $n = 1,2,\dots$.

A class of events $\mathcal{M}$ is called a monotone class of events if it satisfies the following requirement:

If $A_n \in \mathcal{M}$ for $n = 1,2,\dots$ and $\{A_n\}$ is a monotone sequence, then $\lim_{n\to\infty} A_n \in \mathcal{M}$.

Denote by $\mathcal{B}$ the minimal σ- algebra which contains the algebra $\mathcal{A}$, and denote by $\mathcal{M}$ the minimal monotone class which contains $\mathcal{A}$. We can easily prove that $\mathcal{B}$ and $\mathcal{M}$ coincide. (See **Theorem 1.1 in the Appendix.**)

Now let us consider a random trial and denote by $(\Omega, \mathcal{B}, \underset{\sim}{P})$ the associated probability space. Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be an infinite sequence of mutually independent real random variables. Denote by $\mathcal{B}_n$ the σ- algebra generated by the random variables $\xi_1, \xi_2, \dots, \xi_n$. That is,

$\mathcal{B}_n$ is the minimal σ- algebra which contains the events $\{\omega : \xi_k(\omega) \leqq x\}$ for all $k = 1,2,\ldots, n$ and for all real x.

Denote by $\mathcal{A}_n$ the σ- algebra generated by the random variables $\xi_n, \xi_{n+1},\ldots,$ that is, $\mathcal{A}_n$ is the minimal σ- algebra which contains the events $\{\omega : \xi_k(\omega) \leqq x\}$ for all $k = n, n+1,\ldots$ and for all real x.

Let $\mathcal{A}$ be the intersection of $\mathcal{A}_1, \mathcal{A}_2,\ldots, \mathcal{A}_n,\ldots,$ that is,

(39) $$\mathcal{A} = \bigcap_{n=1}^{\infty} \mathcal{A}_n .$$

The class of events $\mathcal{A}$ is obviously a σ- algebra, the so-called tail σ- algebra.

Now we can formulate the zero-or-one law in the following way:

<u>Theorem 4</u>. <u>If</u> $A \in \mathcal{A}$, <u>then either</u> $\mathbf{P}\{A\} = 0$ <u>or</u> $\mathbf{P}\{A\} = 1$.

<u>Proof</u>. Let

(40) $$\mathcal{M}_o = \{B : B \in \mathcal{B}_n \text{ for some } n = 1,2,\ldots\} .$$

We can easily see that $\mathcal{M}_o$ is an algebra of events, and $\mathcal{A}_1$ is the minimal σ- algebra which contains $\mathcal{M}_o$.

For a given $A \in \mathcal{A}$ let us define

(41) $$\mathcal{M} = \{B : \mathbf{P}\{AB\} = \mathbf{P}\{A\}\mathbf{P}\{B\} \text{ and } B \in \mathcal{A}_1\} .$$

We can easily see that $\mathcal{M}$ is a monotone class of events and by definition $\mathcal{M} \subset \mathcal{A}_1$.

We shall prove that $\mathcal{M}_0 \subset \mathcal{M}$. Since $\mathcal{A}_1$ is the minimal σ- algebra which contains $\mathcal{M}_0$, this implies that $\mathcal{A}_1 \subset \mathcal{M}$. Consequently we have $\mathcal{M} = \mathcal{A}_1$ and therefore $\mathcal{A} \subset \mathcal{M}$. Thus we can conclude that $A \varepsilon \mathcal{M}$ (being $A \varepsilon \mathcal{A}$), that is, $\underset{\sim}{P}\{A\} = \underset{\sim}{P}\{A\}\underset{\sim}{P}\{A\}$. Accordingly, if $A \varepsilon \mathcal{A}$, then either $\underset{\sim}{P}\{A\} = 0$ or $\underset{\sim}{P}\{A\} = 1$.

It remains only to prove that $\mathcal{M}_0 \subset \mathcal{M}$, that is, if $B \varepsilon \mathcal{M}_0$, then $B \varepsilon \mathcal{M}$. If $B \varepsilon \mathcal{M}_0$, then $B \varepsilon \mathcal{B}_n$ for some $n = 1,2,\ldots$. On the other hand if $A \varepsilon \mathcal{A}$, then $A \varepsilon \mathcal{A}_n$ for every $n = 1,2,\ldots$. Thus $B \varepsilon \mathcal{B}_n$ and $A \varepsilon \mathcal{A}_{n+1}$ for some $n = 1,2,\ldots$, which implies that, A and B are independent, that is $\underset{\sim}{P}\{AB\} = \underset{\sim}{P}\{A\}\underset{\sim}{P}\{B\}$. Hence it follows that $B \varepsilon \mathcal{M}$. This completes the proof of the theorem.

For many purposes it is convenient to introduce complex random variables too. A complex random variable ζ is defined as $\xi+i\eta$ where ξ and η are real random variables and i is the square root of -1. If we know the joint distribution of ξ and η, then we can determine the probability that $\zeta = \xi+i\eta$ belongs to any Borel set of the complex plane.

If $\underset{\sim}{E}\{\xi\}$ and $\underset{\sim}{E}\{\eta\}$ exist, then we say that the expectation $\underset{\sim}{E}\{\zeta\}$ exists and it is defined by

$$\underset{\sim}{E}\{\zeta\} = \underset{\sim}{E}\{\xi\} + i\underset{\sim}{E}\{\eta\} . \tag{42}$$

Otherwise we say that $\underset{\sim}{E}\{\zeta\}$ does not exist. Evidently $\underset{\sim}{E}\{\zeta\}$ exists if and only if $\underset{\sim}{E}\{|\zeta|\} < \infty$.

If ξ is a real random variable and $g(x) = g_1(x) + ig_2(x)$ where $g_1(x)$ and $g_2(x)$ are finite Borel measurable functions, then $g(\xi)$ is a complex random variable. If $\underset{\sim}{P}\{\xi \leq x\} = F(x)$, then

$$\underset{\sim}{E}\{g(\xi)\} = \int_{-\infty}^{\infty} g(x)dF(x) \tag{43}$$

provided that the integral in (43) is absolutely convergent.

By choosing various functions $g(x)$ we can define by (43) various transforms of $F(x)$ which have great importance in the theory of probability.

In what follows we shall assume that ξ is a real random variable and its distribution function is $\underset{\sim}{P}\{\xi \leq x\} = F(x)$.

Let $g(x) = e^{-sx}$ where s is a real or complex number. The expectation

$$\underset{\sim}{E}\{e^{-s\xi}\} = \int_{-\infty}^{\infty} e^{-sx}dF(x) = \phi(s) \tag{44}$$

is a function of s , if it exists, and it is called the Laplace-Stieltjes transform of ξ , or the Laplace-Stieltjes transform of $F(x)$.

If $Re(s) = 0$, then $\phi(s)$ always exists. If $\phi(s)$ exists for $s = \sigma_1 + i\tau_1$ and $s = \sigma_2 + i\tau_2$, where $\sigma_1 < \sigma_2$, then it exists in the strip $\sigma_1 < Re(s) < \sigma_2$ and $\phi(s)$ is a regular function of s in this strip.

If $\phi(s)$ exists for $Re(s) = c$, then $F(x)$ is uniquely determined by $\phi(s)$ given for $Re(s) = c$. This is a consequence of the inversion formulas given below.

In 1922 P. Lévy [110],[111 p.166] proved the following inversion formula.

Theorem 5. *If $\phi(s)$ is the Laplace-Stieltjes transform of a non-decreasing function $F(x)$ defined on $(-\infty, \infty)$ for which $F(+\infty)$ and $F(-\infty)$ are finite, that is,*

$$\phi(s) = \int_{-\infty}^{\infty} e^{-sx} dF(x) , \tag{45}$$

then (45) is absolutely convergent for $Re(s) = 0$ and

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{-iT}^{iT} \frac{e^{sb} - e^{sa}}{s} \phi(s) ds = \frac{F(b+0) + F(b-0)}{2} - \frac{F(a+0) + F(a-0)}{2} \tag{46}$$

for any real a and b.

Proof. Let us suppose that $a < b$. For any fixed a and b let

$$I_T = \frac{1}{2\pi i} \int_{-iT}^{iT} \frac{e^{sb} - e^{sa}}{s} \phi(s) ds = \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{itb} - e^{ita}}{it} \phi(it) dt . \tag{47}$$

If we put (45) into (47) and interchange the order of integration, then we obtain that

$$I_T = \int_{-\infty}^{\infty} J_T(x) dF(x) \tag{48}$$

where

$$J_T(x) = \frac{1}{\pi} \int_{T(x-b)}^{T(x-a)} \frac{\sin u}{u} du . \tag{49}$$

Since

$$(50) \qquad 0 < \int_0^c \frac{\sin u}{u}\,du \leq \int_0^\pi \frac{\sin u}{u}\,du = 1.8519\ldots$$

for any $c > 0$ and

$$(51) \qquad \int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2}$$

by Dirichlet's integral formula, it follows that $J_T(x)$ is a bounded function of x and

$$(52) \qquad \lim_{T\to\infty} J_T(x) = J(x) = \begin{cases} 1 & \text{if } a < x < b\,, \\ \frac{1}{2} & \text{if } x = a \text{ or } x = b\,, \\ 0 & \text{if } x < a \text{ or } x > b. \end{cases}$$

Hence by (48)

$$(53) \qquad \lim_{T\to\infty} I_T = \int_{-\infty}^{\infty} J(x)dF(x) = \frac{F(b+0) + F(b-0)}{2} - \frac{F(a+0) + F(a-0)}{2}$$

for $a < b$. This completes the proof of the theorem.

The following inversion formula is given by D. V. Widder [215] p. 242.

Theorem 6. *If $\phi(s)$ is the Laplace-Stieltjes transform of a non-decreasing function $F(x)$ defined on $(-\infty, \infty)$ for which $F(+\infty)$ and $F(-\infty)$ are finite, that is,*

$$(54) \qquad \phi(s) = \int_{-\infty}^{\infty} e^{-sx} dF(x)$$

<u>and if</u> (54) <u>is absolutely convergent for</u> $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, <u>then</u>

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{sa}}{s} \phi(s)ds = \frac{F(a+0) + F(a-0)}{2} - F(-\infty) \tag{55}$$

<u>whenever</u> $\sigma_1 < c < \sigma_2$ <u>and</u> $c > 0$, <u>and</u>

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{sa}}{s} \phi(s)ds = \frac{F(a+0) + F(a-0)}{2} - F(+\infty) \tag{56}$$

<u>whenever</u> $\sigma_1 < c < \sigma_2$ <u>and</u> $c < 0$.

<u>Proof</u>. The proof is based on Dirichlet's integral formula (51) and follows on similar lines as the proof of the previous theorem.

Now we shall show that if $F(x)$ is a distribution function, then $F(x)$ is uniquely determined by its Laplace-Stieltjes transform $\phi(s)$ given for $\mathrm{Re}(s) = c$. If $F(x)$ is a distribution function, then $F(+\infty) = 1$, $F(-\infty) = 0$ and $F(x+0) = F(x)$ for every x . For every a we can determine $[F(a+0) + F(a-0)]/2$ by (46) if $c = 0$ $(b \to +\infty)$, by (55) if $c > 0$, and by (56) if $c < 0$. If $x = a$ is a continuity point of $F(x)$, then $F(a) = [F(a+0) + F(a-0)]/2$. If $x = a$ is a discontinuity point of $F(x)$, then $F(a) = F(a+0)$, and we can find a sequence of continuity points $\{a_n\}$ of $F(x)$ such that $a_n > a$ for all n and $\lim_{n\to\infty} a_n = a$. Then $F(a) = F(a+0) = \lim_{n\to\infty} F(a_n)$.

If $\xi_1, \xi_2, \ldots, \xi_n$ are real random variables and $s_1, s_2, \ldots, s_n$ are complex or real numbers, then we can define the expectation

$$(57)\qquad \phi(s_1, s_2, \ldots, s_n) = \underset{\sim}{E}\{e^{-s_1\xi_1 - s_2\xi_2 - \cdots - s_n\xi_n}\}$$

if it exists. The function $\phi(s_1, s_2, \ldots, s_n)$ is the multi-dimensional Laplace-Stieltjes transform of the random variables $\xi_1, \xi_2, \ldots, \xi_n$. If

$$(58)\qquad \underset{\sim}{P}\{\xi_1 \leq x_1, \xi_2 \leq x_2, \ldots, \xi_n \leq x_n\} = F(x_1, x_2, \ldots, x_n) ,$$

then

$$(59)\quad \phi(s_1, s_2, \ldots, s_n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-s_1x_1 - s_2x_2 - \cdots - s_nx_n} d_{x_1} d_{x_2} \cdots d_{x_n} F(x_1, x_2, \ldots, x_n)$$

is the multi-dimensional Laplace-Stieltjes transform of the joint distribution function $F(x_1, x_2, \ldots, x_n)$.

The expectation (57) always exists if $Re(s_1) = Re(s_2) = \ldots = Re(s_n) = 0$ and (58) is uniquely determined by $\phi(s_1, s_2, \ldots, s_n)$.

We note that if $\xi_1, \xi_2, \ldots, \xi_n$ are mutually independent random variables, then

$$(60)\quad \underset{\sim}{E}\{e^{-s_1\xi_1 - s_2\xi_2 - \cdots - s_n\xi_n}\} = \underset{\sim}{E}\{e^{-s_1\xi_1}\}\underset{\sim}{E}\{e^{-s_2\xi_2}\}\ldots \underset{\sim}{E}\{e^{-s_n\xi_n}\}$$

for all those $s_1, s_2, \ldots, s_n$ for which the expectations exist.

In a similar way as the Laplace-Stieltjes transform we can define various other transforms of a real random variable ξ. Let $\underset{\sim}{P}\{\xi \leq x\} = F(x)$.

The expectation

(61) $$\underset{\sim}{E}\{z^{\xi}\} = \int_{-\infty}^{\infty} z^{x} dF(x) ,$$

if it exists, is called the generating function of ξ .

The expectation

(62) $$\underset{\sim}{E}\{e^{i\omega\xi}\} = \int_{-\infty}^{\infty} e^{i\omega x} dF(x) ,$$

if it exists, is called the characteristic function of ξ , or, the Fourier-Stieltjes transform of $F(x)$.

The transforms (61) and (62) are merely variants of the Laplace-Stieltjes transform (44) . If in (44) we put $s = -\log z$, then we obtain (61) and if in (44) we put $s = -i\omega$, then we obtain (62).

For nonnegative random variables ξ we occasionally consider the Mellin-Stieltjes transform

(63) $$\underset{\sim}{E}\{\xi^{s}\} = \int_{+0}^{\infty} x^{s} dF(x) = \mu(s) .$$

If (63) is convergent for $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, then

(64) $$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{a^{-s}}{s} \mu(s) ds = F(\infty) - \frac{F(a+0) + F(a-0)}{2}$$

whenever $\sigma_1 < c < \sigma_2$ and $c > 0$, and

(65) $$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{a^{-s}}{s} \mu(s) ds = F(+0) - \frac{F(a+0) + F(a-0)}{2}$$

whenever $\sigma_1 < c < \sigma_2$ and $c < 0$. In (64) $F(\infty) = 1$.

For nonnegative random variables ξ we occasionally consider also the Stieltjes transform

$$\underset{\sim}{E}\{\frac{1}{s+\xi}\} = \int_{+0}^{\infty} \frac{1}{s+x}\, dF(x) \,. \tag{66}$$

The integral (66) converges everywhere on the complex plane except the negative real axis.

For any positive a we have

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi i} \int_{0}^{a} [\mu(-u-i\varepsilon)-\mu(-u+i\varepsilon)]du = \tag{67}$$

$$= \frac{F(a+0)+F(a-0)}{2} - \frac{F(+0)+F(0)}{2} \,.$$

Here $F(a+0) = F(a)$ for all $a \geqq 0$.

<u>Two Theorems of Helly.</u> In 1912 <u>E. Helly</u> [73] discovered several important theorems for a sequence of nondecreasing functions. These theorems are extremely useful in the theory of probability because we can apply them to a sequence of distribution functions. Our next aim is to prove these theorems.

Let $G_n(x)$ $(n = 1,2,\dots)$ be a sequence of nondecreasing functions defined on the interval $(-\infty, \infty)$. Let $G(x)$ be also a nondecreasing function defined on $(-\infty, \infty)$. Denote by C the set of all those real numbers x for which $G(x)$ is continuous.

<u>Definition.</u> <u>We say that the sequence</u> $\{G_n(x)\}$ <u>converges weakly to</u> $G(x)$ <u>if</u>

$$\lim_{n\to\infty} G_n(x) = G(x) \tag{68}$$

<u>for every</u> $x \in C$. <u>We shall write also</u> $G_n(x) \Rightarrow G(x)$.

<u>Lemma 1.</u> <u>We have</u>

(69) $$\lim_{n\to\infty} G_n(x) = G(x)$$

<u>for every</u> $x \varepsilon C$ <u>if and only if</u>

(70) $$\lim_{n\to\infty} G_n(x) = G(x)$$

<u>for every</u> $x \varepsilon D$ <u>where</u> D <u>is a set dense everywhere on the real line.</u>

<u>Proof.</u> The necessity of the condition is obvious. The set of discontinuity points of $G(x)$ is at most countable. Thus C is dense everywhere. Let us prove the sufficiency of the condition. For any $x \varepsilon C$ let us choose two sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ in such a way that $a_k \varepsilon D$, $b_k \varepsilon D$ and $a_k \leqq x \leqq b_k$ for $k = 1,2,\ldots,$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = x$. Then

(71) $$G_n(a_k) \leqq G_n(x) \leqq G_n(b_k) .$$

If $n \to \infty$ in (71), then we obtain that

(72) $$G(a_k) \leqq \liminf_{n\to\infty} G_n(x) \leqq \limsup_{n\to\infty} G_n(x) \leqq G(b_k) .$$

Since

(73) $$\lim_{k\to\infty} G(a_k) = \lim_{k\to\infty} G(b_k) = G(x)$$

it follows from (72) that $\lim_{n\to\infty} G_n(x) = G(x)$ for any $x \varepsilon C$. This completes the proof of the lemma.

<u>Theorem 7</u>. <u>Let</u> $G_1(x), G_2(x), \ldots, G_n(x), \ldots$ <u>be a sequence of nondecreasing functions defined on the interval</u> $(-\infty, \infty)$. <u>Let us assume that</u> $G_n(\infty) < K$ <u>and</u> $G_n(-\infty) = 0$ <u>for all</u> $n = 1,2,\ldots$. <u>Then the sequence</u> $\{G_n(x)\}$ ______________________ <u>contains a subsequence</u> $\{G_{n_k}(x)\}$ <u>which converges weakly to a nondecreasing function</u> $G(x)$.

<u>Proof</u>. Let $D = \{a_1, a_2, \ldots, a_n, \ldots\}$ be a countable set which is dense everywhere in $(-\infty, \infty)$. For example, D may be chosen as the set of rational numbers. Since $0 \leqq G_n(a_1) \leqq K$ for $n = 1,2,\ldots$, by the Bolzano-Weierstrass theorem $\{G_n(a_1)\}$ contains a convergent subsequence $\{G_{n_k^{(1)}}(a_1)\}$. Since $0 \leqq G_{n_k^{(1)}}(a_2) \leqq K$ for $n = 1,2,\ldots$ by the Bolzano-Weierstrass theorem $\{G_{n_k^{(1)}}(a_2)\}$ also contains a convergent subsequence $\{G_{n_k^{(2)}}(a_2)\}$. Obviously $\{G_{n_k^{(2)}}(a_1)\}$ is also convergent. Continuing this procedure for every $r = 1,2,\ldots$ we can find a subsequence $\{G_{n_k^{(r)}}(x)\}$ such that it is convergent for $x = a_1, a_2, \ldots, a_r$ and the r-th sequence is a subsequence of the $r-1$ st one. By Cantor's method of diagonals, we can conclude that if $G_{n_k}(x) = G_{n_k^{(k)}}(x)$ for $k = 1,2,\ldots$, then $\{G_{n_k}(x)\}$ is convergent for all $x \in D$. Let us define

$$(74) \qquad G(x) = \lim_{k \to \infty} G_{n_k}(x)$$

for $x \in D$. Obviously $G(x)$ is nondecreasing on the set D . Thus $G(x+0)$ and $G(x-0)$ exist for every x and are uniquely determined by $G(x)$ for $x \in D$. If $x \notin D$, then let us define $G(x)$ as any real number satisfying the inequalities $G(x-0) \leq G(x) \leq G(x+0)$. Then $G(x)$ is a nondecreasing function of x defined on the interval $(-\infty,\infty)$ and $\lim_{n\to\infty} G_n(x) = G(x)$ for $x \in D$. By Lemma 1, it follows that $G_n(x) \Rightarrow G(x)$ which was to be proved.

We note that $0 \leq G(-\infty)$ and $G(+\infty) \leq K$ necessarily hold.

Theorem 8. *Let $\{G_n(x)\}$ be a sequence of nondecreasing functions, defined on the interval $(-\infty,\infty)$, which converges weakly to a nondecreasing function $G(x)$. Let us assume that $G_n(\infty) < K$ and $G_n(-\infty) = 0$ for all $n = 1,2,\ldots$. If*

$$\lim_{n\to\infty} G_n(-\infty) = G(-\infty) \quad \text{and} \quad \lim_{n\to\infty} G_n(+\infty) = G(+\infty) \tag{75}$$

and if $h(x)$ is a continuous and bounded function of x on the interval $(-\infty,\infty)$, then we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h(x)dG_n(x) = \int_{-\infty}^{\infty} h(x)dG(x) . \tag{76}$$

Proof. Let $|h(x)| < M$ for $-\infty < x < \infty$. Denote by C the set of continuity points of $G(x)$. For any $\varepsilon > 0$ let us choose $a \in C$ and $b \in C$ such that $G(a) < \varepsilon/M$ and $G(+\infty) - G(b) < \varepsilon/M$. Let us choose a sufficiently large number of points $x_k \in C$ $(k = 0,1,\ldots, m)$ such that $a = x_0 < x_1 < \ldots < x_m = b$ and

$$|h(x) - h(x_k)| < \frac{\varepsilon}{K} \tag{77}$$

if $x_k \leq x \leq x_{k+1}$ $(k = 0,1,\ldots, m-1)$. This can be achieved because $h(x)$ is uniformly continuous on the interval $[a,b]$. Let us choose N so large that if $n > N$, then

$$|G_n(x_k) - G(x_k)| < \frac{\varepsilon}{Mm} \tag{78}$$

for $k = 0,1,\ldots, m$ and $|G_n(\infty) - G(\infty)| < \varepsilon/M$.

Now we shall show that

$$\left|\int_{-\infty}^{\infty} h(x)dG_n(x) - \int_{-\infty}^{\infty} h(x)dG(x)\right| < 10\varepsilon \tag{79}$$

if $n > N$.

Let us define

$$h_\varepsilon(x) = h(x_k) \quad \text{for} \quad x_k \leqq x < x_{k+1} \tag{80}$$

and $k = 0,1,\ldots, m-1$. With this notation we can write that

$$\begin{aligned} \left|\int_{-\infty}^{\infty} hdG_n - \int_{-\infty}^{\infty} hdG\right| &\leqq \left|\int_{-\infty}^{a} hdG_n - \int_{-\infty}^{a} hdG\right| + \left|\int_{a}^{b} hdG_n - \int_{a}^{b} h_\varepsilon dG_n\right| \\ &+ \left|\int_{a}^{b} h_\varepsilon dG_n - \int_{a}^{b} h_\varepsilon dG\right| + \left|\int_{a}^{b} h_\varepsilon dG - \int_{a}^{b} hdG\right| + \left|\int_{b}^{\infty} hdG_n - \int_{b}^{\infty} hdG\right| . \end{aligned} \tag{81}$$

On the right hand side of (81) the first term is $\leqq M[G_n(a) + G(a)] \leqq 3\varepsilon$, the second term is $\leqq \varepsilon$, the fourth term is $\leqq \varepsilon$ and the fifth term is $\leqq M[G_n(\infty) - G_n(b) + G(\infty) - G(b)] \leqq 3\,\varepsilon$. We can easily prove that the third term on the right- hand side of (81) is $\leqq 2\,\varepsilon$, whence (79) follows. To prove the last statement let us observe that

$$(82)\quad \begin{aligned} \int_a^b h_\varepsilon dG_n - \int_a^b h_\varepsilon dG &= \sum_{k=0}^{m-1} h(x_k)[G_n(x_{k+1}) - G_n(x_k)] - \\ - \sum_{k=0}^{m-1} h(x_k)[G(x_{k+1}) - G(x_k)] &= \sum_{k=0}^{m-1} h(x_k)[G_n(x_{k+1}) - G(x_{k+1})] \\ &- \sum_{k=0}^{m-1} h(x_k)[G_n(x_k) - G(x_k)] \ . \end{aligned}$$

By (78) it follows immediately that the absolute value of (82) is $\leq 2\varepsilon$. This completes the proof of (79) which implies (76).

Finally, we note that if $\lim_{x\to+\infty} h(x) = \lim_{x\to-\infty} h(x) = 0$, then (76) is true without making the assumption (75).

Continuity Theorems. Let $\{F_n(x)\}$ is a sequence of distribution functions, and let

$$(83)\qquad \phi_n(s) = \int_{-\infty}^{\infty} e^{-sx} dF_n(x)$$

be the Laplace-Stieltjes transform of $F_n(x)$. If $\mathrm{Re}(s) = 0$, then (83) is absolutely convergent.

Theorem 9. *If $\{F_n(x)\}$ converges weakly to a distribution function $F(x)$ and*

$$(84)\qquad \phi(s) = \int_{-\infty}^{\infty} e^{-sx} dF(x)$$

is the Laplace-Stieltjes transform of $F(x)$, then

$$(85)\qquad \lim_{n\to\infty} \phi_n(s) = \phi(s)$$

We note that if $G_n(x) \Longrightarrow G(x)$ and (75) holds too, then we say that $G_n(x)$ converges weakly and completely to $G(x)$.

for $Re(s) = 0$, *and the convergence is uniform on* $Re(s) = 0$.

Proof. This theorem is a particular case of Theorem 8. For if $Re(s) = 0$, then $h(x) = e^{-sx}$ is a continuous function of x and $|h(x)| \leq 1$ for $-\infty < x < \infty$. Since $|e^{-sx}| \leq 1$ for every s with $Re(s) = 0$, it follows that $\phi_n(s) \to \phi(s)$ uniformly on $Re(s) = 0$.

Under certain conditions the converse of Theorem 9 is also valid. The following theorem was found by H. Cramér [503]. He improved an earlier version found in 1922 by P. Lévy [111 p. 197], [110] .

Theorem 10. *If*

(86) $$\lim_{n\to\infty} \phi_n(s) = \phi(s)$$

for every s *for which* $Re(s) = 0$ *and*

(87) $$\lim_{s\to 0} \phi(s) = 1 ,$$

then $\{F_n(x)\}$ *converges weakly to a distribution function* $F(x)$ *and*

(88) $$\int_{-\infty}^{\infty} e^{-sx} dF(x) = \phi(s)$$

for $Re(s) = 0$.

Proof. By Theorem 7 it follows that every subsequence of $\{F_n(x)\}$ contains a subsequence $\{F_{n_k}(x)\}$ which converges weakly to a nondecreasing function $F(x)$ and $0 \leq F(-\infty)$ and $F(\infty) \leq 1$. Without loss of generality we may assume that $F(x+0) = F(x)$ for every x . We shall prove that $F(\infty) - F(-\infty) \geq 1$, whence it follows that $F(x)$ is a distribution function.

We can easily see that for any $\varepsilon > 0$ we have the inequality

(89) $$F_{n_k}\left(\frac{2}{\varepsilon}\right) - F_{n_k}\left(-\frac{2}{\varepsilon}\right) \geqq 2\left|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\phi_{n_k}(iu)du\right| - 1 .$$

If $k \to \infty$, then $\phi_{n_k}(s) \to \phi(s)$ for $Re(s) = 0$ and $|\phi_{n_k}(s)| \leq 1$ for $Re(s) = 0$. Thus if $x = 2/\varepsilon$ and $x = -2/\varepsilon$ are continuity points of $F(x)$, then by (89) it follows that

(90) $$F(\infty)-F(-\infty) \geqq F\left(\frac{2}{\varepsilon}\right) - F\left(-\frac{2}{\varepsilon}\right) \geqq 2\left|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\phi(iu)du\right| - 1 .$$

Since by (87) $\lim_{s \to 0} \phi(s) = 1$, it follows that the extreme right member of (90) tends to 1 if $\varepsilon \to 0$. This implies that $F(\infty) - F(-\infty) \geqq 1$, that is, $F(x)$ is a distribution function.

Accordingly, we proved that $F_{n_k}(x) \Rightarrow F(x)$ where $F(x)$ is a distribution function. We can apply Theorem 8 to the sequence $\{F_{n_k}(x)\}$ to obtain that

(91) $$\lim_{k\to\infty}\int_{-\infty}^{\infty} e^{-sx}dF_{n_k}(x) = \int_{-\infty}^{\infty} e^{-sx}dF(x)$$

for $Re(s) = 0$. By (86) and (91) we can conclude that

(92) $$\int_{-\infty}^{\infty} e^{-sx}dF(x) = \phi(s) \quad \text{for} \quad Re(s) = 0 ,$$

that is, $\phi(s)$ is necessarily the Laplace-Stieltjes transform of a distribution function. The distribution function $F(x)$ is uniquely determined by $\phi(s)$. (See Theorem 5.) Thus the distribution function $F(x)$ does not depend on the particular sequence $\{F_{n_k}(x)\}$. For every subsequence of $\{F_n(x)\}$ contains a subsequence which converges weakly to the same distribution function $F(x)$. This implies that the whole sequence $\{F_n(x)\}$ converges weakly to $F(x)$.

This last statement can be proved by contradiction. Let us suppose that $x = a$ is a continuity point of $F(x)$ and $\lim_{n\to\infty} F_n(a) \neq F(a)$. We shall show that this assumption leads to a contradiction. If $\lim_{n\to\infty} F_n(a) \neq F(a)$, then for some $\varepsilon > 0$ there are infinitely many m_i $(i = 1,2,\ldots)$ such that $|F_{m_i}(a) - F(a)| > \varepsilon$. By Theorem 7 the sequence of distribution functions $\{F_{m_i}(x)\}$ contains a subsequence $\{F_{m_i'}(x)\}$ which converges weakly to a distribution function. By our previous result this distribution function is necessarily $F(x)$. Thus $\lim_{n\to\infty} F_{m_i'}(a) = F(a)$. However, this contradicts to $|F_{m_i'}(a) - F(a)| > \varepsilon$ for $i = 1,2,\ldots$. This contradiction proves the last statement.

Note. If we suppose that $\{F_n(x)\}$ is a sequence of distribution functions of nonnegative random variables and if

$$(93)\qquad \lim_{n\to\infty} \phi_n(s) = \phi(s)$$

for $s \varepsilon I$ where I is an interval of positive length on the imaginary axis $\mathrm{Re}(s) = 0$, which contains the point $s = 0$, and if $\lim_{s\to 0} \phi(s) = 1$, then $\{F_n(x)\}$ converges weakly to a distribution function $F(x)$ of a nonnegative random variable and

$$(94)\qquad \int_{-0}^{\infty} e^{-sx}dF(x) = \lim_{n\to\infty} \phi_n(s)$$

for $\mathrm{Re}(s) \geqq 0$.

This last result was found in 1950 by A. Zygmund [226].

Finally, we shall mention another theorem for the weak convergence of distribution functions. This theorem is based on the convergence of the moments of the distribution functions.

We define the r-th moment $(r = 0,1,2,\ldots)$ of a distribution function $F(x)$ as

$$M_r = \int_{-\infty}^{\infty} x^r dF(x) \tag{95}$$

provided that

$$\int_{-\infty}^{\infty} |x|^r dF(x) < \infty . \tag{96}$$

If we know

$$\psi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x}\, dF(x) , \tag{97}$$

the characteristic function of $F(x)$, for $-\infty < \omega < \infty$, then we can determine the r-th moment of $F(x)$ by

$$M_r = \frac{1}{i^r} \left(\frac{d^r \psi(\omega)}{d\omega^r}\right)_{\omega=0} \tag{98}$$

for $r = 0,1,\ldots$ provided that M_r exists.

If M_r $(r = 1,2,\ldots)$ exists, then

$$\Lambda_r = \frac{1}{i^r} \left(\frac{d^r \log\psi(\omega)}{d\omega^r}\right)_{\omega=0} \tag{99}$$

also exists and is called the r-th semiinvariant of $F(x)$. (See T. N. Thiele [200].)

We can prove easily that

$$(100)\quad \Lambda_r = r! \sum_{\nu=1}^{r} (-1)^{\nu-1}(\nu-1)! \sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_r=\nu \\ \alpha_1+2\alpha_2+\cdots+r\alpha_r=r}} \frac{M_1^{\alpha_1} M_2^{\alpha_2} \cdots M_r^{\alpha_r}}{\alpha_1!\alpha_2!\cdots\alpha_r!(2!)^{\alpha_2}\cdots(r!)^{\alpha_r}}$$

for $r = 1,2,\ldots$ where $\alpha_i = 0,1,2,\ldots,$ and

$$(101)\quad M_r = r! \sum_{\alpha_1+2\alpha_2+\cdots+r\alpha_r=r} \frac{\Lambda_1^{\alpha_1} \Lambda_2^{\alpha_2} \cdots \Lambda_r^{\alpha_r}}{\alpha_1!\alpha_2!\cdots\alpha_r!(2!)^{\alpha_2}\cdots(r!)^{\alpha_r}}$$

for $r = 1,2,\ldots$ where $\alpha_i = 0,1,2,\ldots$.

We mention here that if

$$(102)\quad \limsup_{m\to\infty} \frac{(M_{2m})^{1/2m}}{m} < \infty$$

or if

$$(103)\quad \sum_{m=1}^{\infty} \frac{1}{(M_{2m})^{1/2m}} = \infty ,$$

then the sequence of moments $\{M_r\}$ determines $F(x)$ uniquely. This is not always the case. In 1894 T. J. Stieltjes [191] gave several examples for infinitely many distribution functions which have the same sequence of moments.

Theorem 11. *Let $\{F_n(x)\}$ be a sequence of ditribution functions for which the moments*

$$(104)\quad M_r(n) = \int_{-\infty}^{\infty} x^r dF_n(x)$$

exist for all $r = 0,1,2,\ldots$. *Furthermore, let* $F(x)$ *be a distribution* function for which the moments

$$M_r = \int_{-\infty}^{\infty} x^r dF(x) \tag{105}$$

exist for all $r = 0, 1, 2, \ldots$. If

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} x^r dF_n(x) = \int_{-\infty}^{\infty} x^r dF(x) \tag{106}$$

for all $r = 0,1,2,\ldots$ and if $F(x)$ is uniquely determined by the sequence of moments $M_0, M_1, M_2, \ldots$, then

$$\lim_{n \to \infty} F_n(x) = F(x) \tag{107}$$

in every continuity point of $F(x)$.

Proof. By Theorem 7 the sequence of distribution functions $\{F_n(x)\}$ is weakly compact, that is, every infinite subsequence of $\{F_n(x)\}$ contains a subsequence $\{F_{n_j}(x)\}$ which converges weakly to a nondecreasing function $G(x)$ as $j \to \infty$ and $0 \leqq G(-\infty)$ and $G(+\infty) \leqq 1$. Without loss of generality we may assume that $G(x+0) = G(x)$ for every x . We shall prove that $G(+\infty)-G(-\infty) \geqq 1$, whence it follows that $G(x)$ is a distribution function.

By Theorem 3 we have

$$F_{n_j}(a) - F_{n_j}(-a) \geqq 1 - \frac{M_2(n_j)}{a^2} \tag{108}$$

for all $a > 0$. If $x = a$ and $x = -a$ are continuity points of $G(x)$ and if we let $j \to \infty$ in (108), then we obtain that

$$G(a) - G(-a) \geqq 1 - \frac{M_2}{a^2} . \tag{109}$$

If $a \to \infty$, then the right-hand side of (109) tends to 1, and therefore $G(+\infty) - G(-\infty) \geqq 1$. Accordingly $G(x)$ is a distribution function.

Now we shall prove that

$$(110) \qquad \lim_{j \to \infty} \int_{-\infty}^{\infty} x^r dF_{n_j}(x) = \int_{-\infty}^{\infty} x^r dG(x)$$

for all $r = 0,1,2,\dots$.

If $x = a$ and $x = -a$ are continuity points of $G(x)$, then by Theorem 8 we have

$$(111) \qquad \lim_{j \to \infty} \int_{-a}^{a} x^r dF_{n_j}(x) = \int_{-a}^{a} x^r dG(x) .$$

On the other hand we have

$$(112) \qquad \left| \int_{|x| \geqq a} x^r dF_{n_j}(x) \right| \leqq \int_{|x| \geqq a} |x|^r dF_{n_j}(x) \leqq \frac{1}{a^r} \int_{|x| \geqq a} x^{2r} dF_{n_j}(x) \leqq \frac{M_{2r}(n_j)}{a^r}$$

for all $a > 0$ and $j = 1,2,\dots$. If $j \to \infty$, then the extreme right member in (112) tends to M_{2r}/a^r . By (111) and (112) we can conclude that for any $\varepsilon > 0$

$$(113) \qquad \left| \int_{-\infty}^{\infty} x^r dF_{n_j}(x) - \int_{-a}^{a} x^r dG(x) \right| < \varepsilon$$

if j and a are sufficiently large. This proves (110).

By (106) and (110) it follows that

(114) $$\int_{-\infty}^{\infty} x^r dG(x) = \int_{-\infty}^{\infty} x^r dF(x)$$

for all $r = 0,1,2,\ldots$. Hence $G(x) \equiv F(x)$. Thus we proved that every subsequence of $\{F_n(x)\}$ contains a subsequence which converges weakly to the same distribution function $F(x)$. This implies that the whole sequence $\{F_n(x)\}$ converges weakly to $F(x)$. This completes the proof of the theorem.

We note that if $\Lambda_r(n)$ denotes the r-th semiinvariant of $F_n(x)$ and Λ_r the r-th semiinvariant of $F(x)$, then obviously the condition (106) can be replaced by

(115) $$\lim_{n \to \infty} \Lambda_r(n) = \Lambda_r$$

for all $r = 1,2,\ldots$.

In the particular case when $F(x) = \Phi(x)$, the normal distribution function, in 1890 P. L. Chebyshev [616] proved that $\Phi(x)$ is uniquely determined by its sequence of moments and used Theorem 11 in his investigations in the theory of probability. However, he did not prove this theorem. The theorem was proved only in 1898 by A. A. Markov [580] (See also J. V. Uspensky [204 pp. 383-388].) Under the condition (102) Theorem 11 was proved in 1920 by G. Pólya [596] . In the general case Theorem 11 was proved in 1931 by M. Fréchet and J. Shohat [523] .

42. Infinitely Divisible Distributions and Stable Distributions. The notion of infinitely divisible distributions and stable distributions play an important role in finding all the possible limiting distributions of suitably normalized sums of mutually independent random variables and in finding all the possible distributions for stochastic processes with independent increments.

The definition of infinitely divisible distributions and stable distributions is based on the notion of convolution.

The convolution of two distribution functions $G(x)$ and $H(x)$ is defined as

$$(1) \qquad G(x)*H(x) = H(x)*G(x) = \int_{-\infty}^{\infty} G(x-y)dH(y) = \int_{-\infty}^{\infty} H(x-y)dG(y) .$$

Definition 1. A distribution function $F(x)$ is called infinitely divisible if for every $n = 1,2,\ldots,$ it can be represented as the n-th iterated convolution of a distribution function with itself, that is, if for every $n = 1,2,\ldots$ there exists a distribution function $F_n(x)$ such that

$$(2) \qquad F(x) = F_n(x)*\ldots*F_n(x)$$

n times.

Definition 2. A distribution function $F(x)$ is called stable if for every $a_1, a_2, b_1 > 0, b_2 > 0$ there exist two constants a and $b > 0$ such that

(3) $$F(a_1+b_1x) * F(a_2+b_2x) = F(a+bx)$$

holds.

Obviously every stable distribution function is infinitely divisible, whereas the converse is not true in general.

Let us mention a few examples. The normal distribution function

(4) $$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

is infinitely divisible and stable. The Cauchy distribution function

(5) $$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x$$

is infinitely divisible and stable. The gamma distribution function

(6) $$G_a(x) = \begin{cases} \frac{1}{\Gamma(a)} \int_0^x e^{-u} u^{a-1} du & \text{for } x \geqq 0 , \\ 0 \text{ for } x < 0 , \end{cases}$$

where $a > 0$, is infinitely divisible, but it is not stable. The Poisson distribution function

(7) $$F(x) = \sum_{0 \leqq j \leqq x} e^{-a} \frac{a^j}{j!} ,$$

where $a > 0$, is infinitely divisible, but it is not stable.

Our aim is to find the most general form of an infinitely divisible distribution function and the most general form of a stable distribution function. We shall find explicitely the Laplace-Stieltjes transform

$$\phi(s) = \int_{-\infty}^{\infty} e^{-sx} dF(x) , \tag{8}$$

where $Re(s) = 0$, for an infinitely divisible distribution function $F(x)$ and, in particular, for a stable distribution function $F(x)$.

Infinitely Divisible Distribution Functions. Let us suppose that $F(x)$ is an infinitely divisible distribution function. Let the Laplace-Stieltjes transform of $F(x)$ be defined by (8) for $Re(s) = 0$. For every $n = 1,2,\ldots$ there exists a distribution function $F_n(x)$ such that (2) is satisfied. Let us denote by

$$\phi_n(s) = \int_{-\infty}^{\infty} e^{-sx} dF_n(x) \tag{9}$$

the Laplace- Stieltjes transform of $F_n(x)$, which exists if $Re(s) = 0$.

By (2) we have

$$\phi(s) = [\phi_n(s)]^n \tag{10}$$

for $n = 1,2,\ldots$.

Now we shall show that $\phi_n(s)$ is uniquely determined by (10) for $Re(s) = 0$ and $n = 1,2,\ldots$.

This follows from the following auxiliary theorem.

Lemma 1. *If* $\phi(s)$ *is the Laplace-Stieltjes transform of an infinitely divisible distribution function, then* $\phi(s)$ *never vanishes on* $Re(s) = 0$.

Proof. Since $\phi(0) = 1$ and $\phi(s)$ is continuous on $Re(s) = 0$, there is an $a > 0$ such that $|\phi(s)| > 0$ for $s = iu$ and $|u| \leqq a$. Hence by (10)

$$\lim_{n\to\infty} \phi_n(s) = \lim_{n\to\infty} |\phi(s)|^{\frac{1}{n}} = 1 \tag{11}$$

for $s = iu$ and $|u| \leqq a$. If we write $\phi(s) = |\phi(s)|\, e^{i\theta(s)}$, then it follows from (11) that

$$\lim_{n\to\infty} \phi_n(s) = \lim_{n\to\infty} |\phi(s)|^{\frac{1}{n}}\, e^{\frac{i\theta(s)}{n}} = 1 \tag{12}$$

for $s = iu$ and $|u| \leqq a$.

Since $|\phi_n(s)| \leqq 1$ for $Re(s) = 0$ and

$$1-Re(\phi_n(2s)) \leqq 4[1-Re(\phi_n(s))] \tag{13}$$

for $Re(s) = 0$, it follows from (12) that

$$\lim_{n\to\infty} \phi_n(s) = 1 \tag{14}$$

for $s = iu$ and $|u| \leqq 2a$. By doubling the interval $|u| \leqq a$ as many times as we like, we obtain that

$$\lim_{n\to\infty} \phi_n(s) = 1 \tag{15}$$

for $Re(s) = 0$. This excludes the possibility that $\phi(s) = 0$ for some $Re(s) = 0$. If we would have $\phi(s) = 0$ for some s with $Re(s) = 0$, then by (10) it would follow that $\phi_n(s) = 0$ for the same s and for all $n = 1,2,\ldots$. This, however, would contradict to (15). Hence the lemma follows.

Since $\phi_n(0) = 1$ and $\phi_n(s)$ is continuous for $\text{Re}(s) = 0$, we can write that

$$\phi_n(s) = [\phi(s)]^{\frac{1}{n}} \tag{16}$$

for $\text{Re}(s) = 0$ and $n = 1,2,\ldots$ where the right-hand side is the principal branch for which $\phi_n(0) = 1$. Since $\phi(s)$ never vanishes for $\text{Re}(s) = 0$, the Laplace-Stieltjes transform $\phi_n(s)$ is uniquely determined for all $\text{Re}(s) = 0$.

Theorem 1. The function $\phi(s)$ defined for $\text{Re}(s) = 0$ is the Laplace-Stieltjes transform of an infinitely divisible distribution function $F(x)$ if and only if $\log \phi(s)$ can be represented in the form

$$\log \phi(s) = -\mu s + \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG(x) \tag{17}$$

where μ is a real constant, $G(x)$ is a nondecreasing function for which $G(-\infty) = 0$, $G(+\infty)$ is finite, and the integrand at $x = 0$ is defined by

$$\left[\left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\right]_{x=0} = \frac{s^2}{2}. \tag{18}$$

The representation of $\log \phi(s)$ by the formula (17) is unique.

Proof. First we shall prove that the condition is necessary, that is, if $\phi(s)$ is the Laplace-Stieltjes transform of an infinitely divisible distribution function, then $\log \phi(s)$ can be represented in the form (17). We assume that $\log\phi(0) = 0$ and $\log\phi(s)$ is continuous on $\text{Re}(s) = 0$.

If $\phi_n(s)$ is defined by (9) for $n = 1,2,\ldots$, then by (15) we have

(19) $$\lim_{n\to\infty} \phi_n(s) = 1$$

for all s with $\mathrm{Re}(s) = 0$. Thus by Theorem 41.10 it follows that

(20) $$\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

If we refer to Theorem 41.9, then we can conclude that in (19) the convergence is uniform on $\mathrm{Re}(s) = 0$.

Let

(21) $$I_n(s) = n[\phi_n(s)-1] = n\int_{-\infty}^{\infty} (e^{-sx}-1)\,dF_n(x)$$

for $\mathrm{Re}(s) = 0$. Then

(22) $$\lim_{n\to\infty} I_n(s) = \log\phi(s)$$

and

(23) $$\lim_{n\to\infty} \mathrm{Re}[I_n(s)] = \log|\phi(s)|$$

for every s with $\mathrm{Re}(s) = 0$. The convergence is uniform in every finite interval of $\mathrm{Re}(s) = 0$.

Define

(24) $$G_n(x) = n\int_{-\infty}^{x} \frac{y^2}{1+y^2}\,dF_n(y) .$$

Then

Since $|I_n(s)-\log\phi(s)| \leq |\log\phi(s)|^2/n$ for $\mathrm{Re}(s) = 0$, and for sufficiently large n values, it follows that

$$I_n(s) = \int_{-\infty}^{\infty} (e^{-sx}-1)\,\frac{1+x^2}{x^2}\,dG_n(x) \tag{25}$$

for $\operatorname{Re}(s) = 0$.

The function $G_n(x)$ is nondecreasing in the interval $(-\infty,\infty)$, $G_n(-\infty) = 0$ and $G_n(\infty) \leqq n$. We shall prove that actually $\{G_n(\infty)\}$ is bounded. However, first we deduce some inequalities which will be useful. Since

$$\frac{\sin x}{x} \leqq 1-\frac{x^2}{8} \text{ for } 0 \leqq x \leqq 2 \text{ and } \frac{\sin x}{x} \leqq \frac{1}{2} \text{ for } x \geqq 2 , \tag{26}$$

it follows that

$$\frac{8}{\delta^3}\int_0^{\delta}(1-\cos y\,u)du = \frac{8}{\delta^2}\left(1-\frac{\sin \delta y}{\delta y}\right) \geqq \begin{cases} y^2 & \text{if } |y| \leqq \frac{2}{\delta} \\ \frac{4}{\delta^2} & \text{if } |y| \geqq \frac{2}{\delta} \end{cases} \tag{27}$$

for $\delta > 0$. From (27) we obtain that

$$\frac{8}{\delta^3}\int_0^{\delta}(1-\cos yu)du \geqq \frac{y^2}{1+y^2} \tag{28}$$

for $0 \leqq \delta \leqq 2$ and

$$\frac{2}{\delta}\int_0^{\delta}(1-\cos yu)du \geqq 1 \geqq \frac{y^2}{1+y^2} \tag{29}$$

if $|y| \geqq 2/\delta$.

If we use (28) with $\delta = 2$, then we can write that

$$(30) \qquad G_n(\infty) = n \int_{-\infty}^{\infty} \frac{y^2}{1+y^2}\, dF_n(y) \leq \int_0^2 [n \int_{-\infty}^{\infty} (1-\cos yu)\, d_y F_n(y)]du =$$

$$= - \int_0^2 \mathrm{Re}[I_n(\ iu)]du \ .$$

Since $\lim\limits_{n\to\infty} I_n(\ iu) = \log|\phi(\ iu)|$ uniformly for $u \in [0,2]$, it follows Re

that $\{G_n(\infty)\}$ is bounded.

By Theorem 41.7 we can find a subsequence $\{G_{n_k}(x)\}$ of $\{G_n(x)\}$ and a nondecreasing function $G(x)$ for which $0 \leq G(-\infty)$ and $G(\infty) < \infty$ such that $G_{n_k}(x) \Rightarrow G(x)$. Now we shall prove that

$$(31) \qquad \lim_{k\to\infty} G_{n_k}(\infty) = G(\infty) \quad \text{and} \quad \lim_{k\to\infty} G_{n_k}(-\infty) = G(-\infty)$$

also hold. We shall prove that for any $\varepsilon > 0$

$$(32) \qquad \int_{|x|>a} d_x G_n(x) < \varepsilon$$

if $a > 0$ and n are sufficiently large. This implies (31).

If we put $\delta = 2/a$ in (29), then we obtain the following inequality

$$(33) \qquad \int_{|x|>a} dG_n(x) = n \int_{|y|>a} \frac{y^2}{1+y^2}\, dF_n(y) \leq a \int_0^{2/a} [n \int_{|y|>a} (1-\cos yu) dF_n(u)]dy \leq$$

$$\leq - a \int_0^{2/a} \mathrm{Re}[I_n(\ iu)]du \ .$$

Now $\lim\limits_{n\to\infty} \mathrm{Re}[I_n(\ iu)] = \log|\phi(\ iu)|$ uniformly for $u \in [0,\ 2/a]$ and $\log|\phi(\ iu)| \to 0$ as $u \to 0$. Hence it follows that (32) is valid if $a > 0$ and n are sufficiently large.

By (25) we can write that

$$I_{n_k}(s) = \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dG_{n_k}(x) - \mu_{n_k} s \tag{34}$$

$$\frac{1+x^2}{x^2}$$

where

$$\mu_{n_k} = \int_{-\infty}^{\infty} \frac{1}{x}\, d\, G_{n_k}(x) = n_k \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, d\, F_{n_k}(x) \,. \tag{35}$$

By (22) we have

$$\lim_{k\to\infty} I_{n_k}(s) = \log \phi(s) \tag{36}$$

for $\operatorname{Re}(s) = 0$. If we define the integrand in (34) for $x = 0$ by (18), then the integrand in (34) is a continuous and bounded function of x in the interval $(-\infty,\infty)$. Consequently by Theorem 41.8 we obtain that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\, d\, G_{n_k}(x) = \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG(x) \tag{37}$$

for $\operatorname{Re}(s) = 0$. By (36) and (37) it follows that in (34) $\lim_{k\to\infty} \mu_{n_k} = \mu$ necessarily exists and μ is finite. Accordingly, we have

$$\log \phi(s) = -\mu s + \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG(x) \tag{38}$$

for $\operatorname{Re}(s) = 0$ where μ is a real constant, and $G(x)$ is a nondecreasing function of x for which $G(-\infty) = 0$ and $G(\infty) < \infty$. This proves that (17) necessarily holds.

Now we shall prove that $\phi(s)$ uniquely determines $G(x)$ for every continuity point and the constant μ .

Let

$$\psi(s) = 2 \log\phi(s) + \int_{is-1}^{is+1} \log\phi(iu)du \tag{39}$$

for $Re(s) = 0$, that is,

$$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} dH(x) \tag{40}$$

for $Re(s) = 0$ where

$$H(x) = 2 \int_{-\infty}^{x} (1 - \frac{\sin y}{y}) \frac{1+y^2}{y^2} dG(y) . \tag{41}$$

The function $H(x)$ is nondecreasing, $H(-\infty) = 0$ and $H(\infty) < \infty$. By Theorem 41.5 the function $H(x)$ is uniquely determined for each of its continuity points by $\psi(s)$ and therefore by $\phi(s)$. Since

$$(1 - \frac{\sin x}{x}) \frac{1+x^2}{x^2} > 0 \tag{42}$$

for all x , the function $G(x)$ is also uniquely determined for each of its continuity points.

Finally, the constant μ is uniquely determined by forming the difference between the integral in (38) and $\log \phi(s)$.

This proves the uniqueness of the representation (17).

It remains to prove that the condition is sufficient too, that is, if $\log \phi(s)$ is given by (17), then $\phi(s)$ is the Laplace-Stieltjes transform of an infinitely divisible distribution function. It is sufficient to prove that $\log \phi(s)$ is the logarithm of the Laplace-Stieltjes transform of a distribution function for any real constant μ and for any nondecreasing function $G(x)$ for which $G(-\infty) = 0$ and $G(\infty)$ is finite. For in this case if we replace μ by μ/n and $G(x)$ by $G(x)/n$ then we obtain that $\log \phi(s)/n$ is the logarithm of the Laplace-Stieltjes transform of a distribution function for all $n = 1,2,\ldots$. This proves the infinitely divisibility.

To prove that $\phi(s)$ defined by (17) is a Laplace-Stieltjes transform of a distribution function we observe that for any choice of the real numbers $x_0 < x_1 < x_2 < \ldots < x_m$ the function

$$(43) \quad \log \psi_m(s) = -\mu s + \sum_{k=1}^{m} \left(e^{-sx_k} - 1 + \frac{sx_k}{1+x_k^2}\right) \frac{1+x_k^2}{x_k^2} [G(x_k) - G(x_{k-1})]$$

defined for $\mathrm{Re}(s) = 0$ is the logarithm of the Laplace-Stieltjes transform of a distribution function.

For every $m = 1,2,\ldots$ we can choose $\psi_m(s)$ in such a way that $\lim_{m\to\infty} \log \psi_m(s) = \log \phi(s)$ for $\mathrm{Re}(s) = 0$. Since $\lim_{s\to 0} \phi(s) = 1$, it follows by Theorem 41.10 that $\phi(s)$ is necessarily the Laplace-Stieltjes transform of a distribution function. This completes the proof of the theorem.

From Theorem 1 it follows easily another representation of $\log \phi(s)$.

Theorem 2. The function $\phi(s)$ defined for $\mathrm{Re}(s) = 0$ is the Laplace-Stieltjes transform of an infinitely divisible distribution function $F(x)$ if and only if $\log \phi(s)$ can be represented in the form

$$(44) \qquad \log \phi(s) = -\mu s + \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{-0} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dM(x) + \int_{+0}^{+\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dN(x)$$

where μ is a real constant, σ^2 is a nonnegative constant, $M(x)$ is a nondecreasing function of x in the interval $(-\infty, 0)$, $N(x)$ is a nondecreasing function of x in the interval $(0, \infty)$ and these functions satisfy the requirements

$$(45) \qquad \lim_{x \to -\infty} M(x) = \lim_{x \to +\infty} N(x) = 0 ,$$

and

$$(46) \qquad \int_{-\varepsilon}^{-0} x^2 dM(x) + \int_{+0}^{\varepsilon} x^2 dN(x) < \infty$$

for some $\varepsilon > 0$. The representation (44) is unique.

Proof. Let us suppose that $\log \phi(s)$ is given by (17). Let us define

$$(47) \qquad M(x) = \int_{-\infty}^{x} \frac{1+y^2}{y^2} dG(y)$$

for $x < 0$,

$$(48) \qquad N(x) = -\int_{x}^{\infty} \frac{1+y^2}{y^2} dG(y)$$

for $x > 0$, and

$$(49) \qquad \sigma^2 = G(+0) - G(-0) .$$

In this case the function $M(x)$ is nondecreasing in $(-\infty, 0)$, $N(x)$ is nondecreasing in $(0, +\infty)$. The functions $M(x)$ and $N(x)$ are continuous at those and only those points at which $G(x)$ is continuous.

Obviously $M(-\infty) = N(+\infty) = 0$ and

$$(50) \qquad \int_{-\varepsilon}^{-0} x^2 dM(x) + \int_{+0}^{\varepsilon} x^2 dN(x) < \infty$$

for any finite $\varepsilon > 0$.

By Theorem 1 we can conclude that μ, σ^2 and the functions $M(x)$ and $N(x)$ for their continuity points are uniquely determined by $\phi(s)$.

Conversely, any two functions $M(x)$ and $N(x)$ satisfying the conditions in Theorem 2, and any real constant μ and nonnegative constant σ^2 determine the logarithm of the Laplace-Stieltjes transform of an infinitely divisible distribution function by formula (44). We note that

Several examples for infinitely divisible distributions have been known for a long time. Such are the normal distribution, the Cauchy distribution, the Poisson distribution and the gamma distribution. In 1929 B. De Finetti [241], [242], [243] found a class of infinitely divisible distributions in his studies of stochastic processes with independent increments. In 1932 A. N. Kolmogorov [280], [281] determined the most general form of $\log \phi(s)$ for infinitely divisible distribution functions with a finite variance. In 1934 P. Lévy [288] determined the most general form of $\log \phi(s)$ for arbitrary infinitely divisible distribution functions. The formula of P. Lévy is given by (44) in this section. Formula (17) for $\log \phi(s)$ was deduced in 1937 by A. Ya. Khintchine [273], [277]. For a comprehensive study of infinitely divisible distributions we refer to B. V. Gnedenko and A. N. Kolmogorov [260].

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if (46) holds for some $\varepsilon > 0$, then it holds for every $\varepsilon > 0$.

Stable Distribution Functions. Let $\phi(s)$ be the Laplace-Stieltjes transform of a stable distribution function $F(x)$. Our aim is to determine the most general form of $\phi(s)$. By Definition 1 it follows that $\phi(s)$ is the Laplace-Stieltjes transform of a stable distribution function $F(x)$ if and only if for every $b_1 > 0$ and $b_2 > 0$ there exist two constants a and $b > 0$ such that

$$(51)\qquad \log \phi\left(\frac{s}{b_1}\right) + \log \phi\left(\frac{s}{b_2}\right) = \log \phi\left(\frac{s}{b}\right) + as$$

for $Re(s) = 0$. On the other hand every stable distribution function is necessarily infinitely divisible, and therefore $\log \phi(s)$ can be expressed in the form of (44). The problem is to determine what conditions should we impose on μ, σ^2, $M(x)$ and $N(x)$ in order that $\log \phi(s)$ satisfy (51). This problem was solved in 1936 by A. Ya. Khintchine and P. Lévy [279]. See also A.Ya.Khintchine [275].

From Theorem 2 we can deduce the following result.

Theorem 3. The function $\phi(s)$ defined for $Re(s) = 0$ is the Laplace-Stieltjes transform of a stable distribution function if and only if $\log \Phi(s)$ can be represented in the form

$$(52)\qquad \log \phi(s) = -\mu s + \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{-0} \left(e^{-sx}-1 + \frac{sx}{1+x^2}\right)dM(x) + \int_{+0}^{+\infty} \left(e^{-sx}-1 + \frac{sx}{1+x^2}\right)dN(x)$$

where either μ is a real constant, σ^2 is a nonnegative constant, $M(x) \equiv 0$ for $x < 0$ and $N(x) \equiv 0$ for $x > 0$, or μ is a real constant, $\sigma^2 = 0$ and

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in 1925 by P. Lévy [111] and some details were proved

(53) $$M(x) = \frac{c_1}{|x|^\alpha} \ \underline{\text{for}} \ x < 0 \ , \ N(x) = -\frac{c_2}{x^\alpha} \ \underline{\text{for}} \ x > 0 \ ,$$

__where__ $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 > 0$, __and__ $0 < \alpha < 2$.

__Proof.__ Without loss of generality we may assume that in (52) $M(x+0) = M(x)$ for $x < 0$ and $N(x+0) = N(x)$ for $x > 0$. Then $M(x)$ for $x < 0$ and $N(x)$ for $x > 0$ are uniquely determined by $\phi(s)$. In this case by (51) we obtain that for every $b_1 > 0$ and $b_2 > 0$ there exists a $b > 0$ such that $M(x)$ and $N(x)$ satisfy the relations

(54) $$M(b_1 x) + M(b_2 x) = M(bx) \quad \text{for} \quad x < 0$$

and

(55) $$N(b_1 x) + N(b_2 x) = N(bx) \quad \text{for} \quad x > 0 \ .$$

Furthermore, we have

(56) $$\sigma^2\left(\frac{1}{b_1^2} + \frac{1}{b_2^2} - \frac{1}{b^2}\right) = 0 \ .$$

It is not necessary to impose any restriction on μ .

First, let us suppose that $M(x) \equiv 0$ for $x < 0$ and $N(x) \equiv 0$ for $x > 0$. Then (54) and (55) are satisfied with any $b > 0$, and (56) is satisfied with $b = b_1 b_2 / \sqrt{b_1^2 + b_2^2}$ if $\sigma^2 > 0$ and with any $b > 0$ if $\sigma^2 = 0$. In this case

(57) $$\log \phi(s) = -\mu s + \frac{\sigma^2 s^2}{2}$$

for $\text{Re}(s) = 0$ where μ is a real constant and σ^2 is a nonnegative constant.

If $\sigma^2 > 0$, then

$$(58) \qquad F(x) = \Phi(\frac{x-\mu}{\sigma})$$

is a normal distribution function, whereas if $\sigma^2 = 0$,then

$$(59) \qquad F(x) = \begin{cases} 1 & \text{for } x \geq \mu , \\ 0 & \text{for } x < \mu. \end{cases}$$

In both cases, (58) and (59), $F(x)$ is a stable distribution function.

Second, let us suppose that $M(x) \neq 0$ for some $x < 0$ or $N(x) \neq 0$ for some $x > 0$. We shall show that in this case $\sigma^2 = 0$ and $M(x)$ and $N(x)$ are given by (53).

Let $N(x) \neq 0$ for some $x > 0$. By (55) it follows that for every $b_1 > 0, b_2 > 0, \ldots, b_k > 0$ there exists a $b > 0$ such that

$$(60) \qquad N(b_1 x) + N(b_2 x) + \ldots + N(b_k x) = N(bx)$$

for all $x > 0$. If $b_1 = b_2 = \ldots = b_k = 1$, then let us denote by $\rho(k)$ the corresponding b , that is, for every $k = 1,2,\ldots,$ there exists a $\rho(k) > 0$ such that

$$(61) \qquad kN(x) = N(\rho(k)x)$$

for all $x > 0$.

Now we shall show that

$$(62) \qquad \rho(k\ell) = \rho(k)\rho(\ell)$$

for $k = 1,2,\ldots$ and $\ell = 1, 2,\ldots$.

On the one hand $k\ell N(x) = N(\rho(k\ell)x)$, and on the other hand $k\ell N(x) =$ $= kN(\rho(\ell)x) = N(\rho(k)\rho(\ell)x)$ for $x > 0$. Thus

$$N(\rho(k\ell)x) = N(\rho(k)\rho(\ell)x) \tag{63}$$

for all $k = 1,2,\ldots$, $\ell = 1,2,\ldots$, and $x > 0$. If $N(x) \neq 0$ for some $x > 0$, then it follows from (63) that (62) holds.

This last statement follows from the fact that if $N(ax) = N(x)$ for all $x > 0$, where $a > 0$ and $a \neq 1$, then $N(x) \equiv 0$. For we have $N(x) =$ $N(a^n x)$ for $n = 0, \pm 1, \pm 2,\ldots$. If $a > 1$ and $n \to \infty$, then $N(a^n x) \to N(\infty) =$ 0 for all $x > 0$. If $a < 1$ and $n \to -\infty$, then $N(a^n x) \to N(\infty) = 0$, and therefore $N(x) = 0$ for all $x > 0$. Thus if $N(x) \neq 0$ for some $x > 0$, then necessarily $a = 1$.

By assumption $\rho(k) > 0$ for all $k = 1,2,\ldots$. It follows from (62) that $\rho(1) = 1$ and $\rho(1) > \rho(2) > \ldots > \rho(k) > \ldots$. For if $N(x) \neq 0$ for some $x > 0$, then $kN(x) > (k+1)N(x)$, that is, $N(\rho(k)x) > N(\rho(k+1)x)$. Hence $\rho(k) > \rho(k+1)$ follows for all $k = 1,2,\ldots$.

The only solution of the functional equation which satisfies the above requirements is given by

$$\rho(k) = k^{-1/\alpha} \tag{64}$$

where $1/\alpha$ is a positive constant, or which is the same, α is a positive constant. (Below we shall provide a proof of this statement.)

By (61) and (64) we obtain that

, and therefore $N(x) = 0$

$$kN(x) = N(k^{-1/\alpha}x) \tag{65}$$

for all $x > 0$ and $k = 1,2,\ldots$. Hence it follows that

$$N((\tfrac{k}{\ell})^{-1/\alpha}) = \frac{k}{\ell} N(1) \tag{66}$$

for $k = 1,2,\ldots$ and $\ell = 1,2,\ldots$. Let x be any given positive number. For every $\ell = 1,2,\ldots$ let us choose a positive integer k such that

$$(\tfrac{k}{\ell})^{-1/\alpha} > x \quad \text{and} \quad \lim_{\ell \to \infty} (\tfrac{k}{\ell})^{-1/\alpha} = x . \tag{67}$$

Then by (66) we obtain that

$$N(x+0) = N(x) = N(1)x^{-\alpha} \tag{68}$$

for every $x > 0$ and α is a positive constant. If $N(x) \not\equiv 0$ for $x > 0$, then $N(1) = -c_2$ where $c_2 > 0$. The condition

$$\int_{+0}^{1} x^2 dN(x) < \infty \tag{69}$$

implies that $\alpha < 2$.

Accordingly if $N(x) \not\equiv 0$ for $x > 0$, then

$$N(x) = -\frac{c_2}{x^\alpha} \tag{70}$$

for $x > 0$ where $0 < \alpha < 2$ and $c_2 > 0$.

Furthermore, by (56) it follows that

$$\sigma^2(k - \frac{1}{(\rho(k))^2}) = 0 \tag{71}$$

for $k = 1,2,\ldots$, that is, $\sigma^2(k-k^{2/\alpha}) = 0$. Thus $\sigma^2 = 0$ necessarily holds.

In exactly the same way we can prove that if $M(x) \neq 0$ for some $x < 0$, then

$$M(x) = \frac{c_1}{|x|^\alpha} \tag{72}$$

for $x < 0$ where $0 < \alpha < 2$ and $c_1 > 0$. Furthermore, $\sigma^2 = 0$.

This completes the proof of the theorem.

Finally, we shall give the solution of the functional equation (62).

Lemma 1. Let us suppose that $\rho(1), \rho(2),\ldots, \rho(k),\ldots$ is a decreasing sequence of positive numbers and

$$\rho(k\ell) = \rho(k)\rho(\ell) \tag{73}$$

for $k = 1,2,\ldots$ and $\ell = 1,2,\ldots$. Then we have

$$\rho(k) = k^{-c} \tag{74}$$

for $k = 1,2,\ldots$, where c is a positive constant.

Proof. By (73) we have $\rho(1) = 1$. If a and b are positive integers, then let us define

$$\rho(\tfrac{a}{b}) = \frac{\rho(a)}{\rho(b)} . \tag{75}$$

If $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ where a_1, a_2, b_1, b_2 are positive integers, then

$$\rho(\frac{a_1}{b_1}) > \rho(\frac{a_2}{b_2}) \ . \tag{76}$$

This follows from the inequalities $a_1 b_2 < a_2 b_1$, $\rho(a_1 b_2) > \rho(a_2 b_1)$, $\rho(a_1)\rho(b_2) > \rho(a_2)\rho(b_1)$.

Thus we extended the definition of $\rho(x)$ from positive integers to positive rational numbers in such a way that $\rho(x)$ is a positive decreasing function of x on the set of positive rational numbers and

$$\rho(xy) = \rho(x)\rho(y) \tag{77}$$

for positive rational numbers x and y .

Let $x > 0$ be a rational number. For each $n = 1,2,\ldots$ let us choose an integer r_n such that

$$(1+ \frac{1}{n})^{r_n} \leq x < (1+ \frac{1}{n})^{r_n+1} \tag{78}$$

is satisfied. Then $\lim_{n\to\infty} r_n/n = \log x$.

Since $(1+ \frac{1}{n})^n$ $(n = 1,2,\ldots)$ is an increasing sequence for which $1 < (1+ \frac{1}{n})^n < e$, it follows that

$$\lim_{n\to\infty} \rho((1 + \frac{1}{n})^n) = \lim_{n\to\infty} [\rho(1+ \frac{1}{n})]^n = C \tag{79}$$

exists and $0 < C < 1$. By (78) we have

$$[\rho(1+ \frac{1}{n})]^{r_n+1} \leq \rho(x) \leq [\rho(1+ \frac{1}{n})]^{r_n} \ . \tag{80}$$

If $n \to \infty$ in (80), then by (79) we obtain that both extreme members tend to $C^{\log x}$. Thus

$$\rho(x) = C^{\log x} = x^{-c} \tag{81}$$

for every positive rational number x where $0 < C < 1$ or $c = -\log C$ is a positive real number. This completes the proof of Lemma 1.

From Theorem 3 we can easily deduce an explicit expression for $\log \phi(s)$ which was found in 1936 by A. Ya. Khintchine and P. Lévy [279].

Theorem 4. *The function $\phi(s)$ defined for $\mathrm{Re}(s) = 0$ is the Laplace-Stieltjes transform of a stable distribution function if and only if $\log \phi(s)$ can be represented in the form*

$$\log \phi(s) = -ms - c|s|^{\alpha}[1 + \beta \frac{s}{|s|} d(s,\alpha)] \tag{82}$$

where m is a real constant, $c \geq 0$, $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$ and

$$d(s, \alpha) = \begin{cases} \tan \frac{\alpha\pi}{2} & \text{for } \alpha \neq 1, \\ -\frac{2}{\pi} \log|s| & \text{for } \alpha = 1. \end{cases} \tag{83}$$

In formula (82) $s/|s| = 0$ if $s = 0$.

Proof. By Theorem 3 we have for $\mathrm{Re}(s) = 0$ that

$$\log \phi(s) = -\mu s + \frac{\sigma^2 s^2}{2} + c_1 \int_{-\infty}^{-0} (e^{-sx} - 1 + \frac{sx}{1+x^2}) \frac{\alpha dx}{|x|^{\alpha+1}} + \tag{84}$$

$$+ c_2 \int_{+0}^{+\infty} (e^{-sx} - 1 + \frac{sx}{1+x^2}) \frac{\alpha dx}{x^{\alpha+1}},$$

in 1925 by P. Lévy [111] and was completely proved where either μ is a real constant $\sigma^2 \geq 0$, and $c_1 = c_2 = 0$, or μ is a real constant, $\sigma^2 = 0$, $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 > 0$ and $0 < \alpha < 2$.

If $\omega > 0$, then

$$(85) \quad \int_0^\infty \left(e^{i\omega x} - 1 - \frac{i\omega x}{1+x^2}\right) \frac{\alpha dx}{x^{\alpha+1}} = \begin{cases} -\Gamma(1-\alpha)\omega^\alpha e^{-\frac{i\alpha\pi}{2}} - \dfrac{i\omega\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & \text{for } 0 < \alpha < 1 , \\ -i\omega\log\omega - \dfrac{\omega\pi}{2} + i\omega(1-C) & \text{for } \alpha = 1 , \\ -\Gamma(1-\alpha)\omega^\alpha e^{-\frac{i\alpha\pi}{2}} - \dfrac{i\omega\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & \text{for } 1 < \alpha < 2 , \end{cases}$$

where $C = 0.577215 \ldots$ is Euler's constant and $\Gamma(1-\alpha)$ is the gamma function. By using the relation

$$(86) \qquad \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha} ,$$

where $\alpha \neq 0, \pm 1, \pm 2, \ldots$, we can express $\Gamma(1-\alpha)$ by $\Gamma(\alpha)$ in (85).

If $\sigma^2 \geq 0$ in (84), and $c_1 = c_2 = 0$, then (84) reduces to (82) where $m = \mu$, $c = \sigma^2/2$, and $\alpha = 2$.

If $\sigma^2 = 0$ in (84), and $c_1 + c_2 > 0$, $c_1 \geq 0$, $c_2 \geq 0$, then by (85) we obtain that (84) reduces to (82) where $0 < \alpha < 2$,

$$(87) \qquad m = \begin{cases} \mu - (c_2 - c_1) \dfrac{\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & \text{for } 0 < \alpha < 1 \text{ or } 1 < \alpha < 2 , \\ \mu + (c_2 - c_1)(1 - C) & \text{for } \alpha = 1 , \end{cases}$$

and C is Euler's constant,

(88) $$c = \frac{(c_1 + c_2)\pi}{2\Gamma(\alpha)\sin\frac{\alpha\pi}{2}} \quad \text{for } 0 < \alpha < 2,$$

and, in particular, $c = (c_1 + c_2)\frac{\pi}{2}$ for $\alpha = 1$, and

furthermore,

(89) $$\beta = \frac{c_2 - c_1}{c_1 + c_2} \quad \text{for } 0 < \alpha < 2 .$$

Conversely, if m, c, α, β satisfy the requirements of Theorem 4, then the parameters μ, σ^2, c_1, c_2, α in (84) are uniquely determined and they also satisfy the requirements. These follow from the relations (87), (88), (89).

Finally, we note that (82) can also be expressed in the following way:

If $0 < \alpha < 1$, then

(90) $$\log \phi(s) = -ms - c|s|^{\alpha}[1+\beta \frac{s}{|s|} \tan \frac{\alpha\pi}{2}] =$$
$$= -ms + c_1 \int_{-\infty}^{0} (e^{-sx}-1) \frac{\alpha dx}{|x|^{\alpha+1}} + c_2 \int_{0}^{\infty} (e^{-sx}-1) \frac{\alpha dx}{x^{\alpha+1}}$$

for $\mathrm{Re}(s) = 0$ where c_1 and c_2 can be obtained by (88) and (89). This can be proved by using the integral formula

(91) $$\int_0^{\infty} (e^{i\omega x}-1) \frac{\alpha dx}{x^{\alpha+1}} = -\Gamma(1-\alpha)\omega^{\alpha} e^{-\frac{i\alpha\pi}{2}}$$

which is valid for $\omega > 0$ and $0 < \alpha < 1$.

If $\alpha = 1$, then

(92)
$$\log \phi(s) = -ms-c|s|[1-\beta \frac{2s \log|s|}{\pi|s|}] =$$
$$= -\mu s+c_1 \int_{-\infty}^{0} (e^{-sx}-1+ \frac{sx}{1+x^2}) \frac{dx}{x^2} + c_2 \int_{0}^{\infty} (e^{-sx}-1+ \frac{sx}{1+x^2}) \frac{dx}{x^2}$$

for $Re(s) = 0$ where μ, c_1 and c_2 can be obtained by (87), (88) and (89). This can be proved by using (85) for $\alpha = 1$.

If $1 < \alpha < 2$, then

(93)
$$\log \phi(s) = -ms-\cancel{c}|s|^{\alpha}[1+\beta \frac{s}{|s|} \tan \frac{\alpha\pi}{2}] =$$ /c
$$= -ms+c_1 \int_{-\infty}^{0} (e^{-sx}-1+sx) \frac{\alpha dx}{|x|^{1+\alpha}} + c_2 \int_{0}^{\infty} (e^{-sx}-1+sx) \frac{\alpha dx}{x^{1+\alpha}}$$

for $Re(s) = 0$ where c_1 and c_2 can be obtained by (88) and (89). This can be proved by using the integral formula

(94)
$$\int_{0}^{\infty}(e^{i\omega x}-1-i\omega x) \frac{\alpha dx}{x^{\alpha+1}} = -\Gamma(1-\alpha)\omega^{\alpha}e^{-\frac{i\alpha\pi}{2}}$$

which is valid for $\omega > 0$ and $1 < \alpha < 2$.

If $\alpha = 2$, then

(95)
$$\log \phi(s) = -ms-c|s|^2$$

for $Re(s) = 0$.

In this section we established that if a real random variables ξ has a stable distribution, and

$$(96) \qquad E\{e^{-s\xi}\} = \phi(s) ,$$

which necessarily exists for $\mathrm{Re}(s) = 0$, then $\log \phi(s)$ can be expressed in the form (82). Accordingly, in the most general case, the distribution of ξ depends on four parameters α, β, c, m where $\alpha \varepsilon (0, 2]$ is the so called "characteristic exponent", $\beta \varepsilon [-1, 1]$ can be considered as a measure of asymmetry, $c \geq 0$ is a scale parameter, and m is a location parameter.

If ξ has a stable distribution function and $\log \phi(s)$ is given by (82), then we say that ξ has a stable distribution of type $S(\alpha,\beta,c,m)$, and write

$$(97) \qquad \xi \sim S(\alpha,\beta,c,m) .$$

If ξ has a stable distribution $S(\alpha,\beta,c,m)$ then $-\xi$ has also a stable distribution, namely

$$(98) \qquad -\xi \sim S(\alpha,-\beta,c,-m) .$$

If ξ has a stable distribution $S(\alpha,\beta,c,m)$ and a and $b > 0$ are real numbers, then $(\xi-a)/b$ has also a stable distribution, namely

$$(99) \qquad \frac{\xi-a}{b} \sim S(\alpha,\beta, \frac{c}{b^{\alpha}} , \frac{m-a}{b})$$

for $\alpha \neq 1$ and

$$(100) \qquad \frac{\xi-a}{b} \sim S(1,\beta, \frac{c}{b} , \frac{m-a}{b} + \frac{2c\beta}{b\pi} \log b)$$

for $\alpha = 1$.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be mutually independent real random variables having the same stable distribution $S(\alpha,\beta,c,m)$. In this case

$$(101) \qquad \frac{\xi_1+\xi_2+\ldots+\xi_n-A_n}{B_n} \sim S(\alpha,\beta,c,m)$$

if

$$(102) \qquad A_n = \begin{cases} m\,(n - n^{\frac{1}{\alpha}}) & \text{for } \alpha \neq 1, \\ \dfrac{2c\beta\; n\,\log n}{\pi} & \text{for } \alpha = 1, \end{cases}$$

and $B_n = n^{\frac{1}{\alpha}}$. For by (82)

$$(103) \qquad \xi_1+\xi_2+\ldots+\xi_n \sim S(\alpha,\beta,nc,nm),$$

and therefore by (99)

$$(104) \qquad \frac{\xi_1+\xi_2+\ldots+\xi_n-A_n}{B_n} \sim S\left(\alpha,\beta,\frac{nc}{B_n^{\alpha}},\frac{nm-A_n}{B_n}\right)$$

if $\alpha \neq 1$ and

$$(105) \qquad \frac{\xi_1+\xi_2+\ldots+\xi_n-A_n}{B_n} \sim S\left(1,\beta,\frac{nc}{B_n},\frac{nm}{B_n}+\frac{2cn\beta}{B_n\pi}\log B_n-\frac{A_n}{B_n}\right)$$

if $\alpha = 1$. If we choose A_n according to (102) and $B_n = n^{1/\alpha}$, then (104) and (105) imply (101).

We note that if $c = 0$, then $\mathbf{P}\{\xi = m\} = 1$. If $c > 0$ then ξ is a continuous random variable which has a continuous density function $f(x;\alpha,\beta,c,m)$. If $c > 0$, $0 < \alpha < 1$ and $\beta = 1$, then $\mathbf{P}\{\xi \geq m\} = 1$

and $f(x;\ \alpha,\beta,c,m) > 0$ for $x > m$. If $c > 0$, $0 < \alpha < 1$ and $\beta = -1$, then $P\{\xi \leq m\} = 1$ and $f(x;\ \alpha,\beta,c,m) > 0$ for $x < m$. If $c > 0$, and either $0 < \alpha < 1$ and $-1 < \beta < 1$ or $1 \leq \alpha \leq 2$ and $-1 \leq \beta \leq 1$, then $f(x;\ \alpha,\beta,c,m) > 0$ for every x. If $\beta = 0$, then $\xi - m$ has the same distribution as $m - \xi$. If $\alpha = 2$, then β is irrelevant, but $\xi - m$ has a symmetric distribution in this case too.

If a random variable ξ has a stable distribution and $\log \phi(s)$ is given by (82), then we use the notation $F(x;\ \alpha,\beta,c,m)$ for denoting the distribution function of ξ. If $c > 0$ then ξ has a continuous density function which we denote by $f(x\ ;\ \alpha,\beta,c,m)$.

In some particular cases $F(x;\ \alpha,\beta,c,m)$ and $f(x;\ \alpha,\beta,c,m)$ have been known for a long time.

If $\alpha = 2$, when β is irrelevant, and $c > 0$, then

$$F(x;\ 2,0,c,m) = F\left(\frac{x-m}{\sqrt{2c}}\right) \tag{106}$$

where

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du \tag{107}$$

is the normal distribution function, and

$$f(x;\ 2,0,c,m) = f\left(\frac{x-m}{\sqrt{2c}}\right) \frac{1}{\sqrt{2c}} \tag{108}$$

where

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \tag{109}$$

is the normal density function. In this case

(110) $$\phi(s) = e^{-ms+cs^2}$$

for any complex s.

The normal distribution function was studied first in 1782 by P. S. Laplace [106]. In 1809 C. F. Gauss [59] proved that the normal distribution is stable. There are extensive tables for the normal distribution function and for the normal density function.

If $\alpha = 1$, $\beta = 0$ and $c > 0$, then

(111) $$F(x;\, 1,0,c,m) = F\left(\frac{x-m}{c}\right)$$

where

(112) $$F(x) = \frac{1}{2} + \frac{1}{\pi} \text{ arc tan } x$$

is the Cauchy distribution function, and

(113) $$f(x;\, 1,0,c,m) = f\left(\frac{x-m}{c}\right) \frac{1}{c}$$

where $f(x) = 1/\pi(1+x^2)$ for $-\infty < x < \infty$ is the Cauchy density function. In (112) we define $-\frac{\pi}{2} < \text{arc tan } x < \frac{\pi}{2}$. In this case

(114) $$\phi(s) = e^{-ms+c|s|}$$

for $\text{Re}(s) = 0$.

The distribution function (112) was found for the first time in 1827 by A. Cauchy [231] as a particular case of symmetric stable distributions.

If $\alpha = \frac{1}{2}$, $\beta = 1$, $c = 1$ and $m = 0$, then

by S.D. Poisson [154] and in 1853

$$(115)\qquad F(x;\tfrac{1}{2},1,1,0) = \begin{cases} 2[1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1/\sqrt{x}} e^{-\frac{u^2}{2}}]du & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

and

$$(116)\qquad f(x;\tfrac{1}{2},1,1,0) = \begin{cases} \frac{1}{\sqrt{2\pi x^3}} e^{-\frac{1}{2x}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

In this case

$$(117)\qquad \phi(s) = e^{-\sqrt{2s}}$$

for $\mathrm{Re}(s) \geq 0$. The distribution (115) has been studied by P. Lévy [292]. The transform (117) has been found by G. Doetsch[245].

If $\alpha = \frac{1}{2}$, $-1 \leq \beta \leq 1$, $c = 1$ and $m = 0$, then we have

$$(118)\qquad f(x;\tfrac{1}{2},\beta,1,0) = \mathrm{Re}\{\frac{z}{\pi x}[\sqrt{\pi}\, e^{-z^2} + 2iw(z)]\}$$

for $x > 0$ where

$$(119)\qquad w(z) = e^{-z^2}\int_0^z e^{u^2} du$$

and $z = [(1+\beta)+i(1-\beta)]/\sqrt{8x}$. If, in particular, $\beta = 0$, then in (118) we can write that

$$(120)\qquad w(\frac{1+i}{\sqrt{8x}}) = \frac{(1+i)\sqrt{\pi}}{2} e^{\frac{-i}{4x}}[C(\frac{1}{4x}) + iS(\frac{1}{4x})]$$

where $S(x)$ and $C(x)$ are the Fresnel integrals defined by

$$(121) \qquad S(x) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \sin u^2 du \quad \text{and} \quad C(x) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{x}} \cos u^2 du .$$

The formula (118) has been found by V. M. Zolotarev [338]. The function $w(z)$ has been tabulated by K. A. Karpov [90].

Now we shall consider the problem of finding $f(x;\alpha,\beta,c,m)$ in the general case. It is sufficient to find $f(x;\alpha,\beta,c,0)$ for some particular $c > 0$ because the general case can be obtained by linear transformation from this particular case. If $c > 0$, then $|\Phi(s)|$ is integrable on the line $\mathrm{Re}(s) = 0$ and by Fourier inversion we obtain that

$$(122) \qquad f(x;\alpha,\beta,c,0) = \frac{1}{\pi c^{1/\alpha}} \int_0^{\infty} e^{-u^\alpha} \cos(uc^{-1/\alpha}x - u^\alpha \beta \tan \frac{\alpha\pi}{2})du$$

for $\alpha \neq 1$ and

$$(123) \qquad f(x;1,\beta,c,0) = \frac{1}{\pi c} \int_0^{\infty} e^{-u} \cos(\frac{ux}{c} + \frac{2u\beta}{\pi} \log \frac{u}{c})du .$$

Obviously,

$$(124) \qquad f(x;\alpha,\beta,c,0) = f(-x;\alpha,-\beta,c,0)$$

holds for every x. Thus it is sufficient to find $f(x;\alpha,\beta,c,0)$ for $x > 0$, $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$ and some particular $c > 0$.

In what follows we shall determine the density function $f(x;\alpha,\beta,c,0)$ for $0 < \alpha < 1$, $1 < \alpha \leq 2$, $-1 \leq \beta \leq 1$ and some particularly chosen $c > 0$.

First, for any α and β satisfying the inequalities $0 < \alpha < 1$ or

$1 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$ let us determine a real γ such that

(125) $$\tan \frac{\gamma\pi}{2} = \beta \tan \frac{\alpha\pi}{2}$$

and $-1 < \gamma < 1$. Then let us define

(126) $$c = \cos \frac{\gamma\pi}{2} .$$

We note that the inequality

(127) $$|\gamma| \leq 1 - |1-\alpha|$$

always holds.

If c is defined by (126) then let us write

(128) $$f(x;\ \alpha,\beta,c,0) = h(x;\ \alpha,\gamma)$$

for $\alpha \neq 1$. If $\alpha \neq 1$, then by (122) we have

(129) $$h(x;\ \alpha,\gamma) = \frac{1}{\pi} \mathrm{Re}\{\int_0^\infty e^{ixu-u^\alpha e^{\frac{\gamma\pi i}{2}}} du\}$$

for every x .

Since

(130) $$h(x;\ \alpha,\gamma) = h(-x;\ \alpha,-\gamma)$$

holds for every x , it is sufficient to determine $h(x;\ \alpha,\gamma)$ for $x > 0$.

By (129) and (130) we can write also that

$$(131) \qquad h(x;\ \alpha,\gamma) = \frac{1}{\pi}\,\mathrm{Re}\,\{\int_0^\infty e^{-ixu-u^\alpha e^{-\frac{\gamma\pi i}{2}}}\,du\}$$

for every x and $\alpha \neq 1$.

<u>Theorem 5.</u> <u>If</u> $0 < \alpha < 1$, <u>then</u>

$$(132) \qquad h(x;\ \alpha,\gamma) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\Gamma(k\alpha+1)}{k!\,x^{k\alpha+1}} \sin\frac{k(\alpha+\gamma)\pi}{2}$$

<u>for</u> $x > 0$.

<u>Proof.</u> If $x > 0$ and if in (131) we use the substitution $z = ixu$, then we obtain that

$$(133) \qquad h(x;\ \alpha,\gamma\ ) = \frac{1}{\pi x}\,\mathrm{Re}\,\{\frac{1}{i}\int_0^{i\infty} e^{-z-z^\alpha x^{-\alpha} e^{-\frac{(\alpha+\gamma)\pi i}{2}}}\,dz\}\ .$$

If $0 < \alpha < 1$, then the integrand in (133) tends to 0 as $\mathrm{Re}(z) \to +\infty$ and therefore by Cauchy's integral theorem we can replace the path of integration in (133) by the positive real axis. By using the exponential expansion and the integral representation of the gamma function we obtain that

$$(134) \qquad \begin{aligned} h(x;\ \alpha,\gamma) &= \frac{1}{\pi x}\,\mathrm{Re}\,\{\frac{1}{i}\int_0^\infty e^{-u-u^\alpha x^{-\alpha} e^{-\frac{(\alpha+\gamma)\pi i}{2}}}\,du\,\} = \\ &= \frac{1}{\pi x}\,\mathrm{Re}\,\{\frac{1}{i}\sum_{k=0}^{\infty}\frac{(-1)^k e^{-\frac{k(\alpha+\gamma)\pi i}{2}}}{k!\;x^{k\alpha}}\int_0^\infty e^{-u}\,u^{k\alpha}\,du\,\} = \\ &= \frac{1}{\pi x}\sum_{k=0}^{\infty}\frac{(-1)^{k-1}\,\Gamma(k\alpha+1)}{k!\;x^{k\alpha}}\,\sin\frac{k(\alpha+\gamma)\pi}{2} \end{aligned}$$

for $x > 0$ and $0 < \alpha < 1$. This proves (132).

Theorem 6. If $1 < \alpha < 2$, then

$$(135) \qquad h(x;\alpha,\gamma) = \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\,\Gamma(\frac{k}{\alpha}+1)x^{k-1}}{k!} \sin\frac{k(\alpha+\gamma)\pi}{2\alpha}$$

for $x > 0$.

Proof. If $x > 0$ and if in (131) we use the substitution $z = e^{-\frac{\gamma\pi i}{2}} u^{\alpha}$, then we obtain that

$$(136) \qquad h(x;\alpha,\gamma) = \frac{1}{\pi}\,\mathrm{Re}\,\{ \frac{e^{\frac{\gamma\pi i}{2\alpha}}}{\alpha} \int_L e^{-z-ixz^{\frac{1}{\alpha}} e^{\frac{\gamma\pi i}{2\alpha}}} z^{\frac{1}{\alpha}-1}\,dz\}$$

where $L = \{z : z = e^{-\frac{\gamma\pi i}{2}} u^{\alpha}$ and $0 \leq u < \infty\}$. If $1 < \alpha < 2$, then the integrand in (136) tends to 0 as $\mathrm{Re}(z) \to +\infty$ and therefore by Chaucy's integral theorem we can replace the path of integration in (136) by the positive real axis. By an exponential expansion we obtain that

$$(137) \qquad \begin{aligned} h(x;\alpha,\gamma) &= \frac{1}{\pi}\,\mathrm{Re}\,\{ \frac{e^{\frac{\gamma\pi i}{2\alpha}}}{\alpha} \int_0^{\infty} e^{-u-ixu^{\frac{1}{\alpha}} e^{\frac{\gamma\pi i}{2\alpha}}} u^{\frac{1}{\alpha}-1}\,du\} \\ &= \frac{1}{\pi}\,\mathrm{Re}\,\{ \frac{e^{\frac{\gamma\pi i}{2\alpha}}}{\alpha} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} i^{k-1} x^{k-1} e^{\frac{(k-1)\gamma\pi i}{2\alpha}}}{(k-1)!} \int_0^{\infty} e^{-u} u^{\frac{k}{\alpha}-1}\,du\} \\ &= \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\,\Gamma(\frac{k}{\alpha}+1)x^{k-1}}{k!} \sin\frac{k(\alpha+\gamma)\pi}{2\alpha} \end{aligned}$$

for $x > 0$ and $1 < \alpha < 2$. This proves (135).

For the proofs of Theorem 5 and Theorem 6 we refer to H. Bergström [228], [229], W. Feller [252], [253 pp. 548-549] and Chung-Jeh Chao [232]. See also

A. Wintner [333], P. Humbert [262], and H. Pollard [310].

There is an interesting relation between $h(x;\alpha,\gamma)$ for $\frac{1}{2}<\alpha<1$ and $h(x;\alpha,\gamma)$ for $1<\alpha<2$. This relation was found in 1954 by V. M. Zolotarev [338].

Theorem 7. If $1<\alpha<2$, *then*

$$h(x;\alpha,\gamma)=\frac{1}{x^{\alpha+1}}h\left(\frac{1}{x^{\alpha}},\frac{1}{\alpha},\frac{\alpha+\gamma-1}{\alpha}\right) \tag{138}$$

for $x>0$.

Proof. The proof of (138) follows immediately from (132) and (135).

We can write (138) in the following equivalent form. If $\frac{1}{2}<\alpha<1$, then

$$h(x;\alpha,\gamma)=\frac{1}{x^{\alpha+1}}h\left(\frac{1}{x^{\alpha}},\frac{1}{\alpha},\frac{\alpha+\gamma-1}{\alpha}\right) \tag{139}$$

for $x>0$. That is (138) is valid if $\frac{1}{2}<\alpha<2$ and $x>0$.

Examples. In the particular cases when $\alpha=\frac{2}{3}$ or $\alpha=\frac{3}{2}$ and $\beta=1,0,-1$ we can express $h(x;\alpha,\gamma)$ with the aid of Whittaker functions. If $\mathrm{Re}(m+\frac{1}{2}-k)\geq 0$, then for every z the Whittaker function $W_{k,m}(z)$ is defined by

$$W_{k,m}(z)=\frac{e^{-\frac{z}{2}}z^{k}}{\Gamma(m+\frac{1}{2}-k)}\int_0^{\infty}e^{-t}t^{m-\frac{1}{2}-k}\left(1+\frac{t}{z}\right)^{m-\frac{1}{2}+k}dt\,. \tag{140}$$

(See E. T. Whittaker [331], and E. T. Whittaker and G. N. Watson [208 p. 339].)

If $2m$ is not an integer then the Whittaker function $W_{k,m}(z)$ can also be expressed in the following way:

$$(141) \qquad W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} M_{k,-m}(z)$$

for $|\arg z| < \frac{3}{2}\pi$ and

$$(142) \qquad W_{-k,m}(-z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m+k)} M_{-k,m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m+k)} M_{-k,-m}(z)$$

for $|\arg(-z)| < \frac{3}{2}\pi$, where

$$(143) \qquad M_{k,m}(z) = z^{\frac{1}{2}+m} e^{-\frac{z}{2}}\{1+ \frac{\frac{1}{2}+m-k}{1!(2m+1)} z + \frac{(\frac{1}{2}+m-k)(\frac{3}{2}+m-k)}{2!(2m+1)(2m+2)} z^2+\dots\}$$

and

$$(144) \qquad M_{k,-m}(z) = z^{\frac{1}{2}-m} e^{-\frac{z}{2}}\{1+ \frac{\frac{1}{2}-m-k}{1!(1-2m)} z + \frac{(\frac{1}{2}-m-k)(\frac{3}{2}-m-k)}{2!(1-2m)(2-2m)} z^2+\dots\} .$$

If $2m$ is not a negative integer, then we have

$$(145) \qquad z^{-\frac{1}{2}-m} M_{k,m}(z) = (-z)^{-\frac{1}{2}-m} M_{-k,m}(-z) ,$$

that is,

$$(146) \qquad \begin{aligned} & e^{-z}[1 + \frac{\frac{1}{2}+m-k}{1!(2m+1)} z + \frac{(\frac{1}{2}+m-k)(\frac{3}{2}+m-k)}{2!(2m+1)(2m+2)} z^2 + \dots] = \\ & = [1 - \frac{\frac{1}{2}+m+k}{1!(2m+1)} z + \frac{(\frac{1}{2}+m+k)(\frac{3}{2}+m+k)}{2!(2m+1)(2m+2)} z^2 - \dots] . \end{aligned}$$

By using the above results we can prove that

$$(147)\qquad f(x;\tfrac{2}{3},1,\tfrac{1}{2},0)=\begin{cases}\sqrt{\dfrac{3}{\pi x^2}}\,e^{-\frac{2}{27x^2}}\,W_{\frac{1}{2},\frac{1}{6}}\left(\dfrac{4}{27x^2}\right) & \text{for } x>0\,,\\[2ex] 0 \quad \text{for } x\leqq 0\,,\end{cases}$$

and

$$(148)\qquad f(x;\tfrac{2}{3},0,1,0)=\frac{\sqrt{3}\,e^{\frac{2}{27x^2}}}{6\sqrt{\pi}\,|x|}\,W_{-\frac{1}{2},\frac{1}{6}}\left(\frac{4}{27x^2}\right)\quad \text{for } x\in(-\infty,\infty)\,.$$

Proof of (147). If $\alpha=\frac{2}{3}$ and $\beta=1$, then by (125) $\gamma=\frac{2}{3}$ and by (126) $c=\cos\frac{\pi}{3}=\frac{1}{2}$. Thus by (132) we obtain that for $x>0$

$$(149)\qquad \begin{aligned} f(x;\tfrac{2}{3},1,\tfrac{1}{2},0)=h(x;\tfrac{2}{3},\tfrac{2}{3})&=\frac{\sqrt{3}}{2\pi}\sum_{j=0}^{\infty}\frac{(-1)^j\,\Gamma(2j+\frac{5}{3})}{(3j+1)!\,x^{2j+\frac{5}{3}}}\;-\\ &-\frac{\sqrt{3}}{2\pi}\sum_{j=1}^{\infty}\frac{(-1)^j\,\Gamma(2j+\frac{1}{3})}{(3j-1)!\,x^{2j+\frac{1}{3}}}\end{aligned}$$

$\leftarrow$ where we used that

$$(150)\qquad \sin\frac{2k\pi}{3}=\begin{cases}\dfrac{\sqrt{3}}{2} & \text{if } k=3j+1\\ 0 & \text{if } k=3j\\ -\dfrac{\sqrt{3}}{2} & \text{if } k=3j-1\end{cases}$$

for $j=0,1,2,\dots$. If we use the abbreviation $y=4/27x^2$, then by

(143), (144) and (145) we can write that

$$f(x;\ \tfrac{2}{3},\ 1,\ \tfrac{1}{2},\ 0) = \frac{9e^{-y/2}\ y^{1/2}}{2\sqrt{\pi}} \cdot \tag{151}$$

$$\cdot\left\{ \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})} M_{\frac{1}{2},-\frac{1}{6}}(y) + \frac{\Gamma(-\frac{1}{3})}{\Gamma(-\frac{1}{6})} M_{\frac{1}{2},\frac{1}{6}}(y)\right\} =$$

$$= \frac{9e^{-y/2}\ y^{1/2}}{2\sqrt{\pi}} W_{\frac{1}{2},\frac{1}{6}}(y)$$

for $x > 0$ where $y = 4/27x^2$. The last formula follows from (141). In (151) we used that

$$M_{-\frac{1}{2},\frac{1}{6}}(-y) = M_{\frac{1}{2},\frac{1}{6}}(y) \quad \text{and} \quad M_{-\frac{1}{2},-\frac{1}{6}}(-y) = -M_{\frac{1}{2},-\frac{1}{6}}(y) \tag{152}$$

for $y > 0$, and further that

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}} , \tag{153}$$

$$\Gamma(\tfrac{1}{3})\Gamma(\tfrac{5}{6}) = \Gamma(\tfrac{2}{3})2^{1/2}\sqrt{\pi} , \tag{154}$$

and

$$\Gamma(\tfrac{1}{6})\Gamma(\tfrac{5}{6}) = \frac{\pi}{\sin\frac{\pi}{6}} = 2\pi .$$

If $x \leq 0$, then (147) is obviously true. We note that

$$f(x;\ \tfrac{2}{3},\ -1,\ \tfrac{1}{2},\ 0) = f(-x;\ \tfrac{2}{3},\ 1,\ \tfrac{1}{2},\ 0) \tag{155}$$

holds for all x .

Proof of (148). If $\alpha = \frac{2}{3}$ and $\beta = 0$, then by (125) $\gamma = 0$ and by (126) $c = 1$. Thus by (132) we obtain that for $x > 0$

(156)
$$f(x; \tfrac{2}{3}, 0, 1, 0) = h(x; \tfrac{2}{3}, 0) = \frac{\sqrt{3}}{2\pi} \sum_{j=0}^{\infty} \frac{\Gamma(2j+\frac{5}{3})}{(3j+1)!\,x^{2j+\frac{5}{3}}} - \frac{\sqrt{3}}{2\pi} \sum_{j=1}^{\infty} \frac{\Gamma(2j+\frac{1}{3})}{(3j-1)!\,x^{2j+\frac{1}{3}}}$$

where we used that

(157)
$$\sin\frac{k\pi}{3} = \begin{cases} \frac{\sqrt{3}}{2}(-1)^j & \text{if } k = 3j+1, \\ 0 & \text{if } k = 3j, \\ -\frac{\sqrt{3}}{2}(-1)^j & \text{if } k = 3j-1, \end{cases}$$

for $j = 0,1,2,\ldots$. If we use the abbreviation $y = 4/27x^2$ for $x > 0$, then in a similar way as in (151) we obtain that

(158)
$$f(x; \tfrac{2}{3}, 0, 1, 0) = \frac{3e^{y/2}\,y^{1/2}}{4\sqrt{\pi}} \cdot \left\{ \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{7}{6})} M_{-\frac{1}{2},-\frac{1}{6}}(y) + \frac{\Gamma(-\frac{1}{3})}{\Gamma(\frac{5}{6})} M_{-\frac{1}{2},\frac{1}{6}}(y) \right\} = \frac{3e^{y/2}\,y^{1/2}}{4\sqrt{\pi}} W_{-\frac{1}{2},\frac{1}{6}}(y)$$

for $x > 0$ where $y = 4/27x^2$. Since $f(-x; \frac{2}{3}, 0, 1, 0) = f(x; \frac{2}{3}, 0, 1, 0)$ for all x, (148) follows from (158).

Formulas (147) and (148) have been found by H. Pollard [310] and V. M. Zolotarev [338] respectively.

Next we shall prove that

$$(159)\quad f(x;\ \tfrac{3}{2},\ 1,\ \tfrac{1}{\sqrt{2}},\ 0) = \begin{cases} \dfrac{\sqrt{3}\ e^{2x^3/27}}{6\sqrt{\pi}\ x} W_{-\frac{1}{2},\ \frac{1}{6}}(\frac{4x^3}{27}) & \text{if } x > 0\ , \\[2ex] \dfrac{\sqrt{3}\ e^{-2|x|^3/27}}{\sqrt{\pi}\ |x|} W_{\frac{1}{2},\ \frac{1}{6}}(\frac{4|x|^3}{27}) & \text{if } x < 0\ . \end{cases}$$

Proof of (159). If $\alpha = \frac{3}{2}$ and $\beta = 1$, then by (125) $\gamma = -1/2$ and by (126) $c = \cos\frac{\pi}{4} = 1/\sqrt{2}$. If $\alpha = \frac{3}{2}$ and $\beta = -1$, then $\gamma = 1/2$ and $c = 1/\sqrt{2}$.

If $x > 0$, then by (128), (138) and (156) we obtain that

$$(160)\quad \begin{aligned} &f(x;\ \tfrac{3}{2},\ 1,\ \tfrac{1}{\sqrt{2}},\ 0) = h(x;\ \tfrac{3}{2},\ -\tfrac{1}{2}) = \\ &= \frac{1}{x^{5/2}}\, h\,(\frac{1}{x^{3/2}},\ \frac{2}{3},\ 0) = \frac{1}{x^{5/2}}\, f\,(\frac{1}{x^{3/2}},\ \frac{2}{3},\ 0,\ 1,\ 0) \end{aligned}$$

and the extreme right member in (160) is given by (148).

If $x < 0$, then by (128), (138) and (149) we obtain that

$$(161)\quad \begin{aligned} &f(x;\ \tfrac{3}{2},\ 1,\ \tfrac{1}{\sqrt{2}},\ 0) = f(-x;\ \tfrac{3}{2},\ -1,\ \tfrac{1}{\sqrt{2}},\ 0) = h(-x;\ \tfrac{3}{2},\ \tfrac{1}{2}) = \\ &= \frac{1}{(-x)^{5/2}}\, h\,((\frac{1}{-x})^{3/2},\ \frac{2}{3},\ \frac{2}{3}\ ) = \frac{1}{(-x)^{5/2}}\, f\,((\frac{1}{-x})^{3/2},\ \frac{2}{3},\ 1,\ \frac{1}{2},\ 0) \end{aligned}$$

and the extreme right member in (161) is given by (147). This completes the proof of (159).

Formula (159) was found by V. M. Zolotarev [338].

Finally, we consider the case when $\alpha = 1/3$ and $\beta = 1$. The density function $f(x; \frac{1}{3}, 1, c, 0)$ for $c > 0$ can be expressed with the aid of modified Bessel functions. If $\nu \neq -1, -2, \ldots$, the modified Bessel function of order ν is defined by

$$(162) \qquad I_\nu(z) = \sum_{j=0}^{\infty} \frac{(z/2)^{2j+\nu}}{j!\Gamma(j+\nu+1)}$$

for every z. If $\nu = -1, -2, \ldots$, then $I_\nu(z) = I_{-\nu}(z)$.

The function

$$(163) \qquad K_\nu(z) = \frac{\pi}{2\sin \nu\pi}[I_{-\nu}(z) - I_\nu(z)]$$

is called Basset's function or MacDonald's function.

We have

$$(164) \quad f(x; \frac{1}{3}, 1, \frac{\sqrt{3}}{2}, 0) = \begin{cases} \frac{1}{3\pi x^{3/2}} K_{\frac{1}{3}}(\frac{1}{3\sqrt{3x}}) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 , \end{cases}$$

where

$$(165) \qquad K_{\frac{1}{3}}(x) = \frac{\pi}{\sqrt{3}}[I_{-\frac{1}{3}}(x) - I_{\frac{1}{3}}(x)]$$

or by Airy's integral

$$K_{\frac{1}{3}}(2x^{3/2}) = \sqrt{\frac{3}{x}} \int_0^\infty \cos(t^3 + 3tx)\,dt \tag{166}$$

for $x > 0$. (See A. Erdélyi [49] Vol. 2, p. 22.)

Now we shall prove (164). If $\alpha = 1/3$ and $\beta = 1$, then by (125) $\gamma = 1/3$ and by (126) $c = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

If $x > 0$, then by (128), and (132) we obtain that

$$\begin{aligned} f(x; \tfrac{1}{3}, 1, \tfrac{\sqrt{3}}{2}, 0) = h(x; \tfrac{1}{3}, \tfrac{1}{3}) &= \frac{\sqrt{3}}{2\pi} \sum_{j=0}^\infty \frac{\Gamma(j+1+\frac{1}{3})}{(3j+1)!\,x^{j+1+\frac{1}{3}}} - \\ &- \frac{\sqrt{3}}{2\pi} \sum_{j=1}^\infty \frac{\Gamma(j+1-\frac{1}{3})}{(3j-1)!\,x^{j+1-\frac{1}{3}}} \end{aligned} \tag{167}$$

where we used (157). Since

$$\frac{\Gamma(j+1+\frac{1}{3})}{(3j+1)!} = \frac{\Gamma(\frac{1}{3})\,\Gamma(\frac{2}{3})}{j!\,\Gamma(j+\frac{2}{3})3^{3j+1}} \tag{168}$$

for $j = 0,1,2,\ldots$ and

$$\frac{\Gamma(j+1-\frac{1}{3})}{(3j-1)!} = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{(j-1)!\,\Gamma(j+\frac{1}{3})3^{3j-1}} \tag{169}$$

for $j = 1,2,\ldots$, we obtain from (167) that

$$(170)\qquad f(x;\tfrac{1}{3},1,\tfrac{\sqrt{3}}{2},0)=\frac{1}{(3x)^{3/2}}\left[I_{-\frac{1}{3}}\left(\frac{1}{\sqrt{27x}}\right)-I_{\frac{1}{3}}\left(\frac{1}{\sqrt{27x}}\right)\right]$$

for $x>0$. This proves (164). See also V. M. Zolotarev [338].

We shall close this section by mentioning several useful properties of stable distribution functions.

First we observe that if $\beta=1$ then the Laplace-Stieltjes transform of the stable distribution function is convergent for $\mathrm{Re}(s)\geq 0$, whereas if $\beta=-1$, then the Laplace-Stieltjes transform of the stable distribution function is convergent for $\mathrm{Re}(s)\leq 0$. By symmetry it is sufficient to consider the case of $\beta=1$.

If $0<\alpha<1$, $\beta=1$, $c=\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}$ and $m=0$, then

$$(171)\qquad \begin{aligned}\int_0^\infty e^{-sx}d_xF(x;\alpha,1,c,0)&=\exp\left\{\int_0^\infty(e^{-sx}-1)\frac{\alpha dx}{x^{\alpha+1}}\right\}=\\&=e^{-\Gamma(1-\alpha)s^\alpha}\end{aligned}$$

for $\mathrm{Re}(s)\geq 0$. For if $0<\alpha<1$, $m=0$, and in (90) we choose $c_1=0$ and $c_2=1$, then by (89) $\beta=1$ and by (88) $c=\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}$. For the evaluation of the integral in (171) see Problem 46.5.

If $\alpha=1$, $\beta=1$, $c=\pi/2$ and $m=0$, then

$$(172)\qquad \begin{aligned}\int_{-\infty}^\infty e^{-sx}d_xF(x;1,1,\tfrac{\pi}{2},0)&=\exp\left\{\int_0^\infty\left(e^{-sx}-1+\frac{sx}{1+x^2}\right)\frac{dx}{x^2}+(1-C)s\right\}=\\&=e^{s\log s}\end{aligned}$$

for $\operatorname{Re}(s) \geqq 0$. For if $\alpha = 1$, $m = 0$, and in (92) we choose $c_1 = 0$ and $c_2 = 1$, then by (89), $\beta = 1$, by (88) $c = \pi/2$, and by (87) $\mu = -(1-C)$. For the evaluation of the integral in (172) see Problem 46.5.

If $1 < \alpha < 2$, $\beta = 1$, $c = \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2} = \pi/2\Gamma(\alpha)\sin\frac{\alpha\pi}{2}$, and $m = 0$, then

(173)
$$\int_{-\infty}^{\infty} e^{-sx} d_x F(x;\ \alpha,\ 1,\ c,\ 0) = \exp\left\{\int_0^{\infty} (e^{-sx} - 1 + sx)\frac{\alpha dx}{x^{\alpha+1}}\right\} = e^{-\Gamma(1-\alpha)s^{\alpha}}$$

for $\operatorname{Re}(s) \geqq 0$. For if $1 < \alpha < 2$, $m = 0$ and in (93) we choose $c_1 = 0$ and $c_2 = 1$, then by (89) $\beta = 1$ and by (88) $c = \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}$. For the evaluation of the integral in (173) see Problem 46.5.

From (171) it follows immediately that if $0 < \alpha < 1$ and $\xi \sim S(\alpha,\ 1,\ \cos\frac{\alpha\pi}{2},\ 0)$, then

(174)
$$\underset{\sim}{E}\{e^{-s\xi}\} = e^{-s^{\alpha}}$$

for $\operatorname{Re}(s) \geqq 0$. From (173) it follows immediately that if $1 < \alpha < 2$ and $\xi \sim S(\alpha,\ 1,\ -\cos\frac{\alpha\pi}{2},\ 0)$, then

(175)
$$\underset{\sim}{E}\{e^{-s\xi}\} = e^{s^{\alpha}}$$

for $\operatorname{Re}(s) \geqq 0$.

We note also that if $\xi \sim S(\alpha,\ \beta,\ c,\ m)$ and $\beta = 0$ and $m = 0$, then ξ has a symmetric distribution, that is, if $\xi \sim S(\alpha,\ 0,\ c,\ 0)$, then

$-\xi \sim S(\alpha, 0, c, 0)$ too. If $c \geqq 0$, then we have

$$\int_{-\infty}^{\infty} e^{-sx} d_x F(x;\, \alpha,\, 0,\, c,\, 0) = e^{-c|s|^{\alpha}} \tag{176}$$

for $\operatorname{Re}(s) = 0$. If $c > 0$ and $\alpha = 2$, then (176) is convergent for every s and is equal to e^{cs^2}. If $c = 0$, then (176) is equal to 1 for every s.

Now we shall consider some distributions related to the stable distributions. Let us suppose that $\xi \sim S(\alpha, 1, \cos\frac{\alpha\pi}{2}, 0)$ where $0 < \alpha < 1$. Then ξ is a positive random variable and (174) holds for $\operatorname{Re}(s) \geqq 0$.

The random variable $\xi^{-\alpha}$ has some importance in probability theory. Let

$$G_\alpha(x) = \underset{\sim}{P}\{\xi^{-\alpha} \leqq x\} \tag{177}$$

for $0 < \alpha < 1$. Obviously we have

$$G_\alpha(x) = \begin{cases} 1-F(x^{-1/\alpha};\, \alpha,\, 1,\, \cos\frac{\alpha\pi}{2},\, 0) & \text{for } x > 0, \\ 0 & \text{for } x \leqq 0. \end{cases} \tag{178}$$

The random variable $\xi^{-\alpha}$ has a density function

$$g_\alpha(x) = \begin{cases} \dfrac{1}{\alpha x^{1+\frac{1}{\alpha}}} f(x^{-1/\alpha};\, \alpha,\, 1,\, \cos\frac{\alpha\pi}{2},\, 0) & \text{for } x > 0, \\ 0 & \text{for } x \leqq 0. \end{cases} \tag{179}$$

The Laplace-Stieltjes transform of $G_\alpha(x)$ can be expressed by the Mittag-Leffler function

(180) $$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$$

for $0 < \alpha < 1$. (See G. Mittag-Leffler [135],[136] and A. Erdélyi [49] Vol. 3, p. 206.) We have

(181) $$\int_0^\infty e^{-sx} dG_\alpha(x) = E_\alpha(-s)$$

for every s and $0 < \alpha < 1$. (See H. Pollard [311] and W. Feller [253] p. 428.)

By using Theorem 7 we can deduce from (181) the following result: If $1 < \alpha < 2$, then

(182) $$\int_0^\infty e^{-sx} f(x;\ \alpha,-1,-\cos\frac{\alpha\pi}{2},\ 0)dx = \frac{1}{\alpha} E_{\frac{1}{\alpha}}(-s)$$

for all s. (See V. M. Zolotarev [339].)

By (175) it follows immediately that

(183) $$\int_{-\infty}^\infty e^{-sx} f(x;\ \alpha,-1,-\cos\frac{\alpha\pi}{2},\ 0)dx = e^{(-s)^\alpha}$$

for $\mathrm{Re}(s) \leq 0$ and $1 < \alpha < 2$.

If we use Theorem 7, then we can prove that

(184) $$f(x;\ \alpha,-1,-\cos\frac{\alpha\pi}{2},\ 0) = \frac{1}{\alpha} g_{\frac{1}{\alpha}}(x)$$

for $x > 0$ and $1 < \alpha < 2$ where the right-hand side of (184) is given by (179). By (181) this implies (182). By using the notation (128) we can write (184) in the following form

$$h(x;\ \alpha,\ 2-\alpha) = \frac{1}{x^{\alpha+1}} h(x^{-\frac{1}{\alpha}}, \frac{1}{\alpha}, \frac{1}{\alpha}) \tag{185}$$

for $x > 0$ and $1 < \alpha < 2$. This is indeed true by (138).

According to (181) if $\xi \sim S(\alpha, 1, \cos\frac{\alpha\pi}{2}, 0)$ and $0 < \alpha < 1$, then

$$\underset{\sim}{E}\{e^{-s\xi^{-\alpha}}\} = E_{\alpha}(-s) \tag{186}$$

for every s .

If $0 < \alpha < 1$ and if take into consideration that $E_{\alpha}(-x^{\alpha})$ is a decreasing function of x in the interval $[0, \infty)$ which varies from 1 to 0, then we can easily see that

$$H_{\alpha}(x) = \begin{cases} 1 - E_{\alpha}(-x^{\alpha}) & \text{for } x \geqq 0 , \\ 0 & \text{for } x < 0 \end{cases} \tag{187}$$

is a distribution function. We have

$$\int_0^{\infty} e^{-sx} dH_{\alpha}(x) = \frac{1}{1+s^{\alpha}} \tag{188}$$

for $\text{Re}(s) > -1$. If $\text{Re}(s) > 1$, then by (180) we obtain that

$$\begin{aligned} \int_0^{\infty} e^{-sx} dH_{\alpha}(x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\Gamma(k\alpha)} \int_0^{\infty} e^{-sx}\, x^{k\alpha-1} dx = \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{s^{k\alpha}} = \frac{1}{1+s^{\alpha}} \end{aligned} \tag{189}$$

and (188) can be obtained by analytical continuation.

In his investigations of branching processes V. M. Zolotarev [340] encountered the distribution function

$$(190)\qquad S_\alpha(x) = \begin{cases} 1-\dfrac{\alpha}{\Gamma(\frac{1}{\alpha})}\displaystyle\int_0^\infty e^{-(\frac{x}{u})^\alpha} f(u, \alpha, 1, \cos\frac{\alpha\pi}{2}, 0)\frac{du}{u} & \text{for } x > 0 , \\ 0 \quad \text{for } x < 0 \end{cases}$$

where $0 < \alpha < 1$, and showed that

$$(191)\qquad \int_0^\infty e^{-sx} dS_\alpha(x) = 1 - \frac{s}{(1+s^\alpha)^{1/\alpha}}$$

for $\operatorname{Re}(s) > -1$.

In 1953 Chung-Jeh Chao [232] proved that if $\xi \sim S(\alpha, \beta, c, 0)$ where $\alpha \neq 1$ and $c > 0$, then

$$(192)\qquad P\{\xi \leq 0\} = \frac{1}{2} - \frac{\gamma}{2\alpha}$$

where γ is defined by (125), that is,

$$(193)\qquad \gamma = \frac{2}{\pi} \operatorname{arc\,tan}(\beta \tan \frac{\alpha\pi}{2})$$

and $-\frac{\pi}{2} < \operatorname{arc\,tan} x < \frac{\pi}{2}$. **(See Problem 46.7 .)**

If $\xi \sim S(\alpha, \beta, c, 0)$ where $c > 0$, then ξ has a continuous density function $f(x; \alpha, \beta, c, 0)$ for which we have

$$(194)\qquad f(0; \alpha, \beta, c, 0) = \frac{\Gamma(1+\frac{1}{\alpha})(\cos\frac{\gamma\pi}{2})^{\frac{1}{\alpha}}}{c^{1/\alpha}\,\pi} \cos\frac{\gamma\pi}{2\alpha}$$

whenever $\alpha \neq 1$. The constant γ is defined by (193). Both (192) and (194) are mentioned by V. M. Zolotarev [341].

If $\alpha \neq 1$, then by (122) we obtain that

$$(195) \qquad f(0;\, \alpha,\, \beta,\, c,\, 0) = \frac{1}{c^{1/\alpha}\alpha\pi} \int_0^\infty e^{-u} \cos(u\, \beta \tan \frac{\alpha\pi}{2}) u^{\frac{1}{\alpha}-1}\, du$$

and the evaluation of (195) leads to (194). (See D. Bierens de Hahn [11] p. 505.)

The Moments of Stable Distributions. If $\xi \sim S(2, 0, \frac{1}{2}, 0)$, that is, if ξ has a normal distribution $N(0, 1)$, then

$$(196) \qquad \underset{\sim}{E}\{|\xi|^\delta\} = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}} x^\delta dx = \frac{2^{\frac{\delta}{2}}\, \Gamma(\frac{\delta+1}{2})}{\sqrt{\pi}}$$

for $\mathrm{Re}(\delta) > -1$. If $\xi \sim S(1, 0, 1, 0)$, that is, if ξ has a Cauchy distribution then

$$(197) \qquad \underset{\sim}{E}\{|\xi|^\delta\} = \frac{2}{\pi} \int_0^\infty \frac{x^\delta}{1+x^2} dx = \frac{1}{\cos \frac{\delta\pi}{2}}$$

for $-1 < \delta < 1$.

If $\xi \sim S(\alpha, \beta, c, 0)$ and $\alpha \neq 1$, then we have

$$(198) \qquad \underset{\sim}{E}\{|\xi|^\delta\} = \left(\frac{c}{\cos \frac{\gamma\pi}{2}}\right)^{\frac{\delta}{\alpha}} \frac{\Gamma(1-\frac{\delta}{\alpha})}{\Gamma(1-\delta)} \frac{\cos \frac{\gamma\delta\pi}{2\alpha}}{\cos \frac{\delta\pi}{2}}$$

for $-1 < \delta < 1$ where γ is defined by (193). Formula (198) has been found by Chung-Jeh Chao [232] and V. M. Zolotarev [341]

If $\xi \sim S(\alpha, \beta, c, m)$ and $0 < \alpha < 2$, then

(199) $$E\{|\xi|^{\delta}\} < \infty$$

if $-1 < \delta < \alpha$ and

(200) $$E\{|\xi|^{\delta}\} = \infty$$

if $\delta \geq \alpha$. (See B. V. Gnedenko [259].)

Finally, we shall mention some characteristic properties of stable distribution functions.

Let us suppose that $F(x)$ is a stable distribution function. We exclude the normal distribution $(\alpha = 2)$ and the degenerate distribution $(c = 0)$. Then there exists an α $(0 < \alpha < 2)$ and nonnegative constants c_1, c_2 with sum $c_1 + c_2 > 0$ such that

(201) $$\lim_{x \to \infty} x^{\alpha}[1-F(x)] = c_2$$

and

(202) $$\lim_{x \to \infty} x^{\alpha}F(-x) = c_1 .$$

The constant α is the characteristic exponent of $F(x)$, and c_1 and c_2 are the constants appearing in the representation (84). Thus it follows that necessarily $F(x) = F(x; \alpha, \beta, c, m)$ where α $(0 < \alpha < 2)$ is the constant appearing in (201) and (202), $\beta = (c_2 - c_1)/(c_2 + c_1)$ and c is determined by (88). The constant m is not determined by (201) and (202). (See P. Lévy [113] p. 201.)

If $F(x)$ is a proper stable distribution function (the degenerate case, $c = 0$, is excluded), then $F(x)$ is absolutely continuous and has derivatives of all orders for every x. If $\phi(s)$ denotes the Laplace-Stieltjes transform of $F(x)$, then by (82)

(203) $$|\phi(s)| = e^{-c|s|^{\alpha}}$$

where $c > 0$ and $0 < \alpha < 2$. By Theorem 41.5 we can conclude that

(204) $$F^{(n)}(x) = \frac{i^{n-1}}{2\pi} \int_{-\infty}^{\infty} \phi(iu) e^{iux} u^{n-1} du$$

for all x. (A. Ya. Khintchine [278].)

I. A. Ibragimov and K. E. Chernin [268] proved that every stable distribution function is unimodal. A distribution function $F(x)$ is called unimodal if there exists at least one $x = a$ such that $F(x)$ is convex for $x < a$ and concave for $x > a$.

A proper stable distribution function $F(x)$ with characteristic exponent $\alpha \geqq 1$ is regular on the entire real axis. For $\alpha > 1$ the distribution function $F(x)$ is an entire function.

If $\phi(s)$ is the Laplace-Stieltjes transform of $F(x)$ and we expand $F(x)$ into Taylor series at the point $x = a$, that is,

(205) $$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(a)}{n!} (x-a)^n ,$$

then by (204)

(206) $$\left|\frac{F^{(n)}(a)}{n!}\right| \leqq \frac{\Gamma(\frac{n}{\alpha})}{\pi \alpha n!} c^{-\frac{n}{\alpha}}$$

for $n = 0,1,\ldots$ and hence the radius of convergence at the point $x = a$ is

$$(207) \qquad R(a) = \frac{1}{\limsup\limits_{n\to\infty} \left|\frac{F^{(n)}(a)}{n!}\right|^{\frac{1}{n}}} \begin{cases} = \infty & \text{for } \alpha > 1 , \\ \geqq c & \text{for } \alpha = 1 . \end{cases}$$

This result is due to A. I. Lapin. (See B. V. Gnedenko and A. N. Kolmogorov [260] p. 183.)

A. V. Skorohod [320] proved that if $\alpha < 1$, then

$$(208) \qquad f(x;\alpha,\beta,c,0) = \begin{cases} \frac{1}{x} g_1\left(\frac{1}{x^\alpha}\right) & \text{for } x > 0 , \\ \frac{1}{|x|} g_2\left(\frac{1}{|x|^\alpha}\right) & \text{for } x < 0 , \end{cases}$$

where $g_1(z)$ and $g_2(z)$ are entire functions of z .

43. Limit Laws. Throughout this section we suppose that $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are mutually independent real random variables. We consider a random trial with which a probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$ is associated and we suppose that each $\xi_k(\omega)$ is a finite measurable function of ω defined on Ω and that (41.36) is satisfied for $n = 1,2,\ldots$.

If $\underset{\sim}{E}\{|\xi_k|\} < \infty$, then let us write

$$(1) \qquad a_k = \underset{\sim}{E}\{\xi_k\} = \int_{-\infty}^{\infty} x\, d_x \underset{\sim}{P}\{\xi_k \leqq x\}$$

for $k = 1,2,\ldots,$ and if $\underset{\sim}{E}\{\xi_k^2\} < \infty$, then let us write

$$(2) \qquad b_k^2 = \underset{\sim}{\mathrm{Var}}\{\xi_k\} = \int_{-\infty}^{\infty} (x-a_k)^2 d_x\, \underset{\sim}{P}\{\xi_k \leqq x\}$$

for $k = 1,2,\ldots$.

Most of the results deduced in this section are concerned with the case of mutually independent and identically distributed real random variables. In this particular case we use the notation

(3) $$\underset{\sim}{P}\{\xi_k \leq x\} = F(x) ,$$

(4) $$a = \int_0^\infty x dF(x) ,$$

and

(5) $$b^2 = \int_0^\infty (x-a)^2 dF(x)$$

provided that the integrals in (4) and (5) are absolutely convergent.

The Weak Law of Large Numbers. In 1929 A. Ya. Khintchine [418] proved the weak law of large numbers in the following general form.

Theorem 1. Let $\{\xi_k;\ k = 1,2,\ldots\}$ be a sequence of pairwise independent real random variables with a common distribution function $\underset{\sim}{P}\{\xi_k \leq x\} = F(x)$. If $\underset{\sim}{E}\{|\xi_k|\} < \infty$ and $a = \underset{\sim}{E}\{\xi_k\}$, then for any $\varepsilon > 0$ we have

(6) $$\lim_{n \to \infty} \underset{\sim}{P}\left\{\left|\frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n} - a\right| < \varepsilon\right\} = 1 .$$

Proof. First, we note that in Theorem 1 it is not necessary to assume that the random variables $\{\xi_k\}$ are mutually independent for (6) is valid for pairwise independent variables $\{\xi_k\}$ too. The variables $\{\xi_k\}$ are pairwise independent if (41.36) holds for $n = 2$.

Next, we observe that if $\underset{\sim}{E}\{\xi_k^2\} < \infty$, then (6) is a simple consequence of the inequality (41.31). If we write $b^2 = \underset{\sim}{Var}\{\xi_k\}$ and

(7) $$\eta_n = \frac{\xi_1+\xi_2+\ldots+\xi_n}{n}$$

for $n = 1,2,\ldots,$ then $E\{\eta_n\} = a$ and $Var\{\eta_n\} = b^2/n$. Thus for any $\varepsilon > 0$ by (41.31) we have

(8) $$0 \leqq P\{|\eta_n - a| \geqq \varepsilon\} = P\{(\eta_n - a)^2 \geqq \varepsilon^2\} \leqq \frac{b^2}{\varepsilon^2 n} .$$

If $n \to \infty$, then $b^2/\varepsilon^2 n \to 0$, and (6) follows.

To show that (6) is valid, we shall remove in the above proof the restrictive condition that $Var\{\xi_k\}$ exists. This can be achieved by using the method of truncation.

We shall prove that for any $\varepsilon > 0$ and $\omega > 0$ there exists an $N(\varepsilon, \omega)$ such that

(9) $$P\{|\eta_n - a| \geqq \varepsilon\} \leqq \omega$$

if $n > N(\varepsilon, \omega)$. For a fixed $\delta > 0$ and for $k = 1,2,\ldots,n$ let us define

(10) $$\xi_k^* = \begin{cases} \xi_k & \text{if } |\xi_k| \leqq \delta n \\ 0 & \text{if } |\xi_k| > \delta n \end{cases}$$

for $k = 1,2,\ldots,n$, and write

(11) $$\eta_n^* = \frac{\xi_1^*+\xi_2^*+\ldots+\xi_n^*}{n}$$

Define $A = \{|\eta_n - a| \geqq \varepsilon\}$ and $B = \{\eta_n = \eta_n^*\}$. Then we have $A = AB+A\overline{B} \subset AB+\overline{B}$, and thus $P\{A\} \leqq P\{AB\}+P\{\overline{B}\}$, that is,

$$(12) \qquad P\{|\eta_n - a| \geq \varepsilon\} \leq P\{|\eta_n^* - a| \geq \varepsilon\} + P\{\eta_n \neq \eta_n^*\} .$$

Since $E\{|\xi_k|\} = C < \infty$, it follows that

$$(13) \qquad E\{\xi_k^*\} = a_n^* = \int_{-\delta n}^{\delta n} x dF(x)$$

$(k = 1,2,\dots,n)$ converges to a as $n \to \infty$. Hence for any $\varepsilon > 0$ we have $|a_n^* - a| < \frac{\varepsilon}{2}$, if n is sufficiently large.

If n is so large that $|a_n^* - a| < \frac{\varepsilon}{2}$, then we have

$$(14) \qquad \begin{aligned} & P\{|\eta_n^* - a| \geq \varepsilon\} \leq P\{|\eta_n^* - a_n^*| \geq \frac{\varepsilon}{2}\} \leq \frac{4 Var\{\eta_n^*\}}{\varepsilon^2} \leq \\ & \leq \frac{4 Var\{\xi_1^*\}}{\varepsilon^2 n} \leq \frac{4 E\{\xi_1^{*2}\}}{\varepsilon^2 n} \leq \frac{4\delta}{\varepsilon^2} E\{|\xi_1^*|\} = \frac{4C\delta}{\varepsilon^2} \end{aligned}$$

and here we used (41.31) and (10). On the other hand we have

$$(15) \qquad \begin{aligned} & P\{\eta_n \neq \eta_n^*\} \leq \sum_{k=1}^{n} P\{\xi_k \neq \xi_k^*\} = \sum_{k=1}^{n} P\{|\xi_k| > \delta n\} = \\ & = n \int_{|x|>\delta n} dF(x) \leq \frac{n}{\delta n} \int_{|x|>\delta n} |x| dF(x) < \delta \end{aligned}$$

if n is large enough.

Here we used that

$$(16) \qquad \int_{|x|>\delta n} |x| dF(x) < \delta^2$$

if n is large enough. Since the left-hand side of (16) tends to 0 as $n \to \infty$, it follows that (16) is valid if n is large enough.

Thus by (12), (14) and (15) we obtain that

(17) $$P\{|\eta_n - a| \geq \varepsilon\} \leq (\frac{4C}{\varepsilon^2} + 1)\delta$$

if n is sufficiently large. Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary, the last inequality proves (9) and consequently (6) too.

If $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of real random variables for which $a_k = E\{\xi_k\}$ exists for $k = 1,2,\ldots$, then we say that the weak law of large numbers is valid for the sequence $\{\xi_k\}$ whenever for every $\varepsilon > 0$

(18) $$\lim_{n\to\infty} P\{|\frac{1}{n}\sum_{k=1}^{n}\xi_k - \frac{1}{n}\sum_{k=1}^{n}a_k| < \varepsilon\} = 1 .$$

First, at the end of the seventeenth century J. Bernoulli [348] proved that if $\{\xi_k\}$ is a sequence of mutually independent random variables for which $P\{\xi_k = 1\} = p$ and $P\{\xi_k = 0\} = q$ $(p+q) = 1)$, then (18) holds. In 1837 S. D. Poisson [449] proved that if $\{\xi_k\}$ is a sequence of mutually independent random variables for which $P\{\xi_k = 1\} = p_k$ and $P\{\xi_k = 0\} = q_k$ $(p_k + q_k = 1)$, then (18) holds. In 1867 P. L. Chebyshev [474] proved that if $\{\xi_k\}$ is a sequence of mutually independent discrete random variables for which $E\{\xi_k\} = a_k$ and $Var\{\xi_k\} = b_k$ exist, then

(19) $$P\{|\frac{1}{n}\sum_{k=1}^{n}(\xi_k - a_k)| < \varepsilon\} \geq 1 - \frac{1}{n^2\varepsilon^2}\sum_{k=1}^{n} b_k^2$$

for any $\varepsilon > 0$. From (19) Chebyshev concluded that (18) holds if $b_k^2 \leq B < \infty$ for every k . (See also A.A. Markov [584] pp. 58-62.) Obviously, we can replace the last condition by

(20) $$\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} b_k^2 = 0 .$$

In 1928 A. N. Kolmogorov [421], [423], derived necessary and sufficient conditions for a sequence of mutually independent real random variables to obey the weak law of large numbers. In 1929 A. Ya. Khintchine [418] proved that if $\{\xi_k\}$ is a sequence of pairwise independent, identically distributed real random variables with finite expectations, then (18) holds. For further extensions of the weak law of large numbers we refer to B. V. Gnedenko [531], and B. V. Gnedenko and A. N. Kolmogorov [260].

The Strong Law of Large Numbers. Let $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ be a sequence of real random variables for which $a_k = \mathbf{E}\{\xi_k\}$ $(k = 1,2,\ldots)$ exist. We say that the sequence $\{\xi_k\}$ obeys the strong law of large numbers if

(21) $$\mathbf{P}\{\lim_{n \to \infty} \frac{(\xi_1 - a_1) + (\xi_2 - a_2) + \ldots + (\xi_n - a_n)}{n} = 0\} = 1 .$$

If $(\Omega, \mathcal{B}, \mathbf{P})$ is a probability space and $\xi_k(\omega)$ $(k = 1,2,\ldots)$ are real random variables defined on Ω, then for any choice of a_k $(k = 1,2,\ldots)$ the function

(22) $$\eta_n(\omega) = \frac{1}{n} \sum_{k=1}^{n} (\xi_k(\omega) - a_k)$$

is a random variable. Denote by A the set of points $\omega \in \Omega$ for which the sequence $\{\eta_n(\omega)\}$ is convergent and the limit is 0, that is,

(23) $$A = \{\omega : \lim_{n \to \infty} \eta_n(\omega) = 0\} = \{\omega : \lim_{n \to \infty} \sup |\eta_n(\omega)| = 0\} .$$

We can easily see that A is a random event, that is, $A \in \mathcal{B}$, and thus we can speak about the probability of A. If $\underset{\sim}{P}\{A\} = 1$, then $\{\xi_k\}$ obeys the strong law of large numbers (for the given sequence $\{a_k\}$).

By using Theorem 41.1 we can formulate a useful sufficient condition which ensures that $\underset{\sim}{P}\{A\} = 1$.

<u>Lemma 1.</u> <u>Let</u> $(\Omega, \mathcal{B}, \underset{\sim}{P})$ <u>be a probability space and</u> $\{\eta_n(\omega)\}$ <u>be a sequence of real random variables defined on</u> Ω. <u>If for any</u> $\varepsilon > 0$ <u>and for some positive integers</u> $n_1 < n_2 < \ldots < n_k < \ldots$ <u>we have</u>

$$(24) \qquad \sum_{k=1}^{\infty} \underset{\sim}{P}\{\max_{n_k < i \leq n_{k+1}} |\eta_i(\omega)| > \varepsilon\} < \infty ,$$

<u>then</u>

$$(25) \qquad \underset{\sim}{P}\{\lim_{n \to \infty} \eta_n(\omega) = 0\} = 1 .$$

<u>Proof.</u> Let

$$(26) \qquad A(\varepsilon) = \{\limsup_{n \to \infty} |\eta_n(\omega)| > \varepsilon\}$$

for $\varepsilon \geq 0$. Then $A(\varepsilon) \in \mathcal{B}$ for every $\varepsilon \geq 0$.

If (24) holds for every $\varepsilon > 0$, then by Theorem 41.1 we can conclude that with probability 1 only finitely many events $\{|\eta_n(\omega)| > \varepsilon\}$ $(n = 1, 2, \ldots)$ occur, and this implies that $\underset{\sim}{P}\{A(\varepsilon)\} = 0$ for every $\varepsilon > 0$. Now we shall prove that $\underset{\sim}{P}\{A(0)\} = 0$. Since evidently,

$$A(0) \subset \sum_{r=1}^{\infty} A(\tfrac{1}{r}) , \tag{27}$$

it follows by Boole's inequality that

$$0 \leq P\{A(0)\} \leq \sum_{r=1}^{\infty} P\{A(\tfrac{1}{r})\} = 0 . \tag{28}$$

Accordingly, $P\{\bar{A}\} = P\{A(0)\} = 0$ which proves (25).

We note that, conversely, if $P\{A(0)\} = 0$, then $P\{A(\varepsilon)\} = 0$ for every $\varepsilon > 0$. For, obviously $A(\varepsilon) \subset A(0)$ holds for every $\varepsilon > 0$, and therefore $0 \leq P\{A(\varepsilon)\} \leq P\{A(0)\} = 0$.

Next we shall prove a generalization of the inequality (19) which makes it possible to prove the law of large numbers for mutually independent random variables. This inequality was found in 1928 by A. N. Kolmogorov [421], [423].

Lemma 2. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be mutually independent real random variables with expectations $a_j = E\{\xi_j\}$ and variances $b_j^2 = Var\{\xi_j\}$ $(j = 1,2,\ldots,n)$. For every $\varepsilon > 0$ we have*

$$P\{\max_{1 \leq k \leq n} |\sum_{j=1}^{k} (\xi_j - a_j)| \geq \varepsilon\} \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} b_k^2 . \tag{29}$$

Proof. Let

$$\zeta_k = \sum_{j=1}^{k} (\xi_j - a_j) \tag{30}$$

for $k = 1,2,\ldots,n$. Define n random variables $\chi_1, \chi_2, \ldots, \chi_n$ as follows

$$(31) \qquad \chi_k = \begin{cases} 1 & \text{if } |\zeta_1| < \varepsilon, |\zeta_2| < \varepsilon, \dots, |\zeta_{k-1}| < \varepsilon, |\zeta_k| \geqq \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_1 + \chi_2 + \dots + \chi_n$ is either 0 or 1, and $\chi_1 + \chi_2 + \dots + \chi_n = 1$ if and only if $|\zeta_k| \geqq \varepsilon$ for some $k = 1, 2, \dots, n$.

The left-hand side of (29) can be expressed in the following way:

$$(32) \qquad \begin{aligned} &\mathbf{P}\{\max_{1 \leqq k \leqq n} |\zeta_k| \geqq \varepsilon\} = \mathbf{P}\{\chi_1 + \dots + \chi_n = 1\} = \\ &= \mathbf{E}\{\chi_1 + \dots + \chi_n\} = \sum_{k=1}^{n} \mathbf{E}\{\chi_k\} . \end{aligned}$$

If we multiply (32) by ε^2 then we obtain that

$$(33) \qquad \begin{aligned} \varepsilon^2 \mathbf{P}\{\max_{1 \leqq k \leqq n} |\zeta_k| \geqq \varepsilon\} &= \varepsilon^2 \sum_{k=1}^{n} \mathbf{E}\{\chi_k\} \leqq \sum_{k=1}^{n} \mathbf{E}\{\chi_k \zeta_k^2\} \\ &\leqq \sum_{k=1}^{n} \mathbf{E}\{\chi_k \zeta_n^2\} = \mathbf{E}\{(\sum_{k=1}^{n} \chi_k) \zeta_n^2\} \leqq \mathbf{E}\{\zeta_n^2\} = \sum_{k=1}^{n} b_k^2 \end{aligned}$$

which proves (29). In (33) we used the following inequalities:

$$(34) \qquad \varepsilon^2 \mathbf{E}\{\chi_k\} \leqq \mathbf{E}\{\chi_k \zeta_k^2\}$$

and

$$(35) \qquad \mathbf{E}\{\chi_k \zeta_k^2\} \leqq \mathbf{E}\{\chi_k \zeta_n^2\}$$

for $k = 1, 2, \dots, n$. To prove (34) let us observe that $\varepsilon^2 \chi_k \leqq \zeta_k^2 \chi_k$ for $k = 1, 2, \dots, n$. If $\chi_k = 0$, then this is obvious. If $\chi_k = 1$ then $|\zeta_k| \geqq \varepsilon$, and the inequality holds in this case too. By forming expectations in the inequality just mentioned we get (34). To prove (35) let us write

$$(36)\qquad \chi_k\zeta_n^2 = \chi_k\zeta_k^2 + 2\chi_k\zeta_k(\zeta_n - \zeta_k) + \chi_k(\zeta_n - \zeta_k)^2$$

and form the expectation of (36). The second term on the right-hand side of (36) has expectation $2E\{\chi_k\zeta_k\}E\{\zeta_n - \zeta_k\} = 0$ because $\chi_k\zeta_k$ and $\zeta_n - \zeta_k$ are independent, and $E\{\zeta_n - \zeta_k\} = 0$. The last term on the right-hand side of (36) has expectation ≥ 0. This proves (35). In (33) we also used that $\chi_1 + \ldots + \chi_n \leq 1$. This completes the proof of the inequality (29).

In 1930 A. N. Kolmogorov [424] proved the strong law of large numbers in the following general form.

Theorem 2. *Let $\{\xi_k\}$ be a sequence of mutually independent real random variables for which $a_k = E\{\xi_k\}$ and $b_k^2 = \mathrm{Var}\{\xi_k\}$ exist. If*

$$(37)\qquad \sum_{k=1}^{\infty} \frac{b_k^2}{k^2} < \infty ,$$

then

$$(38)\qquad P\{\lim_{n\to\infty} \frac{(\xi_1 - a_1)+(\xi_2 - a_2)+\ldots+(\xi_n - a_n)}{n} = 0\} = 1 .$$

Proof. Let $\zeta_n = (\xi_1 - a_1)+\ldots+(\xi_n - a_n)$ for $n = 1,2,\ldots$. Then (38) can be expressed as

$$(39)\qquad P\{\lim_{n\to\infty} \frac{\zeta_n}{n} = 0\} = 1 .$$

If for any $\varepsilon > 0$ and for some positive integers $n_1 < n_2 < \ldots < n_k < \ldots$ we have

$$\sum_{k=1}^{\infty} \underset{\sim}{P}\{ \max_{n_k < i \leq n_{k+1}} \left| \frac{\zeta_i}{i} \right| > \varepsilon\} < \infty , \tag{40}$$

then by Lemma 1 we can conclude that (39) is true. In (40) we can write that

$$\begin{aligned} &\underset{\sim}{P}\{ \max_{n_k < i \leq n_{k+1}} \left| \frac{\zeta_i}{i} \right| > \varepsilon\} \leq \underset{\sim}{P}\{ \max_{n_k < i \leq n_{k+1}} | \zeta_i | > n_k \varepsilon\} \leq \\ &\underset{\sim}{P}\{ \max_{1 \leq i \leq n_{k+1}} | \zeta_i | > n_k \varepsilon\} \leq \frac{1}{\varepsilon^2 n_k^2} \sum_{j=1}^{n_{k+1}} b_j^2 \end{aligned} \tag{41}$$

where the last inequality follows from Lemma 2.

Now if

$$\sum_{k=1}^{\infty} \frac{1}{n_k^2} \sum_{j=1}^{n_{k+1}} b_j^2 = \sum_{j=1}^{\infty} b_j^2 \sum_{n_{k+1} \geq j} \frac{1}{n_k^2} < \infty ; \tag{42}$$

then (40) holds and this implies (38).

If we choose $n_k = 2^k$ for $k = 1,2,\ldots,$ then in (42)

$$\sum_{n_{k+1} \geq j} \frac{1}{n_k^2} = \sum_{2^{k+1} \geq j} \frac{1}{2^{2k}} = \frac{4}{3.2^{2k_o}} \leq \frac{16}{3j^2} \tag{43}$$

for $j = 1,2,\ldots$. In (43) $2^{k_o} < j \leq 2^{k_o+1}$.

Accordingly, if (37) is satisfied, then (42) and (40) hold and therefore (38) is true. This completes the proof.

From Theorem 2 A. N. Kolmogorov [424] deduced the following general theorem for mutually independent and identically distributed random variables.

Theorem 3. Let $\{\xi_k\}$ be a sequence of mutually independent and identically distributed real random variables for which $a = \underset{\sim}{E}\{\xi_k\}$ exists. Then

(44) $$\underset{\sim}{P}\{\lim_{n \to \infty} \frac{\xi_1 + \xi_2 + \dots + \xi_n}{n} = a\} = 1 .$$

Proof. First we shall introduce a useful definition. Two sequences of random variables $\{\xi_k\}$ and $\{\xi_k^*\}$ are called equivalent if

(45) $$\sum_{k=1}^{\infty} \underset{\sim}{P}\{\xi_k \neq \xi_k^*\} < \infty .$$

From Theorem 41.1 it follows immediately that if $\{\xi_k\}$ and $\{\xi_k^*\}$ are equivalent sequences, then

(46) $$\underset{\sim}{P}\{\lim_{k \to \infty} (\xi_k - \xi_k^*) = 0\} = 1$$

holds. Furthermore (46) implies that

(47) $$\underset{\sim}{P}\{\lim_{n \to \infty} \frac{(\xi_1 - \xi_1^*) + (\xi_2 - \xi_2^*) + \dots + (\xi_n - \xi_n^*)}{n} = 0\} = 1.$$

By using this observation we shall prove (44) in such a way that we replace the sequence $\{\xi_k\}$ by an equivalent sequence $\{\xi_k^*\}$ for which Theorem 2 is applicable, and then by (47) we can conclude that (44) is valid. To find an equivalent sequence $\{\xi_k^*\}$ we use the method of truncation. Let

$$\xi_k^* = \begin{cases} \xi_k & \text{if } |\xi_k| \leq k , \\ 0 & \text{if } |\xi_k| > k . \end{cases} \tag{48}$$

Since

$$\begin{aligned} \sum_{k=1}^{\infty} P\{|\xi_k| > k\} &= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} P\{j < |\xi_1| \leq j+1\} = \\ &= \sum_{j=1}^{\infty} jP\{j < |\xi_1| \leq j+1\} \leq E\{|\xi_1|\} < \infty , \end{aligned} \tag{49}$$

it follows that $\{\xi_k\}$ and $\{\xi_k^*\}$ are equivalent sequences.

Let $P\{\xi_k \leq x\} = F(x)$. Then

$$a = \int_{-\infty}^{\infty} x dF(x) \tag{50}$$

and

$$a_k^* = E\{\xi_k^*\} = \int_{|x| \leq k} x dF(x) \tag{51}$$

for $k = 1,2,\ldots$. We have $\lim_{k \to \infty} a_k^* = a$, and hence

$$\lim_{n \to \infty} \frac{a_1^* + a_2^* + \ldots + a_n^*}{n} = a . \tag{52}$$

Now each ξ_k^* has a finite variance, namely

$$\mathrm{Var}\{\xi_k^*\} \leq E\{\xi_k^{*2}\} = \int_{|x| \leq k} |x|^2 dF(x) \leq c_1 + 2c_2 + \ldots + kc_k \tag{53}$$

where

$$c_j = \int_{j-1<|x|\leq j} |x|\,dF(x) \tag{54}$$

for $j = 1,2,\ldots$. Furthermore,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\mathrm{Var}\{\xi_k^*\}}{k^2} &\leq \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} jc_j = \sum_{j=1}^{\infty} jc_j \sum_{k=j}^{\infty} \frac{1}{k^2} \leq \\ &\leq 2\sum_{j=1}^{\infty} c_j = 2E\{|\xi_1|\} < \infty .\end{aligned} \tag{55}$$

Here we used that

$$\sum_{k=j}^{\infty} \frac{1}{k^2} \leq \frac{1}{j^2} + \sum_{k=j+1}^{\infty} \frac{1}{k(k-1)} = \frac{1}{j^2} + \frac{1}{j} \leq \frac{2}{j} \tag{56}$$

for $j = 1,2,\ldots$.

The random variables $\xi_1^*, \xi_2^*, \ldots, \xi_k^*, \ldots$ are mutually independent. They have finite expectations $a_k^* = E\{\xi_k^*\}$ $(k = 1,2,\ldots)$ and (55) holds. Thus by Theorem 2 it follows that

$$P\{\lim_{n\to\infty} \frac{(\xi_1^* - a_1^*)+(\xi_2^* - a_2^*)+\ldots+(\xi_n^* - a_n^*)}{n} = 0\} = 1 . \tag{57}$$

By (52) it follows from (57) that

$$P\{\lim_{n\to\infty} \frac{\xi_1^* + \xi_2^* + \ldots + \xi_n^*}{n} = a\} = 1 . \tag{58}$$

By a comparison of (47) and (58) we obtain (44) which was to be proved.

Theorem 3 in the particular case when $P\{\xi_k = 0\} = P\{\xi_k = 1\} = \frac{1}{2}$ for $k = 1,2,\ldots$ was proved in 1909 by <u>É. Borel</u> [16]. In 1917 <u>F. P. Cantelli</u> [18] proved Theorem 3 in the particular case when $P\{\xi_k = 1\} = p$ and

$\underset{\sim}{P}\{\xi_k = 0\} = q$ for $k = 1,2,\ldots,$ where $p+q = 1$. In the general case Theorem 3 was proved in 1929 by A. N. Kolmogorov [424].

Some General Limit Laws. Let $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ be a sequence of mutually independent and identically distributed real random variables. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$. We are interested in studying the asymptotic behavior of ζ_n as $n \to \infty$. To achieve this goal we shall introduce some useful definitions and we shall prove several auxiliary theorems. The results presented here have been found by K. L. Chung and W. H. J. Fuchs [25]. See also K. L. Chung and D. Ornstein [366].

Definition 1. *A real number c is called a possible value of the sequence $\{\zeta_n\}$ if for every $\varepsilon > 0$ there exists an n $(n = 1,2,\ldots)$ such that*

$$\underset{\sim}{P}\{|\zeta_n - c| < \varepsilon\} > 0 . \tag{59}$$

Denote by P the set of all the possible values of $\{\zeta_n\}$.

Definition 2. *A real number c is called a recurrent value of the sequence $\{\zeta_n\}$ if for every $\varepsilon > 0$*

$$\underset{\sim}{P}\{|\zeta_n - c| < \varepsilon \text{ for infinitely many } n = 1,2,\ldots\} = 1 . \tag{60}$$

Denote by R the set of all the recurrent values of $\{\zeta_n\}$.

Theorem 4. *Either R is empty or R is identical with P .*

Proof. Obviously $R \subset P$ because every recurrent value is necessarily a possible value. We shall prove that if $R \neq \Theta$ and $c \in P$, then $c \in R$.

Suppose that $c_1 \varepsilon R$ and $c_2 \varepsilon P$. First we shall show that $c_1 - c_2 \varepsilon R$. This follows from the following inequalities:

$$\begin{aligned}(61)\quad & \underset{\sim}{P}\{|\zeta_n - c_1| < \varepsilon \text{ for only a finite number of } n = 1,2,\ldots\} \geqq \\ & \geqq \underset{\sim}{P}\{|\zeta_k - c_2| < \varepsilon,\ |\zeta_{k+n} - \zeta_k - (c_1 - c_2)| < 2\varepsilon \text{ for only a finite} \\ & \text{number of } n = 1,2,\ldots\} = \underset{\sim}{P}\{|\zeta_k - c_2| < \varepsilon\} \cdot \underset{\sim}{P}\{|\zeta_n - (c_1 - c_2)| < 2\varepsilon \\ & \text{for only a finite number of } n = 1,2,\ldots\}.\end{aligned}$$

Now if $c_2 \varepsilon P$, then there is a $k = 1,2,\ldots$ such that $\underset{\sim}{P}\{|\zeta_k - c_2| < \varepsilon\} > 0$. If $c_1 - c_2 \not\varepsilon R$, then by (60) $\underset{\sim}{P}\{|\zeta_n - (c_1 - c_2)| < 2\varepsilon$ for only a finite number of $n = 1,2,\ldots\} > 0$ and thus by (61) $\underset{\sim}{P}\{|\zeta_n - c_1| < \varepsilon$ for only a finite number of $n = 1,2,\ldots\} > 0$. This contradicts to the hypothesis that $c_1 \varepsilon R$. Consequently, $c_1 - c_2 \varepsilon R$.

If R is not empty, then there is a $c^* \varepsilon R$, and then obviously $c^* \varepsilon P$. By the previous argument $c^* - c^* = 0 \varepsilon R$. If $c \varepsilon P$, then also by the previous argument $0 - c = -c \varepsilon R$. Hence $-c \varepsilon P$. By representing the same argument, we obtain finally that $0-(-c) = c \varepsilon R$. This completes the proof of the theorem.

Next we shall study the structures of the sets P and R.

<u>Theorem 5.</u> <u>Let</u> $\{\xi_k\}$ <u>be a sequence of mutually independent and identically distributed, nonnegative random variables for which</u> $\underset{\sim}{P}\{\xi_k = 0\} < 1$.

If ξ_k *is a lattice variable, then there exists a* $\lambda > 0$ *such that* $P \subset \{\lambda, 2\lambda, \ldots, n\lambda, \ldots\}$ *and there exists an* m *such that* $n\lambda \in P$ *for* $n \geq m$.

If ξ_k *is a non-lattice variable, then* P *is asymptotically dense at* ∞ , *that is, for any* $\varepsilon > 0$ *there exists an* $a = a(\varepsilon)$ *such that the interval* $(x, x+\varepsilon)$ *contains at least one point of* P *whenever* $x > a$.

Proof. First, let us suppose that ξ_k is a nonnegative lattice variable for which $\mathbf{P}\{\xi_k = 0\} < 1$. Then there is a $\lambda > 0$ such that $x \in P$ implies that $x = n\lambda$ $(n = 0,1,2,\ldots)$. Denote by λ the largest positive number with the stated property. Then the $\text{g.c.d}\{n: n\lambda \in P\} = 1$. In this case we can find a finite number of positive integers $a_1, a_2, \ldots, a_s$ such that $\text{g.c.d}\{a_1, a_2, \ldots, a_s\} = 1$ and $a_r\lambda \in P$ for $r = 1,2,\ldots, s$. [The integers $a_1, a_2, \ldots, a_s$ can be obtained in the following way: Let us choose an a_1 such that $a_1\lambda \in P$. Denote by $p_2, p_3, \ldots, p_s$ the prime divisors of a_1 . For every $r = 2,3,\ldots, s$ there is at least one a_r such that $a_r\lambda \in P$ and a_r is not divisible by p_r . The integers $a_1, a_2, \ldots, a_s$ satisfy the required properties.] Then

$$\sum_{r=1}^{s} k_r a_r \lambda \in P \tag{62}$$

for all $k_r = 0,1,2,\ldots$. If $n \geq a_1 a_2 \ldots a_s$, then n can be represented in the form $k_1a_1+\ldots+k_sa_s$ where every k_r $(r = 1,\ldots, s)$ is a nonnegative integer. This proves that $n\lambda \in P$ if $n \geq m = a_1a_2\ldots a_s$.

Second, let us suppose that ξ_k is a nonnegative, non-lattice variable for which $P\{\xi_k = 0\} < 1$. Then there exist an $a \varepsilon P$ and a $b \varepsilon P$ such that $0 < a < b$. In this case $ma + j(b-a) = (m-j)a + jb \varepsilon P$ for every $j = 0,1,\ldots, m$.

If $a \geqq \varepsilon > b-a$ and if m is so large that $(m+1)a < mb$, then every subinterval of length ε of the interval $(ma, ma+a)$ contains at least one point of P . Hence the statement of the theorem follows in this case. The case $\varepsilon > a$ is trivial.

It remains to consider the case when there exists a positive ε such that $\varepsilon \leqq b-a$ whenever $a \varepsilon P$, $b \varepsilon P$ and $a < b$. We shall prove that this is impossible. If the assumption were true, then we would necessarily have $P = \{x_1, x_2, \ldots, x_i, \ldots\}$ where $x_1 \geqq 0$ and $x_{i+1} - x_i \geqq \varepsilon > 0$. In this case we would have that

(63) $$\lim_{n \to \infty} (x_{n+1} - x_n) = d$$

exists where

(64) $$d = \inf_{1 \leqq i < \infty} (x_{i+1} - x_i) \geqq \varepsilon > 0 .$$

For it follows from the previous proof that for every $i = 1,2,\ldots$ we have $x_{n+1} - x_n \leqq x_{i+1} - x_i$ if n is sufficiently large. This implies that

(65) $$\limsup_{n \to \infty} (x_{n+1} - x_n) \leqq d \leqq \liminf_{n \to \infty} (x_{n+1} - x_n) ,$$

which proves (63).

Now if $x \varepsilon P$ and $x > 0$, then $x_i + x \varepsilon P$ for every $i = 1,2,\ldots$. Therefore $x_i + x = x_{k_i}$ where necessarily $k_i > i$. Thus for every

$i = 1,2,\ldots$ we have $x = x_{k_i} - x_i$ and this implies that

(66) $$x = \lim_{i \to \infty} (x_{k_i} - x_i) = d \lim_{i \to \infty} (k_i - i) = dj$$

where the limit exists and j is a positive integer. Accordingly, if the assumption is true, and if $x \in P$ and $x > 0$, then it follows that $x = dj$ where $d > 0$ and $j = 1,2,\ldots$. This implies that ξ_k is a lattice random variable. This contradiction proves the second half of the theorem.

The next theorem follows easily from the previous one.

Theorem 6. *Let $\{\xi_k\}$ be a sequence of mutually independent and identically distributed real random variables for which $P\{\xi_k > 0\} > 0$ and $P\{\xi_k < 0\} > 0$.*

If ξ_k is a lattice variable, then there exists a $\lambda > 0$ such that $P = \{n\lambda: n = 0, \pm 1, \pm 2, \ldots\}$.

If ξ_k is a non-lattice variable, then $P = \{x: -\infty < x < \infty\}$.

Proof. Let us apply the previous theorem to the random variables $\xi_k^+ = \max(0, \xi_k)$ $(k = 1,2,\ldots)$ and $\xi_k^- = \min(0, \xi_k)$ separately. Denote by P^+ the set of possible values for the sequence $\{\xi_k^+\}$ and by P^- the set of possible values for the sequence $\{\xi_k^-\}$.

If ξ_k is a lattice random variable, then there exists a $\lambda > 0$ and a sufficiently large positive integer m such that $P^+ \subset \{\lambda, 2\lambda, \ldots, n\lambda, \ldots\}$ and $n\lambda \in P^+$ if $n \geq m$ and $P^- \subset \{-\lambda -2\lambda, \ldots, -n\lambda, \ldots\}$ and $-n\lambda \in P^-$ if

$n \geq m$. This implies that $P \subset \{n\lambda: n = 0, \pm 1, \pm 2, \ldots\}$. On the other hand for every n $(n = 0, \pm 1, \pm 2, \ldots)$ we have $n\lambda \in P$ because we can choose a sufficiently large integer a such that $n+a \geq m$ and $a \geq m$. Then $(n+a)\lambda \in P^+$ and $-a\lambda \in P^-$. This implies that $n\lambda = (n+a)\lambda - a\lambda \in P$. This proves the first part of Theorem 6.

If ξ_k is a non-lattice variable, then for every positive ε there exists an a such that the interval $(a + \frac{x}{2}, a + \frac{x}{2} + \frac{\varepsilon}{2})$ contains a point of P^+ and the interval $(-a + \frac{x}{2}, -a + \frac{x}{2} + \frac{\varepsilon}{2})$ contains a point of P^-. This implies that the interval $(x, x+\varepsilon)$ contains a point of P for every $\varepsilon > 0$ and for every x. This proves the second part of Theorem 6. **We note that in the latter case it may happen that both ξ_k^+ and ξ_k^- are lattice variables.**

Now we are in a position to characterize the structure of R.

Theorem 7. Let $\{\xi_k\}$ be a sequence of mutually independent and identically distributed real random variables. Denote by R the set of recurrent values. There are three possibilities: (i) R is empty, (ii) $R = \{n\lambda: n = 0, \pm 1, \pm 2, \ldots\}$ where λ is a nonnegative number, (iii) $R = \{x: -\infty < x < \infty\}$.

Proof. If $c \in R$, then necessarily $nc \in R$ for every $n = 0, \pm 1, \pm 2, \ldots$. This shows at once that R is necessarily empty if ξ_k is a nonnegative random variable for which $P\{\xi_k = 0\} < 1$ or if ξ_k is a nonpositive random variable for which $P\{\xi_k = 0\} < 1$. If $P\{\xi_k = 0\} = 1$, then $R = P = \{0\}$. If $P\{\xi_k > 0\} > 0$ and $P\{\xi_k < 0\} > 0$ and R is not empty, then $R = P$. If in the latter case ξ_k is a lattice random variable, then $R = P = \{n\lambda: n = 0, \pm 1, \pm 2, \ldots\}$ where λ is a positive number; if ξ_k is a non-lattice random variable, then $\mathcal{R} = \mathcal{P} = \{x: -\infty < x < \infty\}$. This completes the proof of Theorem 7.

Obviously $\mathcal{R}$ is not empty if and only if $x = 0 \in \mathcal{R}$. The next theorem gives a necessary and sufficient condition for the non-emptiness of $\mathcal{R}$.

Theorem 8. *Let $\{\xi_k\}$ be a sequence of mutually independent and identically distributed real random variables. Denote by $\mathcal{R}$ the set of recurrent values. The set $\mathcal{R}$ is not empty if and only if for some positive ε we have*

$$(67) \qquad M(\varepsilon) = \sum_{n=1}^{\infty} \mathbf{P}\{|\zeta_n| < \varepsilon\} = \infty .$$

Proof. First, we shall prove that if $M(\varepsilon) < \infty$ for some $\varepsilon > 0$, then $M(\varepsilon) < \infty$ for all $\varepsilon > 0$. This follows from the fact that $M(\varepsilon)$ for $0 < \varepsilon < \infty$ is a non decreasing function of ε and from the inequality

$$(68) \qquad M(\varepsilon m) \leq 2m[1 + M(\varepsilon)]$$

which holds for all $\varepsilon > 0$ and $m = 1,2,\ldots$.

Since

$$(69) \qquad M(\varepsilon m) = \sum_{n=1}^{\infty} \mathbf{P}\{|\zeta_n| < \varepsilon m\} \leq \sum_{k=-m+1}^{m} \sum_{n=1}^{\infty} \mathbf{P}\{(k-1)\varepsilon < \zeta_n \leq k\varepsilon\}$$

and

$$(70) \qquad \sum_{n=1}^{\infty} \mathbf{P}\{(k-1)\varepsilon < \zeta_n \leq k\varepsilon\} = \sum_{n=1}^{\infty} \sum_{r=1}^{n} \mathbf{P}\{\zeta_i \notin ((k-1)\varepsilon, k\varepsilon] \text{ for } i = 1,2,\ldots, r-1, \ \zeta_r \in ((k-1)\varepsilon, k\varepsilon], \ \zeta_n \in ((k-1)\varepsilon, k\varepsilon]\} \leq$$

$$\leqq \sum_{r=1}^{\infty} \sum_{n=r}^{\infty} \underset{\sim}{P}\{\zeta_i \notin ((k-1)\varepsilon, k\varepsilon] \text{ for } i = 1,2,\ldots,r-1,\ \zeta_r \varepsilon ((k-1)\varepsilon, k\varepsilon],\ |\zeta_n - \zeta_r| \leqq \varepsilon\}$$

$$= \sum_{r=1}^{\infty} \underset{\sim}{P}\{\zeta_i \notin ((k-1)\varepsilon, k\varepsilon] \text{ for } i = 1,2,\ldots,r-1,\ \zeta_r \varepsilon ((k-1)\varepsilon, k\varepsilon]\}[1+M(\varepsilon)] \leqq$$

$$\leqq 1 + M(\varepsilon) \ ,$$

consequently (68) is indeed true.

Now if $M(\varepsilon) < \infty$ for some $\varepsilon > 0$, then by Theorem 41.1 $\underset{\sim}{P}\{|\zeta_n| < \varepsilon$ for infinitely many $n = 1,2,\ldots\} = 0$, and hence $0 \notin R$. Thus R is empty.

If $M(\varepsilon) = \infty$ for all $\varepsilon > 0$, then

$$(71) \qquad Q(\varepsilon) = \underset{\sim}{P}\{|\zeta_n| \geqq \varepsilon \text{ for all } n = 1,2,\ldots\} = 0$$

for all $\varepsilon > 0$. This follows from the following inequalities

$$(72) \qquad \begin{aligned} 1 &\geqq \underset{\sim}{P}\{|\zeta_n| < \varepsilon \text{ for a finite number of } n = 1,2,\ldots\} = \\ &= \sum_{m=1}^{\infty} \underset{\sim}{P}\{|\zeta_m| < \varepsilon \text{ and } |\zeta_n| \geqq \varepsilon \text{ for all } n > m\} + Q(\varepsilon) \geqq \\ &\geqq \sum_{m=1}^{\infty} \underset{\sim}{P}\{|\zeta_m| < \varepsilon \text{ and } |\zeta_n - \zeta_m| \geqq 2\varepsilon \text{ for all } n > m\} + Q(\varepsilon) \geqq \\ &\geqq Q(2\varepsilon)[1 + \sum_{m=1}^{\infty} \underset{\sim}{P}\{|\zeta_n| < \varepsilon\}] = Q(2\varepsilon)[1 + M(\varepsilon)] \ . \end{aligned}$$

If $M(\varepsilon) = \infty$, then necessarily $Q(2\varepsilon) = 0$.

We shall show that if $M(\varepsilon) = \infty$ for all $\varepsilon > 0$, then $0 \varepsilon R$. Thus it follows that R is not empty, and hence $R = P$.

The last statement follows from the inequalities

$$\mathrm{P}\{|\zeta_n| < \varepsilon \text{ for a finite number of } n = 1,2,\dots\} = Q(\varepsilon) +$$

$$+ \sum_{m=1}^{\infty} \mathrm{P}\{|\zeta_m| < \varepsilon \text{ and } |\zeta_n| \geq \varepsilon \text{ for all } n \geq m\} \leq$$

(73)
$$\leq \sum_{k>1/\varepsilon} \sum_{m=1}^{\infty} \mathrm{P}\{|\zeta_m| < \varepsilon - \frac{1}{k} \text{ and } |\zeta_n| \geq \varepsilon \text{ for all } n \geq m\} \leq$$

$$\leq \sum_{k>1/\varepsilon} \sum_{m=1}^{\infty} \mathrm{P}\{|\zeta_m| < \varepsilon \text{ and } |\zeta_n - \zeta_m| \geq \frac{1}{k} \text{ for all } n \geq m\} =$$

$$= \sum_{k>1/\varepsilon} \sum_{m=1}^{\infty} \mathrm{P}\{|\zeta_m| < \varepsilon\} Q(\frac{1}{k}) = 0 .$$

By (73) $\mathrm{P}\{|\zeta_n| < \varepsilon$ for infinitely many $n = 1,2,\dots\} = 1$ for all $\varepsilon > 0$. Thus $0 \in R$ and R is not empty. In this case $R = P$. This completes the proof of the theorem.

Theorem 9. *Let $\{\xi_k\}$ be a sequence of mutually independent and identically distributed real random variables. If $\mathrm{E}\{|\xi_k|\} < \infty$ and $\mathrm{E}\{\xi_k\} = 0$, then R is not empty.*

Proof. We shall prove that $0 \in R$. By the weak law of large numbers (Theorem 1) it follows that for any $\delta > 0$

$$\lim_{n\to\infty} \mathrm{P}\{|\frac{\zeta_n}{n}| < \delta\} = 1. \tag{74}$$

For any $\varepsilon > 0$ and $m = 1,2,\dots$ we have

$$1+M(\varepsilon) \geq \frac{1}{2m} M(\varepsilon m) \geq \frac{1}{2m} \sum_{n=1}^{[\varepsilon m/\delta]} \underset{\sim}{P}\{|\frac{\zeta_n}{n}| < \frac{\varepsilon m}{n}\} \geq$$
(75)
$$\geq \frac{1}{2m} \sum_{n=1}^{[\varepsilon m/\delta]} \underset{\sim}{P}\{|\frac{\zeta_n}{n}| < \delta\} \to \frac{\varepsilon}{2\delta} \quad \text{as} \quad m \to \infty .$$

Here we used (68) and (74). Since in (75) $\delta > 0$ is arbitrary, it follows that $M(\varepsilon) = \infty$ for all $\varepsilon > 0$. Hence $0 \in R$.

We can utilize the previous results in finding the limiting behavior of the partial sums of mutually independent and identically distributed real random variables $\xi_1, \xi_2, \ldots, \xi_k, \ldots$. Let $\zeta_n = \xi_1+\xi_2+\ldots+\xi_n$ for $n=1,2,\ldots$ and $\zeta_0 = 0$. We shall be interested in studying the random variables

(76) $$\limsup_{n \to \infty} \zeta_n$$

and

(77) $$\eta = \sup_{0 \leq k < \infty} \zeta_k .$$

They are nonnegative random variables which may be ∞ with positive probability.

Let us define the probabilities

(78) $$V(x) = \underset{\sim}{P}\{\limsup_{n \to \infty} \zeta_n \leq x\} = \lim_{n \to \infty} \underset{\sim}{P}\{\sup_{n<k<\infty} \zeta_k \leq x\}$$

and

(79) $$W(x) = \underset{\sim}{P}\{\sup_{0 \leq k < \infty} \zeta_k \leq x\} = \lim_{n \to \infty} \underset{\sim}{P}\{\max_{0 \leq k \leq n} \zeta_k \leq x\}$$

for $-\infty < x < \infty$. Equivalently we can write that

(80) $$V(x) = \underset{\sim}{P}\{\zeta_k > x \text{ for only a finite number of } k = 0, 1, 2, \ldots\}$$

and

(81) $$W(x) = \underset{\sim}{P}\{\zeta_k > x \text{ for none of the subscripts } k = 0, 1, 2, \ldots\} .$$

Lemma 3. *If $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent and identically distributed real random variables for which $\underset{\sim}{P}\{\xi_k = 0\} < 1$, then either $V(x) = 0$ for every x or $V(x) = 1$ for every x.*

Proof. By Theorem 41.4 it follows that for every given x either $V(x) = 0$ or $V(x) = 1$. Since $V(x)$ is a nondecreasing function of x, there are only three possibilities (i) $V(x) = 0$ for every x, (ii) $V(x) = 1$ for every x, and (iii) $V(x) = 0$ for $x < c$ and $V(x) = 1$ for $x \geq c$ where c is a finite real number.

If $\underset{\sim}{P}\{\xi_k = 0\} = 1$, then $V(x) = 0$ for $x < 0$ and $V(x) = 1$ for $x \geq 0$.

If $\underset{\sim}{P}\{\xi_k = 0\} < 1$, then either $V(x) = 0$ for every x or $V(x) = 1$ for every x. This can be proved by using the following inequality

(82) $$V(x) \leq \underset{\sim}{P}\{\xi_1 \leq a\} + \underset{\sim}{P}\{\xi_1 > a\}V(x-a)$$

which holds for every real a.

If $\underset{\sim}{P}\{\xi_1 > a\} > 0$ for some $a > 0$, and $V(x) = 1$, then (82) implies that $V(x-a) = 1$. Hence it follows that either $V(x) \equiv 0$ or $V(x) \equiv 1$.

If $P\{\xi_1 > a\} = 0$ for every $a > 0$, then $P\{\xi_1 \leq 0\} = 1$. Since $P\{\xi_1 = 0\} < 1$ by assumption, there exists an $a < 0$ such that $P\{\xi_1 \leq a\} > 0$. In this case for any $x < 0$ there exists a sufficiently large n such that $P\{\zeta_n \leq x\} > 0$, and consequently we have

$$(83) \qquad 0 < P\{\zeta_n \leq x\} = P\{\zeta_k \leq x \text{ for } k \geq n\} \leq V(x) .$$

That is for any $x < 0$ we have $V(x) > 0$. Thus necessarily $V(x) = 1$ for $x < 0$. Consequently, $V(x) \equiv 1$. This completes the proof of the lemma. We note that a theorem similar to Lemma 3 has been proved by P. Lévy [113 p. 131].

Furthermore, we observe that

$$(84) \qquad W(x) \leq V(x) \leq W(x) + \sum_{k=1}^{\infty} P\{\zeta_k > x\}W(0)$$

holds for all x. The first inequality in (84) is obvious. Since

$$V(x) = W(x) + \sum_{k=1}^{\infty} P\{\zeta_k > x \text{ and } \zeta_n \leq x \text{ for all } n > k\} \leq$$

$$(85) \qquad \leq W(x) + \sum_{k=1}^{\infty} P\{\zeta_k > x \text{ and } \zeta_n - \zeta_k \leq 0 \text{ for all } n \geq k\} =$$

$$= W(x) + \sum_{k=1}^{\infty} P\{\zeta_k > x\}W(0)$$

for all x, it follows that the second inequality is also valid in (84).

If we exclude the trivial case of $P\{\xi_k = 0\} = 1$, then by Lemma 1 it follows that $\limsup_{n\to\infty} \zeta_n$ is either $+\infty$ with probability 1 or $-\infty$ with probability 1. We shall give various criterions to decide which is the case.

The following theorem is an easy consequence of Theorem 9. This theorem was found by K. L. Chung and W. H. J. Fuchs [25].

Theorem 10. *Let $\{\xi_k\}$ be a sequence of mutually independent and identically distributed real random variables for which $E\{|\xi_k|\} < \infty$, $E\{\xi_k\} = 0$ and $P\{\xi_k = 0\} < 1$. If $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots,$ and $\zeta_0 = 0$, then*

$$P\{\sup_{0\le k<\infty} \zeta_k = \infty\} = 1 , \tag{86}$$

and

$$P\{\limsup_{n\to\infty} \zeta_n = \infty\} = 1 . \tag{87}$$

Proof. Denote by R the set of recurrent values of $\{\zeta_n\}$. By Theorem 9 it follows that R is not empty. Since $P\{\xi_k = 0\} < 1$, it follows from Theorem 7 that either $R = \{n\lambda : n = 0, \pm 1, \pm 2, \ldots\}$ where λ is a positive number or $R = \{x : -\infty < x < \infty\}$. In both cases R contains arbitrarily large recurrent values. This proves (86) and (87).

For another proof of Theorem 10 we refer to Y. S. Chow, H. Robbins and D. Siegmund [20].

We shall prove two more theorems found by D. V. Lindley [115] and F. Spitzer [181].

Theorem 11. *Let* ξ_k $(k = 1,2,\ldots)$ *be a sequence of mutually independent and identically distributed real random variables for which* $E\{|\xi_k|\} < \infty$. *Let* $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ *for* $n = 1,2,\ldots$ *and* $\zeta_0 = 0$.

If $E\{\xi_k\} > 0$, *then*

(88)
$$P\{\sup_{0 \leq k < \infty} \zeta_k = \infty\} = 1$$

and

(89)
$$P\{\limsup_{n \to \infty} \zeta_n = \infty\} = 1 .$$

If $E\{\xi_k\} < 0$, *then*

(90)
$$P\{\sup_{0 \leq k < \infty} \zeta_k < \infty\} = 1$$

and

(91)
$$P\{\limsup_{n \to \infty} \zeta_n = -\infty\} = 1 .$$

Proof. Let

(92)
$$a = E\{\xi_k\} .$$

First, let $a > 0$. By the weak law of large numbers (Theorem 1) we have

(93)
$$\lim_{n \to \infty} P\{|\frac{\zeta_n}{n} - a| < \varepsilon\} = 1$$

for any $\varepsilon > 0$. Hence

(94)
$$\lim_{n \to \infty} P\{\zeta_n \leq n(a-\varepsilon)\} = 0 .$$

For any x we have $x < n(a-\varepsilon)$ if $\varepsilon < a$ and if n is sufficiently large. Thus (94) implies that

$$\lim_{n\to\infty} \underset{\sim}{P}\{\zeta_n \leqq x\} = 0 \tag{95}$$

for any x. Since evidently $0 \leqq W(x) \leqq \underset{\sim}{P}\{\zeta_n \leqq x\}$, it follows that $W(x) = 0$ for any x. This proves (88). In this case by (84) we obtain that $V(x) = 0$ for any x. This proves (89).

Second, let $a < 0$. By the strong law of large numbers (Theorem 3) we have

$$\lim_{n\to\infty} \underset{\sim}{P}\{ \sup_{n<k<\infty} \left|\frac{\zeta_k}{k} - a\right| < \varepsilon\} = 1 \tag{96}$$

for any $\varepsilon > 0$. If $a + \varepsilon < 0$, then we have

$$\begin{aligned} &\{ \sup_{n<k<\infty} \left|\frac{\zeta_k}{k} - a\right| < \varepsilon\} \subset \{ \sup_{n<k<\infty} \frac{\zeta_k}{k} < a + \varepsilon\} \\ &\subset \{ \sup_{n<k<\infty} \frac{\zeta_k}{k} < 0 \} \subset \{ \sup_{n<k<\infty} \zeta_k < 0 \}. \end{aligned} \tag{97}$$

By (96) and (97) we obtain that

$$\lim_{n\to\infty} \underset{\sim}{P}\{ \sup_{n<k<\infty} \zeta_k < 0\} = 1 . \tag{98}$$

Evidently for any $x \geqq 0$ we have

$$W(x) \geqq \underset{\sim}{P}\{ \max_{0\leqq k\leqq n} \zeta_k \leqq x\} + \underset{\sim}{P}\{ \sup_{n<k<\infty} \zeta_k < 0\} - 1 . \tag{99}$$

Let now ε be any positive number. For any $\varepsilon > 0$ we can choose n so large that the second term on the right-hand side of (99) is greater than $1-\varepsilon$. This follows from (98). For any fixed n the first term on the right-hand side of (99) tends to 1 if $x \to \infty$. Therefore we can choose x so large that this term is greater than $1-\varepsilon$. Thus it follows from (99) that for any $\varepsilon > 0$

(100) $$W(x) > 1-2\varepsilon$$

if x is large enough. Accordingly $W(\infty) = \lim_{x\to\infty} W(x) = 1$ which proves (90). In this case by (84) we obtain that $V(\infty) = \lim_{x\to\infty} V(x) = 1$. Hence necessarily $V(x) = 1$ for every x. This implies (91).

<u>Theorem 12.</u> <u>Let</u> ξ_k $(k = 1,2,\ldots)$ <u>be mutually independent and identically distributed real random variables for which</u> $P\{\xi_k = 0\} < 1$. <u>Let</u> $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ <u>for</u> $n = 1,2,\ldots$ <u>and</u> $\zeta_0 = 0$. <u>Furthermore, let</u>

(101) $$M = \sum_{n=1}^{\infty} \frac{P\{\zeta_n > 0\}}{n} .$$

<u>If</u> $M = \infty$, <u>then</u>

(102) $$P\{\sup_{0\le k<\infty} \zeta_k = \infty\} = 1 ,$$

<u>and</u>

(103) $$P\{\limsup_{n\to\infty} \zeta_n = \infty\} = 1 .$$

<u>If</u> $M < \infty$, <u>then</u>

(104) $$P\{\sup_{0\le k<\infty} \zeta_k < \infty\} = 1 ,$$

<u>and</u>

(105) $$P\{\limsup_{n\to\infty} \zeta_n = -\infty\} = 1 .$$

Proof. Let

$$W_n(x) = \mathbf{P}\{\max_{0 \le k \le n} \zeta_k \le x\} \tag{106}$$

for $n = 0,1,2,\ldots$ and $-\infty < x < \infty$. Obviously, $W_n(x) = 0$ if $x < 0$. We have

$$W(x) = \lim_{n \to \infty} W_n(x)$$

for any x.

If we let $s \to +\infty$ in formula (15.1), then we obtain that

$$\sum_{n=0}^{\infty} W_n(0)\rho^n = e^{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{P}\{\zeta_n \le 0\}} \tag{107}$$

for $|\rho| < 1$, whence it follows that

$$(1-\rho)\sum_{n=0}^{\infty} W_n(0)\rho^n = e^{-\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{P}\{\zeta_n > 0\}} \tag{108}$$

for $|\rho| < 1$. Since $\lim_{n \to \infty} W_n(0) = W(0)$ exists, by Abel's theorem we obtain that

$$W(0) = \lim_{\rho \to 1} (1-\rho)\sum_{n=0}^{\infty} W_n(0)\rho^n = \begin{cases} e^{-M} & \text{if } M < \infty, \\ 0 & \text{if } M = \infty. \end{cases} \tag{109}$$

Accordingly if $M = \infty$, then $W(0) = 0$ and by (84) $V(0) = 0$. Hence $V(x) = 0$ for every x and this proves (103). Again by (84) it follows that $W(x) \equiv 0$ for every x. This proves (102).

If $M < \infty$, then by (109) $W(0) > 0$, and therefore by (84) $V(0) > 0$.

Hence $V(x) = 1$ for every x, and this proves (105). It remains to prove (104), that is, that $W(\infty) = \lim_{x \to \infty} W(x) = 1$.

For every x we have the obvious inequality

$$W(x) \geq P\{\max_{0 \leq k \leq n} \zeta_k \leq x\} + P\{\sup_{n<k<\infty} \zeta_k \leq x\} - 1 . \tag{110}$$

If $n \to \infty$ in (110), then the second term on the right-hand side tends to $V(x) = 1$, and if $x \to \infty$, then for any fixed n the first term on the right-hand side tends to 1. This implies that $\lim_{x \to \infty} W(x) = 1$, that is, $W(\infty) = 1$. This proves (104).

By the previous three theorems we can conclude immediately that the following corollary is true.

Corollary 1. _Let_ $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ _be a sequence of mutually independent and identically distributed real random variables for which_ $P\{\xi_k = 0\} < 1$. _Let_ $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ _for_ $n = 1,2,\ldots$. _Suppose that_ $E\{|\xi_k|\} < \infty$. _Then_ $E\{\xi_k\} \geq 0$ _if and only if_

$$\sum_{n=1}^{\infty} \frac{P\{\zeta_n > 0\}}{n} = \infty , \tag{111}$$

and $E\{\xi_k\} < 0$ _if and only if_

$$\sum_{n=1}^{\infty} \frac{P\{\zeta_n > 0\}}{n} < \infty. \tag{112}$$

By Theorem 12 it follows that if $P\{\xi_k = 0\} < 1$ and $M < \infty$, then

$W(x)$ is a proper distribution function. The case $\underset{\sim}{P}\{\xi_k = 0\} = 1$ is trivial. The problem arises naturally how to determine $W(x)$ for a given $\underset{\sim}{P}\{\xi_k \leq x\} = F(x)$. For this problem a solution is given by the next theorem due to F. Spitzer [181]. See also S. Täcklind [196] and F. Pollaczek [158].

Theorem 13. Let $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ be a sequence of mutually independent and identically distributed real random variables. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. Define

(113) $$W(x) = \underset{\sim}{P}\{\sup_{0 \leq n < \infty} \zeta_n \leq x\}$$

for $-\infty < x < \infty$. If

(114) $$M = \sum_{n=1}^{\infty} \frac{\underset{\sim}{P}\{\zeta_n > 0\}}{n} < \infty ,$$

then $W(x)$ is a proper distribution function and its Laplace-Stieltjes transform

(115) $$\Omega(s) = \int_0^{\infty} e^{-sx} dW(x)$$

is given by

(116) $$\Omega(s) = e^{-\sum_{n=1}^{\infty} \frac{1}{n}[1 - \underset{\sim}{E}\{e^{-s\zeta_n^+}\}]}$$

for $\mathrm{Re}(s) \geq 0$.

Proof. Obviously $W(x) = 0$ for $x < 0$ and by Theorem 12 we have $W(\infty) = 1$. Thus (115) is convergent for $\mathrm{Re}(s) \geq 0$ and $\Omega(0) = 1$.

Let

$$W_n(x) = \mathop{P}_{\sim}\{\max_{0 \leq k \leq n} \zeta_k \leq x\} \tag{117}$$

and

$$\Omega_n(s) = \int_0^\infty e^{-sx} dW_n(x) \tag{118}$$

for $Re(s) \geq 0$. Since $\lim_{n \to \infty} W_n(x) = W(x)$ for every x , by Theorem 41.9 it follows that $\lim_{n \to \infty} \Omega_n(s) = \Omega(s)$ for $Re(s) = 0$.

By formula (15.1) we have

$$\sum_{n=0}^{\infty} \Omega_n(s)\rho^n = e^{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathop{E}_{\sim}\{e^{-s\zeta_n^+}\}} \tag{119}$$

for $Re(s) \geq 0$ and $|\rho| < 1$. Since by Abel's theorem

$$\Omega(s) = \lim_{\rho \to 1} (1-\rho) \sum_{n=0}^{\infty} \Omega_n(s)\rho^n \tag{120}$$

for $Re(s) \geq 0$, we obtain (116) by (119). This completes the proof of the theorem.

We note that obviously

$$W(0) = \lim_{s \to \infty} \Omega(s) = e^{-M} . \tag{121}$$

The distribution function $W(x)$ can also be obtained by the following theorem found by D. V. Lindley [115].

Theorem 14. Let

$$\mathop{P}_{\sim}\{\xi_k \leq x\} = F(x) \tag{122}$$

for $-\infty < x < \infty$. If $\mathop{P}_{\sim}\{\xi_k = 0\} < 1$ and $M < \infty$, then the distribution function $W(x)$ can be obtained as the unique solution of the integral equation

$$(123)\qquad \int_{-\infty}^{\infty} W(x-y)dF(y) = \begin{cases} W(x) & \text{for } x \geq 0 , \\ 0 & \text{for } x < 0 . \end{cases}$$

Proof. If we use the representation

$$(124)\qquad \sup_{0\leq n<\infty} \zeta_n = [\xi_1 + \sup_{1\leq n<\infty} (\zeta_n - \zeta_1)]^+ ,$$

and if we take into consideration that $\sup_{1\leq n<\infty} (\zeta_n - \zeta_1)$ is independent of ξ_1 and has the distribution function $W(x)$, then we obtain (123). Accordingly, $W(x)$ can be obtained as a solution of the Wiener-Hopf type equation (123).

Now we shall show that $W(x)$ is the unique solution of (123). Let us define a sequence of random variables $\eta_0, \eta_1, \ldots, \eta_n, \ldots$ by the following recurrence formula

$$(125)\qquad \eta_n = [\eta_{n-1} + \xi_n]^+$$

for $n = 1,2,\ldots$ where η_0 is a real random variable which is independent of the sequence $\{\xi_k\}$. By (125) we can write also that

$$(126)\qquad \eta_n = \max(0, \xi_n, \xi_{n-1}+ \xi_n, \ldots, \xi_2+\ldots+ \xi_n, \xi_1+\ldots+ \xi_n+ \eta_0)$$

for $n = 1,2,\ldots$. Thus it follows that

$$(127)\qquad W_n(x)-\mathbf{P}\{\eta_0+ \zeta_n > x\} \leq \mathbf{P}\{\eta_n \leq x\} \leq W_n(x)+\mathbf{P}\{\zeta_n > x\}$$

for every x . If $\mathbf{P}\{\xi_k = 0\} < 1$ and $M < \infty$, then by (105) it follows that

(128) $$\lim_{n\to\infty} \underset{\sim}{P}\{\eta_0 + \zeta_n > x\} = 0$$

for any x and for any η_0. Accordingly, if $\underset{\sim}{P}\{\xi_k = 0\} < 1$ and $M < \infty$, then

(129) $$\lim_{n\to\infty} \underset{\sim}{P}\{\eta_n \leq x\} = \lim_{n\to\infty} W_n(x) = W(x)$$

for every x regardless of the distribution of η_0.

Now let us assume that $W^*(x)$ is any distribution function which satisfies (123). If in (125) we choose η_0 in such a way that $\underset{\sim}{P}\{\eta_0 \leq x\} = W^*(x)$, then by (123) it follows that $\underset{\sim}{P}\{\eta_n \leq x\} = W^*(x)$ for every $n = 1,2,\ldots$. Then by (129) we obtain that necessarily $W^*(x) = W(x)$. This completes the proof of the theorem.

In many cases we can easily solve the integral equation (123) by using the method of factorization.

Let us define

(130) $$\phi(s) = \underset{\sim}{E}\{e^{-s\xi_k}\} = \int_{-\infty}^{\infty} e^{-sx} dF(x)$$

for $\mathrm{Re}(s) = 0$, and suppose that $\underset{\sim}{P}\{\xi_k = 0\} < 1$ and $M < \infty$.

Let us suppose that

(131) $$1-\phi(s) = \phi^+(s)\,\phi^-(s)$$

for $\mathrm{Re}(s) = 0$ where $\phi^+(s)$ satisfies the requirements:

A_1 : $\Phi^+(s)$ is a regular function of s in the domain $\mathrm{Re}(s) > 0$,

A_2 : $\Phi^+(s)$ is a continuous and free from zeros in $\mathrm{Re}(s) \geqq 0$,

A_3 : $\lim_{|s|\to\infty} [\log \Phi^+(s)]/s = 0$ whenever $\mathrm{Re}(s) \geqq 0$,

and $\Phi^-(s)$ satisfies the requirements:

B_1 : $\Phi^-(s)$ is a regular function of s in the domain $\mathrm{Re}(s) < 0$,

B_2 : $\Phi^-(s)$ is continuous in $\mathrm{Re}(s) \leqq 0$, and free from zeros in $\mathrm{Re}(s) < 0$,

B_3 : $\lim_{|s|\to\infty} [\log \Phi^-(s)]/s = 0$ whenever $\mathrm{Re}(s) < 0$.

Such a factorization always exists. We can provide an example by using Theorem 6.1. If $|\rho| < 1$, then by Theorem 6.1 we can write that

$$1-\rho\phi(s) = \Phi^+(s,\rho)\Phi^-(s,\rho) \tag{132}$$

for $\mathrm{Re}(s) = 0$ where we can choose

$$\Phi^+(s,\rho) = \frac{1}{(1-\rho)} e^{\underset{\sim}{T}\{\log[1-\rho\phi(s)]\}} \tag{133}$$

for $\mathrm{Re}(s) \geqq 0$ and

$$\Phi^-(s,\rho) = (1-\rho)e^{\log[1-\rho\phi(s)]-\underset{\sim}{T}\{\log[1-\rho\phi(s)]\}} \tag{134}$$

for $\mathrm{Re}(s) \leqq 0$. Now let

$$\Phi^+(s) = \lim_{\rho\to 1} \Phi^+(s,\rho) = e^{\sum_{n=1}^{\infty} \frac{1}{n}[1-\underset{\sim}{E}\{e^{-s\zeta_n^+}\}]} \tag{135}$$

for $\operatorname{Re}(s) \geqq 0$ and

(136) $$\Phi^-(s) = \lim_{\rho \to 1} \Phi^-(s, \rho) = e^{-\sum_{n=1}^{\infty} \frac{1}{n} \underset{\sim}{E}\{e^{s[-\zeta_n]^+}\}}$$

for $\operatorname{Re}(s) \leqq 0$.

We can easily see that $\Phi^+(s)$ and $\Phi^-(s)$ satisfy (131) for $\operatorname{Re}(s) = 0$ and the requirements A_1, A_2, A_3 and B_1, B_2, B_3 respectively. However, it will be instructive to give another proof.

By using Theorem 19.4 we can represent $\Phi^+(s)$ and $\Phi^-(s)$ in the following way too:

Denote by ρ_1 the first ladder index for the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$. By Theorem 19.4 we have

(137) $$1-\underset{\sim}{E}\{e^{-s\zeta_{\rho_1}}\} = e^{-\sum_{n=1}^{\infty} \frac{1}{n}[\underset{\sim}{E}\{e^{-s\zeta_n^+}\}-\underset{\sim}{P}\{\zeta_n \leqq 0\}]} = e^{-M} \Phi^+(s)$$

for $\operatorname{Re}(s) \geqq 0$. If $\overline{\rho}_1$ denotes the first weak ladder index for the sequence $-\zeta_0, -\zeta_1, \ldots, -\zeta_n, \ldots$, then we have

(138) $$1-\underset{\sim}{E}\{e^{-s\zeta_{\overline{\rho}_1}}\} = e^{-\sum_{n=1}^{\infty} \frac{1}{n}[\underset{\sim}{E}\{e^{s[-\zeta_n]^+}\}-\underset{\sim}{P}\{\zeta_n > 0\}]} = e^{M} \Phi^-(s)$$

for $\operatorname{Re}(s) \leqq 0$.

In (137) ζ_{ρ_1} is a nonnegative random variable. If $s \to 0$ in (137), then we obtain that

$$\mathop{P}_{\sim}\{\zeta_{\rho_1} < \infty\} = 1-e^{-M} . \tag{139}$$

Thus it follows that $\Phi^+(0) = 1$ and

$$1 \leq |\Phi^+(s)| \leq e^M \tag{140}$$

for $Re(s) \geq 0$. The representation (137) shows immediately that $\Phi^+(s)$ is regular in the domain $Re(s) > 0$ and continuous for $Re(s) \geq 0$.

In (138) $-\zeta_{\bar{\rho}_1}$ is a nonnegative random variable. If $s \to 0$ in (138), then we obtain that

$$\mathop{P}_{\sim}\{-\zeta_{\bar{\rho}_1} < \infty\} = 1 . \tag{141}$$

For if $M < \infty$, then necessarily

$$\sum_{n=1}^{\infty} \frac{\mathop{P}_{\sim}\{\zeta_n \leq 0\}}{n} = \infty . \tag{142}$$

By (138) it follows that $\Phi^-(0) = 0$, $|\Phi^-(s)| > 0$ for $Re(s) < 0$ and

$$|\Phi^-(s)| \leq e^{-M} \quad \text{for} \quad Re(s) \leq 0 . \tag{143}$$

The representation (138) shows immediately that $\Phi^-(s)$ is regular in the domain $Re(s) < 0$ and continuous for $Re(s) \leq 0$. On the line $Re(s) = 0$ the zeros of $\Phi^-(s)$ and the zeros of $1-\phi(s)$ coincide and have equal multiplicities.

Now we shall prove that the requirements (131), A_1, A_2, A_3 and B_1, B_2, B_3 determine $\Phi^+(s)$ and $\Phi^-(s)$ up to a constant factor. This is the content of the next theorem.

Theorem 15. *Let us suppose that* $P\{\xi_k = 0\} < 1$ *and* $M < \infty$. *If*

$$1-\phi(s) = \Phi^{+}(s)\,\Phi^{-}(s) \tag{144}$$

for $Re(s) = 0$ *where* $\Phi^{+}(s)$ *and* $\Phi^{-}(s)$ *satisfy the requirements* A_1, A_2, A_3 *and* B_1, B_2, B_3 *respectively, then the Laplace-Stieltjes transform of* $W(x)$ *is given by*

$$\Omega(s) = \frac{\Phi^{+}(0)}{\Phi^{+}(s)} \tag{145}$$

for $Re(s) \geq 0$.

Proof. If $\Phi^{+}(s)$ is given by (135) and $\Phi^{-}(s)$ is given by (136), then all the requirements are satisfied, and by (116) we obtain (145).

Now let us suppose that

$$1-\phi(s) = \Psi^{+}(s)\,\Psi^{-}(s) \tag{146}$$

for $Re(s) = 0$ where $\Psi^{+}(s)$ satisfies A_1, A_2, A_3 and $\Psi^{-}(s)$ satisfies B_1, B_2, B_3 .

Then $\Psi^{+}(s)/\Phi^{+}(s)$ is a regular function of s in the domain $Re(s) > 0$. and continuous and free from zeros in $Re(s) \geq 0$. Similarly $\Phi^{-}(s)/\Psi^{-}(s)$ is a regular function of s in the domain $Re(s) < 0$, and continuous and free from zeros in $Re(s) \leq 0$. For the zeros of $\Phi^{-}(s)$ and $\Psi^{-}(s)$ coincide on the line $Re(s) = 0$ and they cancel out each other in the ratio $\Phi^{-}(s)/\Psi^{-}(s)$. If $Re(s) = 0$, then

(147) $$\frac{\Psi^+(s)}{\Phi^+(s)} = \frac{\Phi^-(s)}{\Psi^-(s)} .$$

By using Morera's theorem (see e.g. W. F. Osgood [782] p. 122) we can easily see that

(148) $$G(s) = \begin{cases} \Psi^+(s)/\Phi^+(s) & \text{for } \operatorname{Re}(s) \geqq 0 , \\ \Phi^-(s)/\Psi^-(s) & \text{for } \operatorname{Re}(s) \leqq 0 \end{cases}$$

is a regular function of s on the whole complex plane, and indeed $G(s)$ is an entire function. By our assumptions

(149) $$\lim_{|s|\to\infty} \frac{\log G(s)}{s} = 0$$

and this implies that $\log G(s)$ is constant on the whole complex plane, that is, $G(s) \equiv C$ where C is a complex constant. (See J. Hadamard [64] pp. 118-119.) Thus

(150) $$\Psi^+(s) = C\Phi^+(s)$$

for $\operatorname{Re}(s) \geqq 0$. Since we excluded the trivial case $\phi(s) \equiv 1$, we have $C \neq 0$. Thus by (150)

(151) $$\frac{\Psi^+(0)}{\Psi^+(s)} = \frac{\Phi^+(0)}{\Phi^+(s)} = \Omega(s)$$

for $\operatorname{Re}(s) \geqq 0$ where $\Phi^+(s)$ is given by (135). Obviously, $\Phi^+(0) = 1$, and thus the last equality in (151) follows from (116) and (135). This completes the proof of the theorem.

For the solution of the integral equation (123) we also refer to

F. Smithes [180], W. L. Smith [179], and F. Spitzer [181].

The Law of the Iterated Logarithm. The law of the iterated logarithm has its origin in a probability problem in the theory of numbers. In 1909 É. Borel [16] considered the following random trial. We choose a point ω at random in the interval $(0, 1)$ and assume that the random point has a uniform distribution over the interval $(0, 1)$. Let us form the binary expansion of ω, that is,

$$\omega = \sum_{k=1}^{\infty} \frac{\xi_k(\omega)}{2^k} . \tag{152}$$

If there is any ambiguity in the expansion (152), then it is immaterial which form we choose. Denote by $\nu_n(\omega)$ the number of ones (or zeros) among the first n digits of the binary expansion of ω. The problem is to determine the asymptotic behavior of $\nu_n(\omega)$ as $n \to \infty$.

To describe the above random trial mathematically let us assume that the associated probability space is $(\Omega, \mathcal{B}, \underset{\sim}{P})$ where Ω is the interval $(0, 1)$, $\mathcal{B}$ is the class of Borel subsets of Ω, and $\underset{\sim}{P}$ is the Lebesque measure. Then $\xi(\omega) = \omega$ defined for $0 < \omega < 1$ is a random variable which has a uniform distribution over the interval $(0, 1)$. In this case $\xi_1(\omega), \xi_2(\omega), \ldots, \xi_k(\omega), \ldots$ defined by (152) is a sequence of mutually independent and identically distributed random variables for which

$$\underset{\sim}{P}\{\xi_k(\omega) = 1\} = \underset{\sim}{P}\{\xi_k(\omega) = 0\} = \frac{1}{2} , \tag{153}$$

and $\nu_n(\omega)$ is a random variable which has the Bernoulli distribution

(154) $$\underset{\sim}{P}\{\nu_n(\omega) = j\} = \binom{n}{j}\frac{1}{2^n}$$

for $j = 0,1,\ldots, n$.

In 1909 É. Borel [16] proved that

(155) $$\underset{\sim}{P}\{\lim_{n\to\infty}\left(\frac{\nu_n(\omega)}{n} - \frac{1}{2}\right) = 0\} = 1 .$$

In 1914 F. Hausdorff [405 pp. 419-422] indicated that

(156) $$\underset{\sim}{P}\{\lim_{n\to\infty} n^{\delta}\left(\frac{\nu_n(\omega)}{n} - \frac{1}{2}\right) = 0\} = 1$$

for all $\delta < \frac{1}{2}$.

In 1914 G. Hardy and J. E. Littlewood [402] proved as a particular case of a more general result that

(157) $$\underset{\sim}{P}\{\limsup_{n\to\infty} n^{\frac{1}{2}}\left|\frac{\nu_n(\omega)}{n} - \frac{1}{2}\right| = \infty\} = 1$$

and

(158) $$\underset{\sim}{P}\{\limsup_{n\to\infty}\frac{\left|\nu_n(\omega) - \frac{n}{2}\right|}{\sqrt{n\log n}} \leq \frac{1}{\sqrt{2}}\} = 1 .$$

See also H. Rademacher [453] and H. Steinhaus [468].

In 1923 A. Ya. Khintchine[413] proved that

(159) $$\underset{\sim}{P}\{\limsup_{n\to\infty}\frac{\left|\nu_n(\omega) - \frac{n}{2}\right|}{\sqrt{n\log\log n}} \leq 1\} = 1$$

and in 1924 A. Ya. Khintchine [414], [415] proved also that

$$\underset{\sim}{P}\{\limsup_{n \to \infty} \frac{|\nu_n(\omega) - \frac{n}{2}|}{\sqrt{n \log \log n}} = \frac{1}{\sqrt{2}}\} = 1 . \tag{160}$$

Actually, A. Ya. Khintchine [415] proved a somewhat more general result, namely that if $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is an infinite sequence of mutually independent and identically distributed random variables for which

$$\underset{\sim}{P}\{\xi_k = 1\} = p \quad \text{and} \quad \underset{\sim}{P}\{\xi_k = 0\} = q \tag{161}$$

where $p+q = 1$ and $0 < p < 1$ and if $\nu_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots,$ then

$$\underset{\sim}{P}\{\limsup_{n \to \infty} \frac{|\nu_n - np|}{\sqrt{2npq \log \log n}} = 1\} = 1 . \tag{162}$$

This result can also be interpreted in the following way. Let $\gamma(n)$ $(n = 1,2,\ldots)$ be an increasing sequence of positive real numbers. Define the events

$$A_n = \{\frac{\nu_n - np}{\sqrt{npq}} > \gamma(n)\} \tag{163}$$

for $n = 1,2,\ldots$ and let

$$A^* = \prod_{n=1}^{\infty} \sum_{i=n}^{\infty} A_i \tag{164}$$

be the event that infinitely many events occur in the sequence $A_1, A_2, \ldots, A_n, \ldots$.

By (162) it follows that if

(165) $$\gamma(n) = c\sqrt{2\log\log n}$$

for $n \geqq 3$, then

(166) $$\underset{\sim}{P}\{A^*\} = \begin{cases} 0 & \text{for } c > 1, \\ 1 & \text{for } c < 1. \end{cases}$$

In 1931 P. Lévy [428] proved that if

(167) $$\gamma(n) = (2\log\log n + c\log\log\log n)^{\frac{1}{2}}$$

for sufficiently large n, then

(168) $$\underset{\sim}{P}\{A^*\} = \begin{cases} 0 & \text{for } c > 3, \\ 1 & \text{for } c \leqq 1. \end{cases}$$

In this result there is a gap for $1 < c \leqq 3$.

By the zero-or-one law which was proved in 1933 by A. N. Kolmogorov [100] (Theorem 4 in Section 41) we can conclude that for any $\gamma(n)$ $(n = 1,2,\ldots)$ we have either $\underset{\sim}{P}\{A^*\} = 0$ or $\underset{\sim}{P}\{A^*\} = 1$. In our case we have

(169) $$\underset{\sim}{P}\{A^*\} = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} \frac{\gamma(n)}{n} e^{-\frac{1}{2}[\gamma(n)]^2} < \infty,$$

and

(170) $$\underset{\sim}{P}\{A^*\} = 1 \quad \text{if} \quad \sum_{n=1}^{\infty} \frac{\gamma(n)}{n} e^{-\frac{1}{2}[\gamma(n)]^2} = \infty.$$

In 1931 P. Lévy [428] formulated a conjecture which was close to the above results. In fact he missed the factor $\gamma(n)$ in (169) and (170). In 1937 P. Lévy [113 p. 266] mentioned the results (169) and (170) without proof and attributed them to A. N. Kolmogorov. In 1937 J. Ville [478 pp. 101-111] proved (169) and in 1942 P. Erdős [377] proved that if $\gamma(n)/\sqrt{n}$ $(n = 1,2,\ldots)$ is an increasing sequence of positive numbers, then (169) and (170) are true. P. Erdős [377] demonstrated that if $p = q = \frac{1}{2}$ in (161) and if

$$\gamma(n) = \frac{1}{\sqrt{2\log\log n}}\left(2\log\log n + \frac{3}{2}\log_3 n + \log_4 n + \right. \tag{171}$$

$$\left. + \ldots + \log_{k-1} n + c \log_k n\right)$$

for sufficiently large n where $k > 3$ and $\log_r n$ $(r = 2,3,\ldots)$ is the r-th iterated logarithm of n, then $\underset{\sim}{P}\{A^*\} = 0$ whenever $c > 1$ and $\underset{\sim}{P}\{A^*\} = 1$ whenever $c \leq 1$.

Now let us consider some generalizations of the previous results. Let us assume that $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent real random variables for which $\underset{\sim}{E}\{\xi_k\} = a_k$ and $\underset{\sim}{Var}\{\xi_k\} = b_k^2$ exist for $k = 1,2,\ldots$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$, $A_n = a_1 + a_2 + \ldots + a_n$ and $B_n^2 = b_1^2 + b_2^2 + \ldots + b_n^2$ for $n = 1,2,\ldots,$ and define

$$\eta_n = \frac{\zeta_n - A_n}{B_n} \tag{172}$$

where B_n is the positive square root of B_n^2 .

Following A. Ya. Khintchine [778] we say that the sequence $\{\xi_k\}$ obeys the law of the iterated logarithm if

(173) $$P\{\limsup_{n\to\infty} \frac{\eta_n}{\sqrt{2 \log \log B_n}} = 1\} = 1 .$$

In 1926 A. Ya. Khintchine [416] proved that (173) is valid if

(174) $$P\{\xi_k = 1\} = p_k \quad \text{and} \quad P\{\xi_k = 0\} = q_k$$

where $p_k + q_k = 1$ and $0 < c_1 < p_k < c_2 < 1$.

In 1929 A. N. Kolmogorov [422] proved that if $\lim_{n\to\infty} B_n = \infty$ and the random variables $|\xi_k|$ $(k = 1,2,\ldots)$ are bounded, namely $|\xi_k| \leq m_k$ where

(175) $$\lim_{k\to\infty} \frac{m_k \sqrt{\log \log B_k}}{B_k} = 0 ,$$

then (173) is valid. In the particular case when $b_k = 1$ and $m_k = m$ for all $k = 1,2,\ldots$ the proof of (173) has been given by A. Ya. Khintchine [97].

In 1937 J. Marcinkiewicz and A. Zygmund [435] constructed an example which demonstrates that if we replace (175) by the weaker condition

(176) $$\limsup_{k\to\infty} \frac{m_k \sqrt{\log \log B_k}}{B_k} < \varepsilon$$

where ε is some fixed positive number, then (173) is not necessarily valid anymore.

In 1941 Ph. Hartman and A. Wintner [404] proved that if $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are mutually independent and identically distributed random variables for which $\mathbf{E}\{\xi_k\} = a$ and $\mathbf{Var}\{\xi_k\} = b^2$ exist, and if $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$, then

$$(177) \qquad \mathbf{P}\{\limsup_{n\to\infty} \frac{\zeta_n - na}{\sqrt{2b^2 n \log\log n}} = 1\} = 1 ,$$

that is, (173) is valid in this case.

In 1966 V. Strassen [470] proved that if $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ where $\{\xi_k\}$ is a sequence of mutually independent and identically distributed real random variables and if

$$(178) \qquad \mathbf{P}\{\limsup_{n\to\infty} \frac{|\zeta_n|}{\sqrt{2n \log\log n}} < \infty\} > 0 ,$$

then $\mathbf{E}\{\xi_k\} = 0$ and $\mathbf{E}\{\xi_k^2\} < \infty$.

In 1941 Ph. Hartman [403] proved that if $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are mutually independent random variables and ξ_k has a normal distribution $N(a_k, b_k^2)$ for $k = 1,2,\ldots$, then $\lim_{n\to\infty} B_n = \infty$ is a sufficient condition for the validity of (173).

The law of the iterated logarithm (173) can also be interpreted in the following way. Let $\gamma(n)$ $(n = 1,2,\ldots)$ be an increasing sequence of positive real numbers and define A^* as the event that infinitely many events occur in the sequence

(179) $$\{\eta_n > \gamma(n) \; ; \; n = 1,2,\ldots\} \; .$$

If

(180) $$\gamma(n) = c\sqrt{2 \log \log B_n}$$

for sufficiently large n values, then (173) is equivalent to the following statement: $P\{A^*\} = 0$ whenever $c > 1$ and $P\{A^*\} = 1$ whenever $c < 1$.

In 1931 P. Lévy [428], [429] studied in some particular cases the problem of finding necessary and sufficient conditions for a sequence $\{\gamma(n)\}$ to imply $P\{A^*\} = 0$ or $P\{A^*\} = 1$.

By the zero-or-one law (Theorem 41.4) it follows that either $P\{A^*\} = 0$, or $P\{A^*\} = 1$.

In 1933 F. P. Cantelli [497] proved that if $\{\xi_k\}$ is a sequence of mutually independent random variables for which $E\{\xi_k\} = 0$, $E\{\xi_k^2\} = 1$, $E\{\xi_k^{2+\delta}\} < \infty$ for some $\delta > 0$ and some other conditions are satisfied too and if we define $\gamma(n)$ by (167), then $P\{A^*\} = 0$ whenever $c > 3$ and $P\{A^*\} = 1$ whenever $c \leq 1$. This is a generalization of a result of P. Lévy [428].

In 1943 W. Feller [384] gave necessary and sufficient conditions for $\{\gamma(n)\}$ to imply $P\{A^*\} = 0$ or $P\{A^*\} = 1$ by imposing a sequence of gradually weakening conditions on the random variables $\{\xi_k\}$. In 1946 W. Feller [387] considered the case of mutually independent and identically distributed random variables $\{\xi_k\}$ for which $P\{\xi_k \leq x\} = F(x)$, $E\{\xi_k\} = 0$, $E\{\xi_k^2\} = 1$ and

$$\log\log a \int_{|x|>a} x^2 dF(x) < C \tag{181}$$

as $a \to \infty$, where $C < \infty$ and proved that $P\{A^*\} = 0$ if and only if

$$\sum_{n=1}^{\infty} \frac{\gamma(n)}{n} e^{-\frac{1}{2}[\gamma(n)]^2} < \infty . \tag{182}$$

He also showed that the theorem is no longer valid if

$$\lim_{a \to \infty} \log\log a \int_{|x|>a} x^2 dF(x) = \infty . \tag{183}$$

In all the results mentioned until now it was assumed that the random variables $\{\xi_k\}$ are independent and have finite variances. However, we can consider any sequence of mutually independent random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ and pose problems analogous to the above ones. Thus let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and let us ask whether there exist constants $c_1, c_2, \ldots, c_n, \ldots$ such that

$$P\{\limsup_{n \to \infty} \frac{\zeta_n}{c_n} = 1\} = 1 , \tag{184}$$

or what conditions should the constants $c_1, c_2, \ldots, c_n, \ldots$ satisfy in order that

$$P\{|\zeta_n| > c_n \text{ for infinitely many } n = 1,2,\ldots\} \tag{185}$$

be 0 or 1.

Such problems were considered for the first time in 1931 by P. Lévy [429]. See also P. Lévy [430], [113 pp. 258-276] and J. Marcinkiewicz [434]. They assumed that

(186) $$C_1 x^{-\alpha} < P\{|\xi_k| > x\} < C_2 x^{-\alpha}$$

holds uniformly for large x and all k where $0 < \alpha < 2$ and C_1 and C_2 are positive constants, and that

(187) $$\lim_{a \to \infty} \int_{-a}^{a} x dP\{\xi_k \leq x\} = 0$$

in the case of $1 \leq \alpha < 2$. They proved that if

(188) $$c_n = [n \log n \, \lambda \, (\log n)]^{1/\alpha}$$

where $\lambda(x)$ is a positive increasing function of x for which $\lim_{x \to \infty} \lambda(2x)/\lambda(x) = 1$, then

(189) $$P\{|\zeta_n| > c_n \text{ for infinitely many } n = 1,2,\ldots\}$$

is 0 or 1 according as the series

(190) $$\sum_{n=1}^{\infty} \frac{1}{n\lambda(n)}$$

converges or diverges. This result has been proved by P. Lévy [429] in the case $0 < \alpha < 1$ and by J. Marcinkiewicz [434] in the case $0 < \alpha < 2$.

In 1946 W. Feller [387] proved that if $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are mutually independent and identically distributed random variables for which $\underset{\sim}{E}\{\xi_k\} = 0$ and $\underset{\sim}{E}\{|\xi_k|^{1+\delta}\} = \infty$ for some $0 < \delta < 1$, then the probability (185) is 0 for $c_n = n$ and 1 for $c_n = n^{1/(1+\delta)}$. For any sequence $\{c_n\}$ for which there exists an ε with $0 \leq \varepsilon < 1$ such that $\{c_n n^{-1/(1+\varepsilon)}\}$ is increasing and $\{c_n/n\}$ is decreasing, the probability (185) is zero or one according as the series

$$\sum_{n=1}^{\infty} \underset{\sim}{P}\{|\xi_1| \geq c_n\} \tag{191}$$

converges or diverges. Furthermore, W. Feller [387] proved also that if $\{\xi_k\}$ are mutually independent and identically distributed random variables for which $\underset{\sim}{E}\{|\xi_k|\} = \infty$, then the probability (185) is 1 for $c_n = n$. For any sequence $\{c_n\}$ for which $\{c_n/n\}$ is increasing, the probability (185) is zero or one according as (191) converges or diverges.

In 1968 W. Feller [389] considered mutually independent and identically distributed symmetric random variables with infinite second moments and gave conditions for the validity of (184). See also B. A. Rogozin [458].

In 1969 W. Feller [391] considered mutually independent random variables $\{\xi_k\}$ for which $\underset{\sim}{E}\{\xi_k\} = 0$ and $\underset{\sim}{E}\{\xi_k^2\} < \infty$ and gave conditions for the validity of (184). See also W. Feller [392] for more refined results.

Finally, we mention the works of V. Strassen [469] and J. Slivka [464]. These authors considered a sequence of mutually independent and identically distributed random variables $\{\xi_k\}$ for which $\underset{\sim}{E}\{\xi_k\} = 0$ and $\underset{\sim}{E}\{\xi_k^2\} = 1$, and studied the asymptotic behavior of the random variable $\nu_n(c)$ defined as the number of subscripts $k = 1,2,\ldots, n$ for which

$$\xi_1 + \xi_2 + \ldots + \xi_k > c(2k \log \log k)^{1/2} . \tag{192}$$

44. Limit Distributions. The main object of this section is to find the solutions of the following problems. Let $R(x)$ be a distribution function. Let $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ be a sequence of mutually independent real random variables having the same distribution function

$$(1) \qquad \mathbf{P}\{\xi_k \leq x\} = F(x)$$

for $k = 1,2,\ldots$. Write $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$.

What conditions should $R(x)$ satisfy in order that there exist a sequence of mutually independent and identically distributed real random variables $\{\xi_k\}$ and real constants A_n $(n = 1,2,\ldots)$ and $B_n > 0$ $(n = 1,2,\ldots)$ such that

$$(2) \qquad \lim_{n \to \infty} \mathbf{P}\left\{\frac{\zeta_n - A_n}{B_n} \leq x\right\} = R(x)$$

in every continuity point of $R(x)$?

What conditions should we impose on $F(x)$ and how should we choose the constants A_n $(n = 1,2,\ldots)$ and $B_n > 0$ $(n = 1,2,\ldots)$ such that

$$(3) \qquad \lim_{n \to \infty} \mathbf{P}\left\{\frac{\zeta_n - A_n}{B_n} \leq x\right\} = R(x)$$

in every continuity point of $R(x)$?

By the works of P. Lévy [111], [113], A. Ya. Khintchine [278], [545], W. Doeblin [508], [510] B. V. Gnedenko [528], [531], and B. V. Gnedenko and A. N. Kolmogorov [260] we can give complete solutions of the above problems.

The solutions are based on some continuity theorems for the Laplace-Stieltjes transforms of infinitely divisible distribution functions. These continuity theorems were proved in 1938 and in 1939 by B. V. Gnedenko [526], [527], [528], [529], [531]. They are the consequences of Theorem 41.9 and Theorem 41.10 in this chapter.

Let $R_n(x)$ $(n = 1,2,\dots)$ be a sequence of infinitely divisible distribution functions. Let

$$(4) \qquad \psi_n(s) = \int_{-\infty}^{\infty} e^{-sx} dR_n(x)$$

for $Re(s) = 0$. By Theorem 42.1 we can write that

$$(5) \qquad \log\psi_n(s) = -\mu_n s + \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG_n(x)$$

for $Re(s) = 0$ where μ_n is a real constant, $G_n(x)$ is a non decreasing function of x for which $G_n(-\infty) = 0$, $G_n(\infty)$ is finite, and the integrand at $x = 0$ is defined by

$$(6) \qquad \left[\left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\right]_{x=0} = \frac{s^2}{2} \quad .$$

Let us suppose that $R_n(x)$ converges weakly to a distribution function $R(x)$, that is,

$$(7) \qquad \lim_{n \to \infty} R_n(x) = R(x)$$

in every continuity point of $R(x)$. It is evident that in this case $R(x)$ is necessarily an infinitely divisible distribution function. Let

$$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} dR(x) \tag{8}$$

for $\operatorname{Re}(s) = 0$. Then by Theorem 42.1 we can write that

$$\log\psi(s) = -\mu s + \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2} dG(x) \tag{9}$$

for $\operatorname{Re}(s) = 0$ where μ is a real constant, $G(x)$ is a nondecreasing function of x for which $G(-\infty) = 0$ and $G(\infty)$ is finite and the integrand at $x = 0$ is defined by (6).

The following theorem gives a necessary and sufficient condition for $R_n(x) \Rightarrow R(x)$.

Theorem 1. Let $R_1(x), R_2(x), \ldots, R_n(x), \ldots$ and $R(x)$ be infinitely divisible distribution functions whose Laplace-Stieltjes transforms are given by (4), (5) and (8), (9). The sequence $\{R_n(x)\}$ converges weakly to $R(x)$ if and only if

$$\lim_{n \to \infty} G_n(x) = G(x) \tag{10}$$

in every continuity point of $G(x)$,

$$\lim_{n \to \infty} G_n(\infty) = G(\infty) \tag{11}$$

and

$$\lim_{n \to \infty} \mu_n = \mu. \tag{12}$$

Proof. In proving this theorem we can use the same method as in the proof of Theorem 42.1 .

First, we shall prove that the conditions (10), (11) and (12) are necessary. Let $R_n(x) \Longrightarrow R(x)$. By Theorem 41.9 it follows that

(13) $$\lim_{n \to \infty} \psi_n(s) = \psi(s)$$

for $Re(s) = 0$ and the convergence is uniform on $Re(s) = 0$. Hence

(14) $$\lim_{n \to \infty} \log\psi_n(s) = \log\psi(s)$$

for $Re(s) = 0$ and the convergence is uniform in every finite interval of $Re(s) = 0$. By using this fact we can prove that the sequence $\{G_n(\infty)\}$ is bounded. Since

(15) $$\frac{x^2}{1+x^2} \leq \int_0^2 (1-\cos xu)du$$

for every x , we have

(16) $$\begin{aligned} G_n(\infty) &= \int_{-\infty}^{\infty} \left(\frac{x^2}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG_n(x) \leq \int_0^2 \left[\int_{-\infty}^{\infty} (1-\cos xu) \frac{1+x^2}{x^2}\, dG_n(x) \right] du \\ &= -\int_0^2 \log|\psi_n(iu)|du \to -\int_0^2 \log|\psi(iu)|du \quad \text{as} \quad n \to \infty . \end{aligned}$$

This implies that $\{G_n(\infty)\}$ is bounded.

We observe that for any $\varepsilon > 0$

(17) $$\int_{|x|>a} dG_n(x) < \varepsilon$$

if a is a sufficiently large positive real number and n is a sufficiently large integer. Since

$$(18) \qquad \frac{x^2}{1+x^2} \leqq a \int_0^{2/a} (1-\cos xu)\,du$$

for $|x| \geqq a > 0$, we have

$$\int_{|x|>a} dG_n(x) = \int_{|x|>a} \left(\frac{x^2}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG_n(x) \leqq a \int_0^{2/a} \left[\int_{|x|>a} (1-\cos xu) \frac{1+x^2}{x^2}\, dG_n(x) \right] du$$

$$(19) \qquad \leqq -a \int_0^{2/a} \log|\psi_n(iu)|\,du \to -a \int_0^{2/a} \log|\psi(iu)|\,du \quad \text{as} \quad n \to \infty .$$

In (19) the last integral tends to 0 as $a \to \infty$ and this proves (17).

Since $G_n(-\infty) = 0$, and $G_n(\infty) < K$, the sequence $\{G_n(x)\}$ is weakly compact, that is, every infinite subsequence of $\{G_n(x)\}$ contains a subsequence $\{G_{n_k}(x)\}$ which converges weakly to a nondecreasing function $G^*(x)$. (See Theorem 41.7) By (17) it follows also that $\lim_{k \to \infty} G_{n_k}(-\infty) = G^*(-\infty)$ and $\lim_{k \to \infty} G_{n_k}(\infty) = G^*(\infty)$. Thus we can apply Theorem 41.8 to obtain that

$$(20) \qquad \lim_{k \to \infty} \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG_{n_k}(x) = \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG^*(x)$$

for $\mathrm{Re}(s) = 0$. By (14) we have

$$(21) \qquad \lim_{k \to \infty} \log \psi_{n_k}(s) = \log \psi(s)$$

for $\operatorname{Re}(s) = 0$. Thus it follows that

$$\lim_{k\to\infty} \mu_{n_k} = \mu^* \tag{22}$$

also exists.

Accordingly, we have

$$\log \psi(s) = -\mu^* s + \int_{-\infty}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG^*(x) \tag{23}$$

for $\operatorname{Re}(s) = 0$.

By Theorem 42.1 the function $\log \psi(s)$ defined for $\operatorname{Re}(s) = 0$ uniquely determines μ^*, and $G^*(x)$ in its continuity points. Thus by (9) we have $\mu^* = \mu$ and $G^*(x) = G(x)$ at the continuity points of $G(x)$. Thus $G_{n_k}(x) \Longrightarrow G(x)$, $G_{n_k}(\infty) \to G(\infty)$, $G_{n_k}(-\infty) \to G(-\infty)$ and $\mu_{n_k} \to \mu$ as $k \to \infty$. Since every infinite subsequence of $\{G_n(x)\}$ contains a subsequence $\{G_{n_k}(x)\}$ which converges weakly and completely to the same limit $G(x)$, it follows that $G_n(x) \Longrightarrow G(x)$, $G_n(\infty) \to G(\infty)$ and $G_n(-\infty) \to G(-\infty)$ as $n \to \infty$. Furthermore, $\mu_n \to \mu$ as $n \to \infty$. This proves that the conditions (10), (11) and (12) are necessary.

Now let us prove that the conditions (10), (11) and (12) are sufficient too. If (10), (11) and (12) are satisfied and $\log \psi_n(s)$ and $\log \psi(s)$ are given by (5) and (9) respectively, then it follows immediately from Theorem 41.8 that

(24) $$\lim_{n\to\infty} \log \psi_n(s) = \log \psi(s)$$

for $Re(s) = 0$, and therefore

(25) $$\lim_{n\to\infty} \psi_n(s) = \psi(s)$$

for $Re(s) = 0$. Finally, by Theorem 41.10 we can conclude that $R_n(x) \Longrightarrow R(x)$. This completes the proof of the theorem.

We can express Theorem 1 in an equivalent form if we use the representation (42.44) for $\log \psi_n(s)$ and $\log \psi(s)$ instead of (5) and (9).

Let us suppose that instead of (5) $\log \psi_n(s)$ is given by the following expression

(26) $$\log \psi_n(s) = -\mu_n s + \frac{\sigma_n^2 s^2}{2} + \int_{-\infty}^{-0} (e^{-sx} - 1 + \frac{sx}{1+x^2}) dM_n(x) + \int_{+0}^{\infty} (e^{-sx} - 1 + \frac{sx}{1+x^2}) dN_n(x)$$

for $Re(s) = 0$ where μ_n is a real constant, σ_n^2 is a nonnegative constant, $M_n(x)$ is a nondecreasing function of x in the interval $(-\infty, 0)$, $N_n(x)$ is a nondecreasing function of x in the interval $(0, \infty)$ and these functions satisfy the requirements

(27) $$\lim_{x\to-\infty} M_n(x) = \lim_{x\to+\infty} N_n(x) = 0$$

and

(28) $$\int_{-\varepsilon}^{-0} x^2 dM_n(x) + \int_{+0}^{\varepsilon} x^2 dN_n(x) < \infty$$

for some $\varepsilon > 0$.

Furthermore, let us suppose that instead of (9) $\log \psi(s)$ is given by the following expression

$$\text{(29)} \quad \log \psi(s) = -\mu s + \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{-0} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dM(x) + \int_{+0}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) dN(x)$$

for $\operatorname{Re}(s) = 0$ where μ is a real constant, σ^2 is a nonnegative constant, $M(x)$ is a nondecreasing function of x in the interval $(-\infty, 0)$, $N(x)$ is a nondecreasing function of x in the interval $(0, \infty)$ and these functions satisfy the requirements

$$\text{(30)} \quad \lim_{x \to -\infty} M(x) = \lim_{x \to +\infty} N(x) = 0$$

and

$$\text{(31)} \quad \int_{-\varepsilon}^{-0} x^2 dM(x) + \int_{+0}^{\varepsilon} x^2 dN(x) < \infty$$

for some $\varepsilon > 0$.

Let us introduce the notation

$$\text{(32)} \quad I_n(\varepsilon) = \int_{-\varepsilon}^{-0} x^2 dM_n(x) + \sigma_n^2 + \int_{+0}^{\varepsilon} x^2 dN_n(x)$$

for $\varepsilon > 0$ and $n = 1, 2, \ldots$.

Theorem 2. *Let $R_1(x), R_2(x), \ldots, R_n(x), \ldots$ and $R(x)$ be infinitely divisible distribution functions whose Laplace-Stieltjes transforms are given by (4), (26) and (8), (29). The sequence $\{R_n(x)\}$ converges weakly to $R(x)$ if and only if*

$$\text{(33)} \quad \lim_{n \to \infty} M_n(x) = M(x)$$

<u>at every continuity point of</u> $M(x)$ <u>in the interval</u> $(-\infty, 0)$,

(34) $$\lim_{n \to \infty} N_n(x) = N(x)$$

<u>at every continuity point of</u> $N(x)$ <u>in the interval</u> $(0, \infty)$,

(35) $$\lim_{n \to \infty} \mu_n = \mu ,$$

<u>and</u>

(36) $$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} I_n(\varepsilon) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} I_n(\varepsilon) = \sigma^2 .$$

<u>Proof</u>. If in Theorem 1 we define

(37) $$M_n(x) = \int_{-\infty}^{x} \frac{1+y^2}{y^2} \, dG_n(y)$$

for $x < 0$,

(38) $$N_n(x) = -\int_{x}^{\infty} \frac{1+y^2}{y^2} \, dG_n(y)$$

for $x > 0$, and

(39) $$\sigma_n^2 = G_n(+0) - G_n(-0) ,$$

furthermore,

(40) $$M(x) = \int_{-\infty}^{x} \frac{1+y^2}{y^2} \, dG(y)$$

for $x < 0$,

$$(41) \qquad N(x) = -\int_x^{\infty} \frac{1+y^2}{y^2}\, dG(y)$$

for $x > 0$, and

$$(42) \qquad \sigma^2 = G(+0) - G(-0) ,$$

then we obtain Theorem 2 and conversely Theorem 2 can be reduced to Theorem 1 by the substitutions (37), (38), (39) and (40), (41), (42).

First, we shall prove that the conditions (33), (34), (35) and (36) are necessary. If $R_n(x) \Longrightarrow R(x)$, then by Theorem 1 we have $G_n(x) \Rightarrow G(x)$, $G_n(\infty) \to G(\infty)$ and $\mu_n \to \mu$ as $n \to \infty$. Thus by (37) we obtain (33), by (38) we obtain (34), and (35) is obvious. It remains to prove (36).

If we take into consideration that

$$(43) \qquad G_n(\varepsilon) - G_n(-\varepsilon) = \int_{-\varepsilon}^{-0} \frac{x^2}{1+x^2}\, dM_n(x) + \sigma_n^2 + \int_{+0}^{\varepsilon} \frac{x^2}{1+x^2}\, dN_n(x)$$

for $\varepsilon > 0$ and that

$$(44) \qquad \frac{x^2}{1+\varepsilon^2} \leqq \frac{x^2}{1+x^2} \leqq x^2$$

for $|x| \leqq \varepsilon$, then obtain that

$$(45) \qquad \frac{I_n(\varepsilon)}{1+\varepsilon^2} \leqq G_n(\varepsilon) - G_n(-\varepsilon) \leqq I_n(\varepsilon)$$

for $\varepsilon > 0$. Now let us suppose that $x = \varepsilon$ and $x = -\varepsilon$ are continuity points of $G(x)$, then

$$(46) \qquad \lim_{n \to \infty} [G_n(\varepsilon) - G_n(-\varepsilon)] = G(\varepsilon) - G(-\varepsilon)$$

and by (45) we obtain that

$$(47) \quad G(\varepsilon)-G(-\varepsilon) \leqq \liminf_{n \to \infty} I_n(\varepsilon) \leqq \limsup_{n \to \infty} I_n(\varepsilon) \leqq (1+\varepsilon^2)[G(\varepsilon)-G(-\varepsilon)] .$$

Since

$$(48) \qquad \lim_{\varepsilon \to 0} [G(\varepsilon) - G(-\varepsilon)] = G(+0) - G(-0) = \sigma^2 ,$$

we obtain (36).

Now let us prove that the conditions (33), (34), (35) and (36) are sufficient too. We shall prove that these conditions imply (10) and (11) in Theorem 1.

Since by (37)

$$(49) \qquad G_n(x) = \int_{-\infty}^{x} \frac{y^2}{1+y^2} \, dM_n(y)$$

for $x < 0$ and by (40)

$$(50) \qquad G(x) = \int_{-\infty}^{x} \frac{y^2}{1+y^2} \, dM(y)$$

for $x < 0$, therefore (33) implies that

$$(51) \qquad \lim_{n \to \infty} G_n(x) = G(x)$$

at every continuity point of $G(x)$ in the interval $(-\infty, 0)$.

By (45) we have

(52) $$G_n(-\varepsilon) + \frac{I_n(\varepsilon)}{1+\varepsilon^2} \leq G_n(\varepsilon) \leq G_n(-\varepsilon) + I_n(\varepsilon)$$

for $\varepsilon > 0$ and therefore by (36) it follows that

(53) $$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} G_n(\varepsilon) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} G_n(\varepsilon) = G(-0) + \sigma^2 = G(+0) .$$

If $0 < \varepsilon < x$, then by (38)

(54) $$G_n(x) - G_n(\varepsilon) = \int_\varepsilon^x \frac{y^2}{1+y^2}\, dN_n(y)$$

and by (41)

(55) $$G(x) - G(\varepsilon) = \int_\varepsilon^x \frac{y^2}{1+y^2}\, dN(y) ,$$

and therefore by (34) we obtain that

(56) $$\lim_{n \to \infty} [G_n(x) - G_n(\varepsilon)] = G(x) - G(\varepsilon)$$

for $0 < \varepsilon < x$ provided that x and ε are continuity points of $G(x)$. Thus by (53) and (56)

(57) $$\limsup_{n \to \infty} G_n(x) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} [G_n(\varepsilon) + G_n(x) - G_n(\varepsilon)] =$$
$$= G(+0) + G(x) - G(+0) = G(x)$$

and

(58) $$\liminf_{n \to \infty} G_n(x) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} [G_n(\varepsilon) + G_n(x) - G_n(\varepsilon)] =$$
$$= G(+0) + G(x) - G(+0) = G(x)$$

if $x > 0$ is a continuity point of $G(x)$. This proves that $G_n(x) \Longrightarrow G(x)$

for $x > 0$.

Finally, since

$$(59)\qquad G_n(\infty) - G_n(x) = \int_x^\infty \frac{y^2}{1+y^2}\, dN_n(y)$$

and

$$(60)\qquad G(\infty) - G(x) = \int_x^\infty \frac{y^2}{1+y^2}\, dN(y)$$

for $x > 0$, it follows from (34) that

$$(61)\qquad \lim_{n\to\infty} [G_n(\infty) - G_n(x)] = G(\infty) - G(x)$$

if $x > 0$ is a continuity point of $G(x)$. This proves that $G_n(\infty) \to G(\infty)$ as $n \to \infty$.

This completes the proof of the theorem.

Note. Theorem 2 remains valid unchangeably if in (26) and in (29) we choose another centering function. Thus we may assume that instead of (26) $\log \psi_m(s)$ is given by

$$(62)\quad \log \psi_m(s) = -\mu_m^* s + \frac{\sigma_m^2 s^2}{2} + \int_{-\infty}^{-0} (e^{-sx} - 1 + s\delta(x))\, dM_n(x) + \int_{+0}^{\infty} (e^{-sx} - 1 + s\delta(x))\, dN_n(x)$$

for $\mathrm{Re}(s) = 0$ where μ_m^* is a real constant, and σ_m^2, $M_n(x)$, $N_n(x)$ satisfy the same requirements as in (26). The function $\delta(x)$ defined for $-\infty < x < \infty$ can be chosen as any bounded and continuous function of x for which $\delta(x) - x = O(x^2)$ as $x \to 0$. (Instead of continuity we may require only that $\delta(x)$ is piecewise continuous and the discontinuity points of $\delta(x)$ are continuity points of $M_m(x)$ or $N_n(x)$ as the case may be.) Beside $\delta(x) = x/(1+x^2)$ the following functions are suitable choices in (62): $\delta(x) = \sin x$ or $\delta(x) = x$ if $|x| < \tau$ and $\delta(x) = 0$ if $|x| \geqq \tau$ where $\tau > 0$ and $M_n(x)$ is continuous at $x = -\tau$ and $N_n(x)$ is continuous at $x = \tau$.

Let us assume also that instead of (29) $\log \psi(s)$ is given by

$$(63) \quad \log \psi(s) = -\mu^* s + \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{-0} (e^{-sx} - 1 + s\delta(x)) dM(x) + \int_{+0}^{\infty} (e^{-sx} - 1 + s\delta(x)) dN(x)$$

for $\mathrm{Re}(s) = 0$ where μ^* is a real constant, and σ^2 , $M(x)$, $N(x)$ satisfy the same requirements as in (29). The function $\delta(x)$ is the same as in (62).

If we suppose that $\log \psi_m(s)$ is given by (62) and $\log \psi(s)$ is given by (63), then Theorem 2 remains valid provided that the condition (35) is replaced by

$$(64) \qquad \lim_{n \to \infty} \mu_n^* = \mu^* .$$

For if in (62) we put

$$(65) \quad \mu_m^* = \mu_m + \int_{-\infty}^{-0} [\delta(x) - \frac{x}{1+x^2}] dM_n(x) + \int_{+0}^{\infty} [\delta(x) - \frac{x}{1+x^2}] dN_n(x) ,$$

then we obtain (26), and if in (63) we put

$$\mu^* = \mu + \int_{-\infty}^{-0} [\delta(x) - \frac{x}{1+x^2}] dM(x) + \int_{+0}^{\infty} [\delta(x) - \frac{x}{1+x^2}] dN(x) \quad , \tag{66}$$

then we obtain (29), and $\mu_m \to \mu$ implies $\mu_m^* \to \mu^*$ and conversely.

In what follows we shall prove some more auxiliary theorems which are needed in solving the problems formulated at the beginning of this section.

Let $R(x)$ be the distribution function of a real random variable and define

$$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} dR(x) \tag{67}$$

for $\operatorname{Re}(s) = 0$. Let

$$a = \int_{|x|<\tau} x dR(x) \tag{68}$$

where τ is some positive real number and let m be a median of $R(x)$, that is, any real number for which

$$R(m) \geq \frac{1}{2} \quad \text{and} \quad 1-R(m-0) \geq \frac{1}{2} \ . \tag{69}$$

<u>Theorem 3</u>. <u>If</u> $\operatorname{Re}(s) = 0$, <u>if</u> $\psi(iu) \neq 0$ for $0 \leq u \leq \delta \leq 2$, <u>and if</u> $\tau > |m|$, <u>then</u>

$$|\psi(s)e^{sa} - 1| \leq \frac{16(1+2\tau)^6(1+|s|\tau)^2}{\delta^3(\tau-|a|)^2(\tau-|m|)^2} \int_0^{\delta} \log \frac{1}{|\psi(iu)|} du \ . \tag{70}$$

Proof. We shall prove (70) in several steps. First we shall prove that

$$|\psi(s)e^{sa}-1| = \left|\int_{-\infty}^{\infty}(e^{-sx}-1)dR(x+a)\right| \leq \tag{71}$$

$$\leq \left\{\frac{|s|^2[1+(\tau+|a|)^2]}{2} + \frac{(2+|s||a|)[1+(\tau-|a|)^2]}{(\tau-|a|)^2}\right\}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}\,dR(x+a)$$

for $Re(s) = 0$.

Let us write

$$\psi(s)e^{sa}-1 = \int_{-\infty}^{\infty}(e^{-sx}-1)dR(x+a) = \int_{|x+a|<\tau}(e^{-sx}-1+sx)dR(x+a) + \tag{72}$$

$$+\int_{|x+a|\geq\tau}(e^{-sx}-1)dR(x+a) - s\int_{|x+a|<\tau}x\,dR(x+a) .$$

In (72) in the last term

$$\int_{|x+a|<\tau}x\,dR(x+a) = \int_{|x|<\tau}(x-a)dR(x) = a-a\int_{|x|<\tau}dR(x) = a\int_{|x|\geq\tau}dR(x) . \tag{73}$$

If $Re(s) = 0$, then $|e^{-sx}-1| \leq 2$ and

$$|e^{-sx}-1+sx| \leq \frac{|s|^2x^2}{2} \tag{74}$$

for all real x. By (73) and (74) we obtain from (72) that

$$|\psi(s)e^{sa}-1| \leq \frac{|s|^2}{2}\int_{|x+a|<\tau}x^2dR(x+a) + (2+|s||a|)\int_{|x+a|\geq\tau}dR(x+a) . \tag{75}$$

Since

$$(76)\quad \int_{|x+a|<\tau} x^2 dR(x+a) \leq [1+(\tau+|a|)^2] \int_{|x+a|<\tau} \frac{x^2}{1+x^2} dR(x+a) \leq [1+(\tau+|a|)^2] \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR(x+a)$$

and

$$(77)\quad \int_{|x+a|\geq\tau} dR(x+a) \leq \frac{1+(\tau-|a|)^2}{(\tau-|a|)^2} \int_{|x+a|\geq\tau} \frac{x^2}{1+x^2} dR(x+a) \leq \frac{1+(\tau-|a|)^2}{(\tau-|a|)^2} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR(x+a) ,$$

we obtain (71) by (75), (76) and (77). From (71) we obtain easily that

$$(78)\quad |\psi(s)e^{sa}-1| \leq \frac{(1+4\tau^2)(1+|s|\tau)^2}{2(\tau-|a|)^2} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR(x+a)$$

for $\mathrm{Re}(s) = 0$. We note that by (68) $|a| < \tau$.

Next we shall prove that if $\tau > |m|$, then

$$(79)\quad \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR(x+a) \leq \{[1+(\tau+|m|)^2] + \frac{[1+(\tau-|m|)^2][1+2\tau(\tau+|m|)]}{(\tau-|m|)^2}\} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR(x+m) .$$

Since

$$(80)\quad (x-a)^2 = (x-m)^2 + 2(m-a)(x-a) - (m-a)^2 \leq (x-m)^2 + 2(m-a)(x-a) ,$$

we can write down that

$$(81)\quad \begin{aligned} \int_{-\infty}^{\infty} \frac{(x-a)^2}{1+(x-a)^2} dR(x) &\leq \int_{|x|<\tau} (x-a)^2 dR(x) + \int_{|x|\geq\tau} dR(x) \leq \int_{|x|<\tau} (x-m)^2 dR(x) + \\ &+ 2(\tau+|m|) \int_{|x|<\tau} (x-a) dR(x) + \int_{|x|\geq\tau} dR(x) = \int_{|x|<\tau} (x-m)^2 dR(x) + \\ &+ [2(\tau+|m|)a+1] \int_{|x|\geq\tau} dR(x) \end{aligned}$$

and further

$$(82)\quad \int_{|x|<\tau} (x-m)^2 dR(x) \leq [1+(\tau+|m|)^2] \int_{|x|<\tau} \frac{(x-m)^2}{1+(x-m)^2} dR(x) \leq [1+(\tau+|m|)^2] \int_{-\infty}^{\infty} \frac{(x-m)^2}{1+(x-m)^2} dR(x)$$

and

$$(83)\quad \int_{|x|\geq\tau} dR(x) \leq \frac{1+(\tau-|m|)^2}{(\tau-|m|)^2} \int_{|x|\geq\tau} \frac{(x-m)^2}{1+(x-m)^2} dR(x) \leq \frac{1+(\tau-|m|)^2}{(\tau-|m|)^2} \int_{-\infty}^{\infty} \frac{(x-m)^2}{1+(x-m)^2} dR(x)$$

for $\tau > |m|$. By (81), (82) and (83) we obtain (79). From (79) it follows easily that

$$(84)\quad \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR(x+a) \leq \frac{(1+4\tau^2)(1+2\tau^2)}{(\tau-|m|)^2} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR(x+m)$$

for $\tau > |m|$.

Now let

$$(85)\quad R^*(x) = \int_{-\infty}^{\infty} R(x+y)dR(y) .$$

We shall prove that

$$(86)\quad \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR(x+m) \leq 2 \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR^*(x) .$$

If we suppose that ξ and η are independent random variables for which $\underset{\sim}{P}\{\xi \leq x\} = R(x)$ and $\underset{\sim}{P}\{\eta \leq x\} = R(x)$, then $\underset{\sim}{P}\{\xi-\eta \leq x\} = R^*(x)$, and (86) can be expressed as

$$(87)\quad \underset{\sim}{E}\left\{\frac{(\xi-m)^2}{1+(\xi-m)^2}\right\} \leq 2\underset{\sim}{E}\left\{\frac{(\xi-\eta)^2}{1+(\xi-\eta)^2}\right\} .$$

For any x we have

(88)
$$P\{\xi-\eta > x\} = P\{(\xi-m) - (\eta-m) > x\} \geq$$
$$P\{\xi-m > x, \eta-m \leq 0\} = P\{\xi-m > x\}P\{\eta \leq m\} \geq \frac{1}{2} P\{\xi-\eta > x\} ,$$

that is

(89)
$$P\{\xi-m > x\} \leq 2P\{\xi-\eta > x\}$$

for all x. If we replace ξ, η, m by $-\xi$, $-\eta$, $-m$ respectively in the above inequality, then we obtain that

(90)
$$P\{m-\xi > x\} \leq 2P\{\eta-\xi > x\}$$

for all x. Thus by (89) and (90) we have

(91)
$$P\{|\xi-m| > x\} \leq 2P\{|\xi-\eta| > x\}$$

for $x \geq 0$, and hence it follows that

(92)
$$P\left\{ \frac{(\xi-m)^2}{1+(\xi-m)^2} > x\right\} \leq 2P\left\{ \frac{(\xi-\eta)^2}{1+(\xi-\eta)^2} > x\right\}$$

for $x \geq 0$. If we integrate (92) from 0 to ∞, then we obtain (97) and therefore (86) too.

If $0 < \delta \leq 2$, then we have

(93)
$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\, dR^*(x) \leq \frac{8}{\delta^3} \int_0^{\delta} [1-|\psi(iu)|^2]du .$$

Since

$$(94) \qquad \int_{-\infty}^{\infty} e^{-sx} dR^*(x) = |\psi(s)|^2$$

for $Re(s) = 0$ and since by (42.28)

$$(95) \qquad \frac{x^2}{1+x^2} \leqq \frac{8}{\delta^3} \int_0^{\delta} (1-\cos xu)du$$

for every x if $0 < \delta \leqq 2$, it follows that

$$(96) \qquad \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dR^*(x) \leqq \frac{8}{\delta^3} \int_0^{\delta} \{\int_{-\infty}^{\infty} (1-\cos xu)dR^*(x)\}du = \frac{8}{\delta^3} \int_0^{\delta} [1-|\psi(iu)|^2]du$$

which was to be proved.

For any real x we have $1-x \leqq e^{-x}$ and hence $1-x \leqq -\log x$ for $x > 0$. Since $\psi(0) = 1$ and $\psi(iu)$ is a continuous function of u, it follows that there exists a positive δ such that $\psi(iu) \neq 0$ for $0 \leqq u \leqq \delta$. Then $\log|\psi(iu)|$ is continuous and bounded in the interval $u\varepsilon[0, \delta]$. Thus

$$(97) \qquad \int_0^{\delta} [1-|\psi(iu)|^2]du \leqq 2 \int_0^{\delta} \log \frac{1}{|\psi(iu)|} du$$

whenever $\psi(iu) \neq 0$ for $0 \leqq u \leqq \delta$.

If we combine (78), (84), (86), (93) and (97), then we obtain that (70) holds for $Re(s) = 0$ whenever $\tau > |m|$, $0 < \delta \leqq 2$ and $\psi(iu) \neq 0$ for $0 \leqq u \leqq \delta$.

We shall mention two more inequalities. Let ξ be a real random variable for which $\underset{\sim}{P}\{\xi \leq x\} = R(x)$ and

$$(98) \qquad \psi(s) = \int_{-\infty}^{\infty} e^{-sx} dR(x)$$

for $\mathrm{Re}(s) = 0$. We have

$$(99) \qquad \underset{\sim}{P}\{|\xi| \geq 2\varepsilon\} \leq \varepsilon \int_{-1/\varepsilon}^{1/\varepsilon} |1-\psi(iu)| du$$

for any $\varepsilon > 0$ and

$$(100) \qquad 1-\mathrm{Re}\{\psi(2iu)\} \leq 4[1-\mathrm{Re}\{\psi(iu)\}]$$

for $-\infty < u < \infty$.

Since

$$(101) \quad \left| \frac{\varepsilon}{2} \int_{-1/\varepsilon}^{1/\varepsilon} \psi(iu) du \right| = \left| \int_{-\infty}^{\infty} \frac{\sin(x/\varepsilon)}{(x/\varepsilon)} dR(x) \right| \leq [1-\underset{\sim}{P}\{|\xi| \geq 2\varepsilon\}] + \frac{1}{2} \underset{\sim}{P}\{|\xi| \geq 2\varepsilon\}$$

and

$$(102) \quad 1-\left| \frac{\varepsilon}{2} \int_{-1/\varepsilon}^{1/\varepsilon} \psi(iu) du \right| \leq \frac{\varepsilon}{2} \left| \int_{-1/\varepsilon}^{1/\varepsilon} [1-\psi(iu)] du \right| \leq \frac{\varepsilon}{2} \int_{-1/\varepsilon}^{1/\varepsilon} |1-\psi(iu)| du$$

for any $\varepsilon > 0$, (99) follows immediately.

Since $1-\cos 2x = 2(1-\cos^2 x) \leq 4(1-\cos x)$ for every x , (100) follows immediately.

We shall need also the following auxiliary theorems.

<u>Lemma 1. Let</u> $R(x)$ <u>be a nondegenerate distribution function. If</u>

$$R(x) = R(a+bx) \tag{103}$$

<u>for every</u> x <u>where</u> a <u>is a real constant and</u> b <u>is a positive real constant, then</u> $a = 0$ <u>and</u> $b = 1$.

<u>Proof</u>. If $0 < b < 1$, then by (103) we obtain that

$$R(x) = R(a(1+b+\ldots+b^{n-1})+b^n x) \tag{104}$$

for $n = 1,2,\ldots$ and for every x . If $n \to \infty$, then $b^n \to 0$ and by (104) $R(x) = R(a/(1-b))$ which is impossible.

If $1 < b < \infty$, and we express (103) in the form

$$R(x) = R(\frac{x}{b} - \frac{a}{b}) , \tag{105}$$

then this case reduces to the previous case and thus we obtain that $1 < b < \infty$ is impossible. Consequently, $b = 1$ must hold.

Finally, we shall prove that $a = 0$. If $a \neq 0$ and $b = 1$, then by (103) we obtain that

$$R(x) = R(x + na) \tag{106}$$

for $n = 0, \pm 1, \pm 2, \ldots$. If $n \to \infty$ and $n \to -\infty$ in (106), then we obtain that $R(x) = R(+\infty) = R(-\infty)$ which is impossible. This implies that $a = 0$ must hold.

Lemma 2. *Let* $\{R_n(x)\}$ *be a sequence of distribution functions, and* a_n $(n = 1,2,\ldots)$ *and* $b_n > 0$ $(n = 1,2,\ldots)$ *be real constants. If*

(107) $$R_n(x) \Longrightarrow R(x)$$

and

(108) $$R_n(a_n + b_n x) \Longrightarrow S(x)$$

where $R(x)$ and $S(x)$ *are nondegenerate distribution functions, then there exist two constants* a *and* $b > 0$ *such that*

(109) $$\lim_{n \to \infty} a_n = a$$

and

(110) $$\lim_{n \to \infty} b_n = b$$

and

(111) $$S(x) = R(a+bx) .$$

Proof. We shall prove that every infinite subsequence of (a_n, b_n) $(n = 1,2,\ldots)$ contains a subsequence (a_{n_j}, b_{n_j}) $(j = 1,2,\ldots)$ for which $a_{n_j} \to a$ where $-\infty < a < \infty$ and $b_{n_j} \to b$ where $0 < b < \infty$ as $j \to \infty$ and that (111) holds. If (111) holds, then a and b do not depend on the particular subsequence of (a_n, b_n) $(n = 1,2,\ldots)$. Thus the whole sequence (a_n, b_n) $(n = 1,2,\ldots)$ is convergent and (109) and (110) hold.

Obviously every infinite subsequence of (a_n, b_n) $(n = 1,2,\ldots)$ contains a subsequence (a_{n_j}, b_{n_j}) $(j = 1,2,\ldots)$ such that $\lim_{j \to \infty} a_{n_j} = a$

and $\lim_{j\to\infty} b_{n_j} = b$ where $-\infty \leqq a \leqq \infty$ and $0 \leqq b \leqq \infty$. Now we shall show that necessarily $-\infty < a < \infty$ and $0 < b < \infty$.

If $b = \infty$, and $c = \sup\{x: \limsup_{j\to\infty}(a_{n_j} + b_{n_j}x) < \infty\}$, then $\limsup_{j\to\infty}(a_{n_j} + b_{n_j}x) = -\infty$ for $x < c$ and $\limsup_{j\to\infty}(a_{n_j} + b_{n_j}x) = +\infty$ for $x > c$. Then by (108) $S(x) = 0$ for $x < c$ and $S(x) = 1$ for $x > c$, that is, $S(x)$ is degenerate. This contradicts to the hypothesis and therefore $0 \leqq b < \infty$.

If $0 \leqq b < \infty$ and $a = \infty$ or $a = -\infty$, then by (108) it follows that $S(x) \equiv 1$ or $S(x) \equiv 0$ which is impossible.

If $b = 0$, then for every x and every $\varepsilon > 0$ we have $a-\varepsilon \leqq a_{n_j} + b_{n_j}x \leqq a+\varepsilon$ whenever j is sufficiently large. Hence $R_{n_j}(a-\varepsilon) \leqq R_{n_j}(a_{n_j} + b_{n_j}x) \leqq R_{n_j}(a+\varepsilon)$ if j is large enough. If $x = a+\varepsilon$ and $x = a-\varepsilon$ are continuity points of $R(x)$ and if we let $j \to \infty$ in the above inequality, then we obtain that $R(a-\varepsilon) \leqq S(x) \leqq R(a+\varepsilon)$. Since x is arbitrary, it follows that $R(a-\varepsilon) = 0$ and $R(a+\varepsilon) = 1$ for any $\varepsilon > 0$ for which $x = a+\varepsilon$ and $x = a-\varepsilon$ are continuity points of $R(x)$. This implies that $R(x)$ is degenerate which contradicts to the hypothesis.

Thus we proved that $-\infty < a < \infty$ and $0 < b < \infty$. Now for any $\varepsilon > 0$ we have $a+bx-\varepsilon < a_{n_j} + b_{n_j}x < a+bx+\varepsilon$ if j is large enough. Thus by (107) and (108) it follows that

$$R(a+bx-\varepsilon) \leqq S(x) \leqq R(a+bx+\varepsilon) \tag{112}$$

provided that x is a continuity point of $S(x)$ and $a+bx+\varepsilon$ and $a+bx-\varepsilon$

are continuity points of $R(x)$. If $\varepsilon \to 0$ in (112), then we obtain that (111) holds whenever x is a continuity point of $S(x)$ and $a+bx$ is a continuity point $R(x)$. However, if two distribution functions are equal on a set which is dense everywhere, then the two distribution functions are identical. This proves (111).

Since by Lemma 1 the constants a and b are uniquely determined by $S(x)$, that is, they do not dependent on the particular subsequence (a_{n_j}, b_{n_j}) $(j = 1,2,\ldots)$, it follows that (a_n, b_n) $(n = 1,2,\ldots)$ is convergent and (109) and (110) hold.

Corollary 1. If in Lemma 2 we have $S(x) = R(x)$, then necessarily $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} b_n = 1$.

This follows from Lemma 1.

Now we are in a position to provide a solution of the problems stated at the beginning of this section.

First we shall characterize the class of distribution functions $R(x)$ which can appear as limiting distributions of suitably normalized sums of mutually independent and identically distributed real random variables.

Let us suppose that $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are mutually independent and identically distributed real random variables for which

$$(113) \qquad \mathbf{P}\{\xi_k \leq x\} = F(x)$$

and

$$(114) \qquad \phi(s) = \int_{-\infty}^{\infty} e^{-sx} dF(x)$$

for $\operatorname{Re}(s) \geqq 0$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$. Let us suppose that there exist constants A_n $(n = 1,2,\ldots)$ and $B_n > 0$ $(n = 1,2,\ldots)$ such that

$$(115) \qquad \lim_{n\to\infty} \mathbf{P}\left\{ \frac{\eta_n - A_n}{B_n} \leqq x \right\} = R(x)$$

in every continuity point of the distribution function $R(x)$. Let

$$(116) \qquad \psi(s) = \int_{-\infty}^{\infty} e^{-sx} dR(x) .$$

for $\operatorname{Re}(s) = 0$.

The following auxiliary theorem contains some information about the asymptotic behavior of B_n as $n \to \infty$.

Lemma 3. *If* $R(x)$ *is a nondegenerate distribution function and* (115) *holds, then*

$$(117) \qquad \lim_{n\to\infty} B_n = \infty$$

and

$$(118) \qquad \lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 1 .$$

Proof. If (115) holds, then we have

(119) $$\lim_{n\to\infty} [\phi(\frac{s}{B_n})]^n e^{sA_n/B_n} = \psi(s)$$

for $\mathrm{Re}(s) = 0$. This implies that

(120) $$\lim_{n\to\infty} \phi(\frac{s}{B_n}) = 1$$

for all $\mathrm{Re}(s) = 0$. The proof of (120) follows on the same lines as the proof of Lemma 42.1. Since $\psi(0) = 1$ and $\psi(s)$ is continuous on $\mathrm{Re}(s) = 0$, it follows that there is an $a > 0$ such that $|\psi(s)| > 0$ for $s = iu$ and $|u| \leqq a$. Hence by (119)

(121) $$\lim_{n\to\infty} |\phi(\frac{s}{B_n})| = \lim_{n\to\infty} |\psi(s)|^{\frac{1}{n}} = 1$$

for $s = iu$ and $|u| \leqq a$. This implies also that (120) holds for $s = iu$ and $|u| \leqq a$. By using the inequality (100) repeatedly, we can conclude that (120) holds for all $\mathrm{Re}(s) = 0$.

Now we shall prove (117) by contradiction. If (117) does not hold, then $\{B_n\}$ contains a bounded infinite subsequence, and by the Bolzano-Weierstrass theorem this latter sequence contains a convergent subsequence $\{B_{n_j}\}$ for which $\lim_{j\to\infty} B_{n_j} = B < \infty$. Then by (120) we obtain that

(122) $$\phi(s) = \lim_{j\to\infty} \phi(\frac{sB_{n_j}}{B_{n_j}}) = \lim_{j\to\infty} \phi(\frac{sB}{B_{n_j}}) = 1$$

for $\mathrm{Re}(s) = 0$. If $B > 0$, then in (122) we use that $\phi(s)$ is uniformly continuous on $\mathrm{Re}(s) = 0$. If $B = 0$, then $\phi(s) = 1$ is obviously true.

If $\phi(s) = 1$ for all $\mathrm{Re}(s) = 0$, then by (119) $|\psi(s)| = 1$ for all $\mathrm{Re}(s) = 0$. In this case $R(x)$ is degenerate. This contradiction proves (117).

If $B_n \to \infty$ as $n \to \infty$, then

$$(123) \qquad \underset{\sim}{P}\left\{ \left| \frac{\xi_{n+1}}{B_{n+1}} \right| \leqq \varepsilon \right\} = \int_{|x| \leqq \varepsilon B_n} dF(x) \to 1$$

as $n \to \infty$ for any $\varepsilon > 0$. Thus by (115) it follows that

$$(124) \qquad \lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\zeta_n - A_{n+1}}{B_{n+1}} \leqq x \right\} = R(x)$$

in every continuity point of $R(x)$. If we compare (115) and (124), then by Corollary 1 we can conclude that (118) holds, and furthermore that

$$(125) \qquad \lim_{n \to \infty} \frac{A_{n+1} - A_n}{B_n} = 0 .$$

The following theorem was discovered in 1925 by P. Lévy [111]. See also A. Ya. Khintchine [278].

Theorem 4. *The distribution function $R(x)$ is the limiting distribution of suitably normalized sums of mutually independent and identically distributed real random variables if and only if $R(x)$ is stable.*

Proof. First we shall prove that the condition is necessary. If

$R(x)$ is degenerate, then $R(x)$ is stable. If $R(x)$ is nondegenerate, then we shall prove that for every $a_1, a_2, b_1 > 0, b_2 > 0$ there exist two constants a and $b > 0$ such that

$$(126)\qquad R(a_1 + b_1 x) * R(a_2 + b_2 x) = R(a+bx)$$

holds. Without loss of generality, we may assume that $b_1 \leq b_2$. In this case by Lemma 3 we have $\lim_{n\to\infty} B_n = \infty$ and $\lim_{n\to\infty} B_n/B_{n+1} = 1$, and for every $n = 1,2,\ldots$ we can find an $m = m(n) \leq n$ such that $m \to \infty$ as $n \to \infty$ and

$$(127)\qquad \lim_{n\to\infty} \frac{B_m}{B_n} = \frac{b_1}{b_2}$$

where $0 < b_1 \leq b_2 < \infty$. Since $B_1/B_n \to 0$ as $n \to \infty$ and $B_n/B_n = 1$, for every sufficiently large n we can find an m such that $1 < m \leq n$ and

$$(128)\qquad \frac{B_{m-1}}{B_n} \leq \frac{b_1}{b_2} \leq \frac{B_m}{B_n}$$

holds. If we choose $m = m(n)$ in such a way, then (127) is satisfied.

Now let us write

$$(129)\qquad \frac{1}{b_1}\left(\frac{\zeta_n - A_n}{B_n} - a_1\right) + \frac{B_m}{b_1 B_n}\left(\frac{\zeta_{n+m} - \zeta_n - A_m}{B_m} - a_2\right) = \frac{\zeta_{n+m} - (A_n + A_m + a_1 B_n + a_2 B_m)}{b_1 B_n}$$

for $n = 1,2,\ldots$ and $m = m(n)$. On the left-hand side of (129) ζ_n and $\zeta_{n+m} - \zeta_n$ are independent and $\zeta_{n+m} - \zeta_n$ has the same distribution as ζ_m. Furthermore $B_m / b_1 B_n \to 1/b_2$ as $n \to \infty$. Thus by (115) it follows that the distribution function of the left-hand side of (129) converges weakly to

$$R(a_1 + b_1 x) * R(a_2 + b_2 x) \tag{130}$$

as $n \to \infty$. If $n \to \infty$, then by (115)

$$\lim_{n \to \infty} P\left\{ \frac{\zeta_{n+m} - A_{n+m}}{B_{n+m}} \leq x \right\} = R(x) \tag{131}$$

in every continuity point of $R(x)$. Now by Lemma 2 we can conclude that there exist two constants a and $b > 0$ such that the distribution function of the right-hand side of (129) converges weakly to

$$R(a+bx) \tag{132}$$

as $n \to \infty$. This proves (126).

We can easily prove that the condition of the theorem is sufficient too. Let $R(x)$ be a stable distribution function of type $S(\alpha,\beta,c,m)$ defined by (42.97). If we suppose that $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are mutually independent random variables having the same distribution function $R(x)$, and

$$A_n = \begin{cases} m(n-n^{1/\alpha}) & \text{for } \alpha \neq 1, \\ \dfrac{2c\beta n \log n}{\pi} & \text{for } \alpha = 1, \end{cases} \tag{133}$$

and

(134) $$B_n = n^{1/\alpha} ,$$

then by (42.101) we have

(135) $$\underset{\sim}{P}\{\frac{\xi_1 + \xi_2 + \ldots + \xi_n - A_n}{B_n} \leqq x\} = R(x)$$

for every $n = 1,2,\ldots$. Thus if $n \to \infty$ in (135) then $R(x)$ appears also as a limiting distribution. This completes the proof of the theorem.

Let $F(x)$ be a distribution function and denote by $F_n(x)$ the n-th iterated convolution of $F(x)$ with itself.

We say that the distribution function $F(x)$ belongs to the domain of attraction of a distribution function $R(x)$ if and only if there exist constants A_n $(n = 1,2,\ldots)$ and $B_n > 0$ $(n = 1,2,\ldots)$ such that

(136) $$\lim_{n \to \infty} F_n(A_n + B_n x) = R(x)$$

in every continuity point of $R(x)$.

If $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent and identically distributed random variables for which $\underset{\sim}{P}\{\xi_k \leqq x\} = F(x)$ and $\zeta_n = \xi_1 + \ldots + \xi_n$ for $n = 1,2,\ldots$, then (136) is equivalent to the requirement that

(137) $$\lim_{n \to \infty} \underset{\sim}{P}\{\frac{\zeta_n - A_n}{B_n} \leqq x\} = R(x)$$

in every continuity point of $R(x)$.

It follows from Theorem 4 that all stable distribution functions and only those have a domain of attraction.

Our next aim is to find necessary and sufficient conditions for $F(x)$ to belong to the domain of attraction of a stable distribution function $R(x)$ and to give a procedure for determining the normalizing constants A_n $(n = 1,2,\dots)$ and $B_n > 0$ $(n = 1,2,\dots)$.

If $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,m)$ defined by (42.97) and

$$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} dR(x) \tag{138}$$

for $\mathrm{Re}(s) = 0$, then by Theorem 42.4 we have

$$\log \psi(s) = -ms - c|s|^{\alpha}\left(1+\beta \frac{s}{|s|} \tan \frac{\alpha\pi}{2}\right) \tag{139}$$

for $\mathrm{Re}(s) = 0$ whenever $0 < \alpha < 1$ or $1 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $c \geq 0$ and m is a real constant, and

$$\log \psi(s) = -ms - c|s|\left(1 - \frac{2\beta s}{\pi|s|} \log|s|\right) \tag{140}$$

for $\mathrm{Re}(s) = 0$ whenever $\alpha = 1$, $-1 \leq \beta \leq 1$, $c \geq 0$ and m is a real constant.

We can write down also that

$$\log \psi(s) = -\mu s + \frac{\sigma^2 s^2}{2} + c_1 \int_{-\infty}^{-0} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{\alpha dx}{|x|^{\alpha+1}} + c_2 \int_{+0}^{\infty} \left(e^{-sx} - 1 + \frac{sx}{1+x^2}\right) \frac{\alpha dx}{x^{\alpha+1}} \tag{141}$$

for $\operatorname{Re}(s) = 0$ where $c_1 \geqq 0$, $c_2 \geqq 0$, $\sigma^2 \geqq 0$ and μ is a real number, and $\sigma^2 = 0$ if $c_1 + c_2 > 0$.

If in (141) $\sigma^2 \geqq 0$ and $c_1 = c_2 = 0$, then we obtain (139) with $\alpha = 2$, $c = \sigma^2/2$ and $m = \mu$.

If in (141) $\sigma^2 = 0$, and $c_1 + c_2 > 0$, $c_1 \geqq 0$, $c_2 \geqq 0$, then we obtain (139) with $0 < \alpha < 1$ and $1 < \alpha < 2$, and (140) with $\alpha = 1$ and with the following parameters

(142) $$\beta = \frac{c_2 - c_1}{c_2 + c_1} \quad \text{for} \quad 0 < \alpha < 2,$$

(143) $$c = \frac{(c_1 + c_2)\pi}{2\Gamma(\alpha)\sin\frac{\alpha\pi}{2}} \quad \text{for} \quad 0 < \alpha < 2,$$

where, in particular, $c = (c_1 + c_2)\frac{\pi}{2}$ for $\alpha = 1$, and

(144) $$m = \begin{cases} \mu - (c_2 - c_1)\dfrac{\alpha\pi}{2\cos\frac{\alpha\pi}{2}} & \text{for } 0 < \alpha < 1 \text{ or } 1 < \alpha < 2, \\ \mu + (c_2 - c_1)(1 - C) & \text{for } \alpha = 1, \end{cases}$$

where $C = 0.577215\ldots$ is Euler's constant.

If we use the centering function

(145) $$\delta(x) = \begin{cases} x & \text{for } |x| < \tau, \\ 0 & \text{for } |x| \geqq \tau, \end{cases}$$

where τ is some positive number, then (141) can be expressed in the following equivalent form

$$(146)\quad \log \psi(s) = -\mu(\tau)s+\frac{\sigma^2 s^2}{2}+c_1\int_{-\infty}^{-0}\left(e^{-sx}-1+s\delta(x)\right)\frac{\alpha dx}{|x|^{\alpha+1}}+c_2\int_{+0}^{\infty}\left(e^{-sx}-1+s\delta(x)\right)\frac{\alpha dx}{x^{\alpha+1}}$$

for $\operatorname{Re}(s)=0$ where

$$(147)\qquad \mu(\tau)=\begin{cases}\mu+(c_2-c_1)\alpha\left[\dfrac{\tau^{1-\alpha}}{1-\alpha}-\dfrac{\pi}{2\cos\frac{\alpha\pi}{2}}\right] & \text{for } 0<\alpha<1 \text{ or } 1<\alpha<2\,,\\ \mu+(c_2-c_1)\log\tau \quad \text{for } \alpha=1\,. \end{cases}$$

By (144) and (147) we can express the relation between m and $\mu(\tau)$ as follows:

$$(148)\qquad \mu(\tau)=\begin{cases} m+(c_2-c_1)\alpha\,\dfrac{\tau^{1-\alpha}}{1-\alpha} & \text{if } 0<\alpha<1 \text{ or } 1<\alpha<2\,,\\ m+(c_2-c_1)[\log\tau-(1-C)] & \text{if } \alpha=1\,,\end{cases}$$

where $C = 0.577215\ldots$ is Euler's constant.

To prove (147) we compare (141) and (146). Then we obtain that

$$(149)\qquad \mu(\tau)=\mu+(c_2-c_1)\alpha\, I(\alpha,\tau)$$

where

$$(150)\qquad I(\alpha,\tau)=\int_0^\tau\frac{x^{2-\alpha}}{1+x^2}\,dx-\int_\tau^\infty\frac{x^{-\alpha}}{1+x^2}\,dx\,.$$

If $0<\alpha<1$, then

$$(151) \qquad I(\alpha,\tau) = \int_0^\tau x^{-\alpha}dx - \int_0^\infty \frac{x^{-\alpha}}{1+x^2}\,dx = \frac{\tau^{1-\alpha}}{1-\alpha} - \frac{\pi}{2\cos\frac{\alpha\pi}{2}} .$$

If $1 < \alpha < 2$, then

$$(152) \qquad I(\alpha,\tau) = \int_0^\infty \frac{x^{2-\alpha}}{1+x^2}\,dx - \int_\tau^\infty x^{-\alpha}dx = -\frac{\pi}{2\cos\frac{\alpha\pi}{2}} + \frac{\tau^{1-\alpha}}{1-\alpha} .$$

If $\alpha = 1$, then

$$(153) \qquad I(1,\tau) = \int_0^\tau \frac{x}{1+x^2}\,dx - \int_\tau^\infty \left[\frac{1}{x} - \frac{x}{1+x^2}\right]dx = \log\tau .$$

Thus (147) follows.

Now let us suppose that $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,m)$. If $c = 0$, then $R(x)$ is degenerate. If $c > 0$, then $R(x)$ is nondegenerate. Let $F(x)$ be a distribution function and denote by $F_n(x)$ the n-th iterated convolution of $F(x)$ with itself.

Theorem 5. *Let $R(x)$ be a nondegenerate stable distribution function of type $S(\alpha,\beta,c,m)$, $F(x)$ a distribution function and A_n $(n = 1,2,\ldots)$ and $B_n > 0$ $(n = 1,2,\ldots)$ constants. We have*

$$(154) \qquad \lim_{n\to\infty} F_n(A_n + B_n x) = R(x)$$

in every continuity point of $R(x)$ if and only if

$$(155) \qquad \lim_{n\to\infty} nF(B_n x) = \frac{c_1}{|x|^\alpha}$$

for $x < 0$,

(156) $$\lim_{n \to \infty} n\,[1-F(B_n x)] = \frac{c_2}{x^\alpha}$$

for $x > 0$,

(157) $$\lim_{n \to \infty} \frac{1}{B_n}\,[n \int_{|x|<\tau B_n} x dF(x) - A_n] = \mu(\tau)$$

and

(158) $$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{n}{B_n^2}\,[\int_{|x|<\varepsilon B_n} x^2 dF(x) - (\int_{|x|<\varepsilon B_n} x dF(x))^2] =$$

$$= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{n}{B_n^2}\,[\int_{|x|<\varepsilon B_n} x^2 dF(x) - (\int_{|x|<\varepsilon B_n} x dF(x))^2] = \sigma^2$$

where $c_1 = c_2 = 0$ *for* $\alpha = 2$, *and* c_1 *and* c_2 *are determined by* (142) *and* (143) *for* $0 < \alpha < 2$, $\sigma^2 = 0$ *if* $c_1 + c_2 > 0$ *and* $0 < \alpha < 2$, *and* $\sigma^2 = 2c$ *if* $c_1 = c_2 = 0$ *and* $\alpha = 2$, τ *is an arbitrary positive number*, *and* $\mu(\tau)$ *is given by* (148) *for* $0 < \alpha < 2$ *and* $\mu(\tau) = m$ *for* $\alpha = 2$.

Proof. Let

(159) $$\phi(s) = \int_{-\infty}^{\infty} e^{-sx} dF(x)$$

and

(160) $$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} dR(x)$$

for $\mathrm{Re}(s) = 0$.

By Lemma 3 we have $\lim_{n \to \infty} B_n = \infty$ and

(161) $$\lim_{n \to \infty} \phi(\frac{s}{B_n}) = 1$$

for all $\operatorname{Re}(s) = 0$.

By Theorem 41.9 and Theorem 41.10 we can conclude that (154) holds if and only if

(162) $$\lim_{n \to \infty} [\phi(\frac{s}{B_n})]^n e^{sA_n/B_n} = \psi(s)$$

for $\operatorname{Re}(s) = 0$. In (162) the convergence is necessarily uniform on $\operatorname{Re}(s) = 0$. Now let us define

(163) $$a_n = \int_{|x|<\tau} x dF(B_n x)$$

for some positive real τ , and let us write

(164) $$\psi_n(s) = \phi(\frac{s}{B_n})e^{sa_n}$$

for $\operatorname{Re}(s) = 0$ and $n = 1,2,\dots$. Then (162) can be expressed as follows

(165) $$\lim_{n \to \infty} [\psi_n(s)]^n e^{-sna_n + sA_n/B_n} = \psi(s) .$$

The distribution function $R(x)$ is stable and therefore it is necessarily infinitely divisible. Thus $\psi(s) \neq 0$ for $\operatorname{Re}(s) = 0$ and we can define $\log \psi(s)$ uniquely as a continuous function of s on $\operatorname{Re}(s) = 0$ for which $\log \psi(0) = 0$. By (161) it follows that $\psi_n(s) \neq 0$ for $\operatorname{Re}(s) = 0$ if n is sufficiently large. Thus $\log \psi_n(s)$ is uniquely determined for $\operatorname{Re}(s) = 0$ and for sufficiently large n values if we define it in a similar way as $\log \psi(s)$. Thus (165) can be expressed as follows

(166) $$\lim_{n \to \infty} \{n \log \psi_n(s) - sna_n + \frac{sA_n}{B_n}\} = \log \psi(s)$$

for $\operatorname{Re}(s) = 0$ and the convergence is uniform in any finite interval of $\operatorname{Re}(s) = 0$. Accordingly, (154) holds if and only if (166) holds for $\operatorname{Re}(s) = 0$.

First we shall deduce necessary conditions for the validity of (166) and then we shall prove that the conditions are sufficient too.

Let us suppose that (166) holds. Then we can prove that

(167) $$\lim_{n \to \infty} n\{\log \psi_n(s) - [\psi_n(s)-1]\} = 0$$

for $\operatorname{Re}(s) = 0$.

Since

(168) $$|\log(1+u)-u| = \left|\sum_{k=2}^{\infty} \frac{u^k}{k}\right| \leqq \frac{1}{2} \sum_{k=2}^{\infty} |u|^k < |u|^2$$

if $|u| < \frac{1}{2}$, it follows from (161) that

(169) $$|\log \psi_n(s) - [\psi_n(s)-1]| \leqq |\psi_n(s)-1|^2$$

for $\operatorname{Re}(s) = 0$ if n is sufficiently large. By (161) we have

(170) $$\lim_{n \to \infty} |\psi_n(s)-1| = 0$$

for $\operatorname{Re}(s) = 0$ and by Theorem 3 we have

$$(171) \qquad |\psi_n(s)-1| \leq \frac{16(1+2\tau)^6(1+|s|\tau)^2}{(\tau-|a_n|)^2(\tau-|m_n|)^2} \int_0^2 \log \frac{1}{|\psi_n(iu)|}\, du$$

for $Re(s) = 0$ if n is sufficiently large and $\tau > |m_n|$ where m_n is a median of the distribution function $F(B_n x)$. If $n \to \infty$, then $m_n \to 0$. For $m_n = m^*/B_n$ where m^* is a median of $F(x)$. Thus (171) holds for any $\tau > 0$ if n is sufficiently large.

If (166) holds and if we form its real part, then we obtain that

$$(172) \qquad \lim_{n\to\infty} n \log|\psi_n(s)| = \log|\psi(s)|$$

for $Re(s) = 0$, and the convergence is uniform for $s = iu$ where $0 \leq u \leq 2$. Thus it follows from (172) that if we multiply (171) by n and let $n \to \infty$, then the right-hand side has a finite limit. This fact together with (170) proves that if (166) holds then

$$(173) \qquad \lim_{n\to\infty} n|\psi_n(s)-1|^2 = 0$$

for $Re(s) = 0$. Finally (167) follows from (169) and (173).

Accordingly, if (166) holds, then (167) holds too, and this implies that

$$(174) \qquad \lim_{n\to\infty} \{n[\psi_n(s)-1] - sna_n + \frac{sA_n}{B_n}\} = \log \psi(s)$$

for $Re(s) = 0$.

We recognize that

$$(175)\quad n[\psi_n(s)-1] - sna_n + \frac{sA_n}{B_n} = n\int_{-\infty}^{\infty}(e^{-sx}-1)dF(B_nx+B_na_n) - sna_n + \frac{sA_n}{B_n}$$

is the logarithm of the Laplace-Stieltjes transform of an infinitely divisible distribution function. If we define

$$(176)\qquad \delta(x) = \begin{cases} x & \text{for } |x| < \tau, \\ 0 & \text{for } |x| \geqq \tau, \end{cases}$$

then (175) can also be expressed as

$$(177)\quad n\int_{-\infty}^{\infty}\left(e^{-sx}-1+s\delta(x)\right)dF(B_nx+B_na_n) + \frac{sA_n}{B_n} - sna_n - sn\int_{-\tau}^{\tau}x\,dF(B_nx+B_na_n) .$$

If $n \to \infty$ then by (174) the expression (177) tends to $\log \psi(s)$ for $\mathrm{Re}(s) = 0$. Let us use the representation (146) for $\log \psi(s)$. Then by Theorem 2 we can conclude that (177) converges to $\log \psi(s)$ for $\mathrm{Re}(s) = 0$ if and only if

$$(178)\qquad \lim_{n\to\infty} nF(B_nx + B_na_n) = \frac{c_1}{|x|^\alpha}$$

for $x < 0$,

$$(179)\qquad \lim_{n\to\infty} n[1-F(B_nx + B_na_n)] = \frac{c_2}{x^\alpha}$$

for $x > 0$,

$$(180) \qquad \lim_{n\to\infty} \Big[n \int_{|x|<\tau} x\,dF(B_n x + B_n a_n) + n a_n - \frac{A_n}{B_n} \Big] = \mu(\tau)$$

and

$$(181) \qquad \lim_{\varepsilon\to 0} \limsup_{n\to\infty} n \int_{|x|<\varepsilon} x^2 dF(B_n x + B_n a_n) =$$

$$= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} n \int_{|x|<\varepsilon} x^2 dF(B_n x + B_n a_n) = \sigma^2 .$$

Accordingly, (178), (179), (180) and (181) are necessary conditions for (154). We can easily prove that they are sufficient too. By Theorem 2 it follows that (178), (179), (180) and (181) imply (174). Now we shall prove that (173) holds in this case too. If we apply Theorem 1, then by (11) it follows from (174) that

$$(182) \qquad \lim_{n\to\infty} n \int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\, dF(B_n x + B_n a_n)$$

exists and is finite. By the inequality (78) we have

$$(183) \qquad |\psi_n(s)-1| \leqq \frac{(1+4\tau^2)(1+|s|\tau)^2}{2(\tau-|a_n|)^2} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\, dF(B_n x + B_n a_n)$$

for $\mathrm{Re}(s) = 0$. By (170), (182) and (183) it follows (173), and thus by (169) we obtain (167). By (174) and (167) we obtain (166) which further implies (154).

Thus we have proved that the conditions (178), (179), (180) and (181) are necessary and sufficient for the validity of (154).

Finally, we shall prove that the two sets of conditions (155), (156), (157), (158) and (178), (179), (180), and (181) are equivalent. First we observe that by (163)

$$(184)\qquad |a_n| \leqq \varepsilon \int_{|x|\leqq\varepsilon} dF(B_n x) + \tau \int_{|x|>\varepsilon} dF(B_n x) \leqq \varepsilon + \tau \int_{|x|>\varepsilon} dF(B_n x)$$

for any $\varepsilon > 0$. Since by (161)

$$(185)\qquad \lim_{n\to\infty} F(B_n x) = \begin{cases} 1 & \text{if } x > 0\,, \\ 0 & \text{if } x < 0\,, \end{cases}$$

it follows from (184) that

$$(186)\qquad \lim_{n\to\infty} a_n = 0\,.$$

This immediately implies that (155) and (178) are equivalent, and furthermore that (156) and (179) are equivalent. Now we shall prove that

$$(187)\qquad \lim_{n\to\infty} n \int_{|x|<\tau} x dF(B_n x + B_n a_n) = \lim_{n\to\infty} n \int_{|x-a_n|<\tau} (x-a_n) dF(B_n x) = 0$$

and this implies that (157) and (180) are equivalent. Since by the definition (163) we have

$$(188)\qquad \int_{|x|<\tau} (x-a_n) dF(B_n x) = a_n \int_{|x|\geqq\tau} dF(B_n x)\,,$$

it follows that

$$(189)\qquad \begin{aligned} \left|\int_{|x-a_n|<\tau} (x-a_n) dF(B_n x)\right| &\leqq \left|\int_{|x|<\tau} (x-a_n) dF(B_n x)\right| + \int_{\tau-|a_n|<|x|<\tau+|a_n|} |x-a_n| dF(B_n x) \leqq \\ &\leqq |a_n| \int_{|x|\geqq\tau} dF(B_n x) + (\tau+a_n) \int_{\tau-|a_n|<|x|<\tau+|a_n|} dF(B_n x)\,. \end{aligned}$$

If we multiply (189) by n and let $n \to \infty$, then by (155) and (156) we have

$$\lim_{n \to \infty} n \int_{|x| \geq \tau} dF(B_n x) = \frac{(c_1 + c_2)}{\tau^{\alpha}} \tag{190}$$

and since $\lim_{n \to \infty} a_n = 0$ also by (155) and (156) it follows that

$$\lim_{n \to \infty} n \int_{\tau - |a_n| < |x| < \tau + |a_n|} dF(B_n x) = 0 . \tag{191}$$

In (189) $|a_n| \to 0$ and $(\tau + |a_n|) \to \tau$ as $n \to \infty$, and thus (187) follows.

It remains to prove that (158) and (181) are equivalent too.

First, we observe that

$$\begin{aligned} &\left| \int_{|x|<\varepsilon} x^2 dF(B_n x + B_n a_n) - \int_{|x|<\varepsilon} (x-a_n)^2 dF(B_n x) \right| = \\ &= \left| \int_{|x-a_n|<\varepsilon} (x-a_n)^2 dF(B_n x) - \int_{|x|<\varepsilon} (x-a_n)^2 dF(B_n x) \right| \leq \\ &\leq (\varepsilon + |a_n|)^2 \int_{\varepsilon - |a_n| \leq |x| \leq \varepsilon + |a_n|} dF(B_n x) . \end{aligned} \tag{192}$$

Next, we observe that if $0 < \varepsilon < \tau$, then

$$\begin{aligned} &\int_{|x|<\varepsilon} (x-a_n)^2 dF(B_n x) - \left[\int_{|x|<\varepsilon} x^2 dF(B_n x) - \left(\int_{|x|<\varepsilon} x dF(B_n x) \right)^2 \right] = \\ &= \left(a_n - \int_{|x|<\varepsilon} x dF(B_n x) \right)^2 - a_n^2 \int_{|x| \geq \varepsilon} dF(B_n x) = \left(\int_{\varepsilon \leq |x| < \tau} x dF(B_n x) \right)^2 - a_n^2 \int_{|x| \geq \varepsilon} dF(B_n x) \end{aligned} \tag{193}$$

and henceforth

$$(194) \qquad n\left|\int_{|x|<\varepsilon}(x-a_n)^2dF(B_nx) - \left[\int_{|x|<\varepsilon}x^2dF(B_nx) - \left(\int_{|x|<\varepsilon}xdF(B_nx)\right)^2\right]\right| \leq$$

$$\leq \left(\tau^2\int_{|x|\geq\varepsilon}dF(B_nx) + a_n^2\right)n\int_{|x|\geq\varepsilon}dF(B_nx) .$$

If we multiply (192) by n and let $n \to \infty$ then the extreme right member tends to 0 by (155) and (156). If in (194) we let $n \to \infty$, then the first factor on the right-hand side tends to 0, and the integral multiplied by n tends to $(c_1+c_2)/\varepsilon^\alpha$ by (155) and (156). Thus

$$(195) \qquad \lim_{n\to\infty}\left[n\int_{|x|<\varepsilon}x^2dF(B_nx+B_na_n) - n\int_{|x|<\varepsilon}(x-a_n)^2dF(B_nx)\right] = 0$$

for any $\varepsilon > 0$ and

$$(196) \qquad \lim_{n\to\infty}\left\{n\int_{|x|<\varepsilon}(x-a_n)^2dF(B_nx) - n\left[\int_{|x|<\varepsilon}x^2dF(B_nx) - \int_{|x|<\varepsilon}xdF(B_nx)\right]\right\} = 0$$

for $0 < \varepsilon < \tau$.

By (181), (195) and (196) we can conclude that (181) is equivalent to the following relation

$$(197) \qquad \begin{aligned} &\lim_{\varepsilon\to 0}\limsup_{n\to\infty} n\left[\int_{|x|<\varepsilon}x^2dF(B_nx) - \left(\int_{|x|<\varepsilon}xdF(B_nx)\right)^2\right] = \\ &= \lim_{\varepsilon\to 0}\liminf_{n\to\infty} n\left[\int_{|x|<\varepsilon}x^2dF(B_nx) - \left(\int_{|x|<\varepsilon}xdF(B_nx)\right)^2\right] = \sigma^2 , \end{aligned}$$

which is the same as (159). This completes the proof of the theorem.

Theorem 5 makes it possible to find necessary and sufficient conditions for $F(x)$ to belong to the domain of attraction of a nondegenerate stable distribution function $R(x)$. First we shall consider the case when $R(x)$ is the normal distribution function defined by

(198) $$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

for $-\infty < x < \infty$.

The following theorem was found in 1935 by A. Ya. Khintchine [545], W. Feller [517], [518] and P. Lévy [560].

Theorem 6. The distribution function $F(x)$ belongs to the domain of attraction of a nondegenerate normal distribution function $R(x)$ if and only if $F(x)$ is nondegenerate and

(199) $$\lim_{x\to\infty} \frac{x^2 \int_{|u|\geq x} dF(u)}{\int_{|u|<x} u^2 dF(u)} = 0 .$$

If (199) is satisfied, then

(200) $$\lim_{n\to\infty} F_n(A_n + B_n x) = \Phi(x)$$

defined by (198) if we choose A_n $(n = 1,2,\ldots)$ in such a way that

(201) $$A_n = n\int_{-\infty}^{\infty} x dF(x)$$

for $n = 1,2,\ldots,$ and if we choose B_n $(n = 1,2,\ldots)$ in such a way that

(202) $$B_n^2 = n\left[\int_{-\infty}^{\infty} x^2 dF(x) - \left(\int_{-\infty}^{\infty} x dF(x)\right)^2\right]$$

for $n = 1,2,\ldots$ whenever

(203) $$\int_{-\infty}^{\infty} x^2 dF(x) < \infty ,$$

<u>and if we choose</u> B_n $(n = 1,2,\ldots)$ <u>in such a way that</u> $B_n > 0$ <u>for</u> $n = 1,2,\ldots,$ $\lim_{n \to \infty} B_n = \infty$ <u>and</u>

(204) $$\lim_{n \to \infty} \frac{n}{B_n^2} \int_{|x|<\varepsilon B_n} x^2 dF(x) = 1$$

<u>for some</u> $\varepsilon > 0$ <u>whenever</u>

(205) $$\int_{-\infty}^{\infty} x^2 dF(x) = \infty .$$

<u>Proof</u>. By Theorem 5 it follows that (200) holds if and only if (155), (156), (157) and (158) are satisfied with $c_1 = 0$, $c_2 = 0$, $\mu(\tau) = 0$ for $\tau > 0$ and $\sigma^2 = 1$. Using this result we can easily prove that (200) holds if and only if for every $\varepsilon > 0$ we have

(206) $$\lim_{n \to \infty} n \int_{|x|>\varepsilon B_n} dF(x) = 0 ,$$

(207) $$\lim_{n \to \infty} \frac{1}{B_n} [n \int_{|x|<\varepsilon B_n} x dF(x) - A_n] = 0 ,$$

and

(208) $$\lim_{n \to \infty} \frac{n}{B_n^2} [\int_{|x|<\varepsilon B_n} x^2 dF(x) - (\int_{|x|<\varepsilon B_n} x dF(x))^2] = 1 .$$

If $c_1 = 0$, $c_2 = 0$, $\mu(\tau) = 0$ for $\tau > 0$ and $\sigma^2 = 1$, then we can immediately see that (155) and (156) imply (206), and (157) implies (207). It remains to show that (158) implies (208). Let

$$(209)\qquad I_n(\varepsilon) = n\Big[\int_{|x|<\varepsilon} x^2 dF(B_n x) - \Big(\int_{|x|<\varepsilon} x dF(B_n x)\Big)^2\Big]$$

for $\varepsilon > 0$. For $0 < \varepsilon' < \varepsilon$ we have

$$(210)\qquad \begin{aligned} I_n(\varepsilon) - I_n(\varepsilon') &= n\Big[\int_{\varepsilon' \leq |x| < \varepsilon} x^2 dF(B_n x) - \Big(\int_{\varepsilon' \leq |x| < \varepsilon} x dF(B_n x)\Big)^2\Big] - \\ &- 2n\Big(\int_{|x|<\varepsilon'} x dF(B_n x)\Big)\Big(\int_{\varepsilon' \leq |x| < \varepsilon} x dF(B_n x)\Big) . \end{aligned}$$

Hence

$$(211)\qquad |I_n(\varepsilon) - I_n(\varepsilon')| \leq n\varepsilon^2 \int_{|x| \geq \varepsilon'} dF(B_n x) + 2n\varepsilon'\varepsilon \int_{|x| \geq \varepsilon'} dF(B_n x) ,$$

and by (206) it follows that

$$(212)\qquad \lim_{n \to \infty} [I_n(\varepsilon) - I_n(\varepsilon')] = 0$$

for $0 < \varepsilon' < \varepsilon$. Now (158) and (212) imply that $\limsup_{n \to \infty} I_n(\varepsilon) = 1$ and $\liminf_{n \to \infty} I_n(\varepsilon) = 1$. For by (212) these limits are independent of ε and by (158) they tend to 1 as $\varepsilon \to 0$. Thus it follows that $\lim_{n \to \infty} I_n(\varepsilon) = 1$ for all $\varepsilon > 0$ and this proves (208).

Conversely, the conditions (206), (207) and (208) evidently imply (155), (156), (157) and (158) with $c_1 = 0$, $c_2 = 0$, $\mu(\tau) = 0$ for $\tau > 0$ and $\sigma^2 = 1$.

Before proving the theorem we shall deduce some relations which we shall need in what follows. Let us introduce the abbreviation

$$(213)\qquad h(x) = \int_{|u|<x} u^2 dF(u)$$

for $x > 0$.

First, we observe that if we exclude the case when $F(x) = 1$ for $x \geq 0$ and $F(x) = 0$ for $x < 0$, then (199) implies that

$$\lim_{x \to \infty} \frac{h(\rho x)}{h(x)} = 1 \tag{214}$$

for any $0 < \rho < \infty$.

We note that if (203) is satisfied, then (214) is obvious. Now we shall prove (214) for $1 \leq \rho < \infty$. From this case it follows immediately that (214) holds for $0 < \rho \leq 1$ too. By (199) it follows that for any $\varepsilon > 0$

$$0 \leq x^2 \int_{|u| \geq x} dF(u) < \varepsilon\, h(x) \tag{215}$$

if x is sufficiently large. Thus if $1 \leq \rho < \infty$, then

$$0 \leq h(\rho x) - h(x) = \int_{x \leq |u| < \rho x} u^2 dF(u) \leq \rho^2 x^2 \int_{|u| \geq x} dF(u) < \varepsilon \rho^2 h(x) \tag{216}$$

if x is sufficiently large. Since $\varepsilon > 0$ is arbitrary, (216) implies (214).

Next, we observe that if we exclude the case when $F(x) = 1$ for $x \geq 0$ and $F(x) = 0$ for $x < 0$, then (199) implies that

$$\lim_{x \to \infty} \frac{x \int_{|u| \geq x} |u|\, dF(u)}{h(x)} = 0 . \tag{217}$$

To prove (217) we note that

$$\int_{|u| \geqq x} |u| dF(u) = \int_x^\infty [1-F(u)+F(-u)]du + x[1-F(x)+F(-x)] \tag{218}$$

for $x \geqq 0$. If (199) holds, then for any $\varepsilon > 0$ (215) is satisfied if x is large enough, and in this case by (218) we obtain that

$$\int_{|u| \geqq x} |u| dF(u) \leqq \varepsilon \int_x^\infty \frac{h(u)}{u^2} du + \varepsilon \frac{h(x)}{x} \leqq \varepsilon \int_{|u| \geqq x} |u| dF(u) + 2\varepsilon \frac{h(x)}{x} \tag{219}$$

for sufficiently large x. Here we used that

$$\int_x^\infty \frac{h(u)}{u^2} du \leqq \frac{h(x)}{x} + \int_{|u| \geqq x} |u| dF(u) \tag{220}$$

for $x > 0$ which follows by integrating by parts. By (219) it follows that if $0 < \varepsilon < 1$ and if x is sufficiently large, then

$$x \int_{|u| \geqq x} |u| dF(u) \leqq \frac{2\varepsilon}{1-\varepsilon} h(x) . \tag{221}$$

Since $0 < \varepsilon < 1$ is arbitrary, this proves (217).

Finally, we observe that if

$$\int_{-\infty}^\infty x^2 dF(x) = \infty , \tag{222}$$

then

$$\lim_{x \to \infty} \frac{\left(\int_{|u|<x} u dF(u) \right)^2}{\int_{|u|<x} u^2 dF(u)} = 0 . \tag{223}$$

To prove (223) let $0 < c < x$. Then we can write that

$$(224)\qquad \Big(\int_{|u|<x} u\,dF(u) - \int_{|u|<c} u\,dF(u)\Big)^2 \leqq \Big(\int_{c\leqq|u|<x} |u|\,dF(u)\Big)^2 \leqq \Big(\int_{|u|<x} u^2 dF(u)\Big)\Big(\int_{|u|\geqq c} dF(u)\Big).$$

Hence it follows that

$$(225)\qquad 0 \leqq \limsup_{x\to\infty} \frac{\Big(\int_{|u|<x} u\,dF(u)\Big)^2}{\int_{|u|<x} u^2 dF(u)} \leqq \int_{|u|>c} dF(u) \to 0 \quad \text{as} \quad c \to \infty .$$

Now we shall prove first that if $F(x)$ is a nondegenerate distribution function for which (199) holds and if we choose A_n $(n = 1,2,\ldots)$ and B_n $(n = 1,2,\ldots)$ according to (201) and (202) or (204), respectively then (206), (207) and (208) are satisfied.

We shall consider the two cases (203) and (205) separately.

First, let us suppose that $F(x)$ is a nondegenerate distribution function for which (199) and (203) are satisfied. In this case

$$(226)\qquad a = \int_{-\infty}^{\infty} x\,dF(x)$$

exists and

$$(227)\qquad b^2 = \int_{-\infty}^{\infty} x^2 dF(x) - \Big(\int_{-\infty}^{\infty} x\,dF(x)\Big)^2$$

is a finite positive number. We note that in this case (199) is automatically satisfied because by (203)

$$(228)\qquad 0 \leqq x^2 \int_{|u|\geqq x} dF(u) \leqq \int_{|u|\geqq x} u^2 dF(u) \to 0 \quad \text{as} \quad x \to \infty .$$

Now by (201) $A_n = na$ and by (202) $B_n^2 = nb^2$ for $n = 1,2,\ldots$.

We want to show that (206), (207) and (208) are satisfied in this case. Since by (203) we have

$$(229)\quad 0 \leqq n \int_{|x|>\varepsilon B_n} dF(x) \leqq \frac{n}{\varepsilon^2 B_n^2} \int_{|x|>\varepsilon B_n} x^2 dF(x) = \frac{1}{\varepsilon^2 b^2} \int_{|x|>\varepsilon B_n} x^2 dF(x) \to 0 \text{ as } n \to \infty ,$$

it follows that (206) holds for $\varepsilon > 0$. Since by (203) we have

$$(230)\quad \left|\frac{n}{B_n} \int_{|x|\geqq\varepsilon B_n} x dF(x)\right| \leqq \frac{1}{\varepsilon b^2} \int_{|x|\geqq\varepsilon B_n} x^2 dF(x) \to 0 \quad \text{as} \quad n \to \infty ,$$

it follows that (207) is satisfied for $\varepsilon > 0$. Since $B_n^2 = nb^2$, (208) trivially holds.

Second, let us suppose that $F(x)$ satisfies (199) and (205). In this case $F(x)$ is automatically nondegenerate, and it follows from (221) that the expectation (226) exists. Now by (201) $A_n = na$ and let us choose $B_n > 0$ in such a way that $\lim_{n \to \infty} B_n = \infty$ and that (204) is satisfied for some $\varepsilon > 0$. If (204) is satisfied for some $\varepsilon > 0$, then by (214) it follows that (204) is satisfied for every $\varepsilon > 0$. By (204) and (199) we obtain that

$$(231)\quad \lim_{n \to \infty} n \int_{|x|>\varepsilon B_n} dF(x) = \lim_{n \to \infty} \frac{B_n^2}{h(\varepsilon B_n)} \int_{|x|>\varepsilon B_n} dF(x) = 0$$

for $\varepsilon > 0$. This proves (206). By (204) and (217) we obtain that

$$(232)\quad \lim_{n \to \infty} \frac{n}{B_n} \int_{|x|\geqq\varepsilon B_n} x dF(x) = \lim_{n \to \infty} \frac{B_n}{h(\varepsilon B_n)} \int_{|x|\geqq\varepsilon B_n} x dF(x) = 0$$

for $\varepsilon > 0$. This proves (207). Finally by (223) it follows that (204) implies (208). This proves that the conditions of the theorem are sufficient.

Next we shall prove that if (206), (207) and (208) are satisfied for some A_n $(n = 1,2,\ldots)$ and $B_n > 0$ $(n = 1,2,\ldots)$ for which $\lim_{n\to\infty} B_n = \infty$, then $F(x)$ is nondegenerate and (199) holds. Furthermore, we can choose A_n according to (201) and B_n according to (202) or (204).

From (208) it follows immediately that $F(x)$ is nondegenerate. Since $\lim_{n\to\infty} B_n = \infty$, for every sufficiently large positive x and for any given $\varepsilon > 0$ there is an n such that $\varepsilon B_n \leqq x < 2\varepsilon B_n$. If $\varepsilon B_n \leqq x < 2\varepsilon B_n$, then we have

$$(233) \qquad 0 \leqq \frac{x^2 \int_{|u|\geqq x} dF(u)}{\int_{|u|<x} u^2 dF(u)} \leqq \frac{4\varepsilon^2 n \int_{|u|\geqq \varepsilon B_n} dF(u)}{\frac{n}{B_n^2}\left[\int_{|u|<\varepsilon B_n} u^2 dF(u) - \left(\int_{|u|<\varepsilon B_n} u dF(u)\right)^2\right]} .$$

If we suppose that (206) and (208) hold and if we let $x \to \infty$ in (233), then $n \to \infty$ and we obtain that the extreme right member in (233) tends to 0 . This proves that (199) is a necessary condition.

We shall consider again two cases, namely the case of (203) and the case of (205).

If (203) holds and if we choose $A_n = na$ and $B_n = nb^2$ for $n = 1,2,\ldots$ where a and b^2 are defined by (226) and (227) respectively, then (206) is satisfied. This follows from (229). The relation (207) is also satisfied.

This follows from (230). Since $B_n^2 = nb^2$, (208) trivially holds.

Now let us suppose that (205) holds. Then by (223) we can conclude that (208) is equivalent to (204) for all $\varepsilon > 0$. Thus it follows that we can choose B_n $(n = 1,2,\ldots)$ in only one way namely so that (204) is satisfied for some $\varepsilon > 0$. Then by (208) it follows that (204) is necessarily satisfied for all $\varepsilon > 0$. The condition (206) does not impose further restrictions on the choice of B_n. We have already seen that if (206) and (208) hold for some $\varepsilon > 0$, then (199) is satisfied, and furthermore that (199) and (204) imply (206) for every $\varepsilon > 0$. Since (206) and (208) imply (199), and since (199) imply that the expectation of $F(x)$ exists, therefore we can choose $A_n = na$ for $n = 1,2,\ldots$ where a is defined by (226). If we suppose that $A_n = na$ for $n = 1,2,\ldots,$ then (207) is satisfied. This follows from (232). This completes the proof of the theorem.

We note that in the particular case when (203) is satisfied, (200) has been proved in 1887 by P. L. Chebyshev [616] and in 1898 by A. A. Markov [579]. This result is the generalization of some more particular results of A.De Moivre [36] and P. S. Laplace [107].

In the above proof we have already used the fact that if (199) is satisfied, then the expectation of $F(x)$ exists. This is a particular case of the following more general theorem due to A. Ya. Khintchine [545] and H. Cramér [503].

Theorem 7. *Suppose that the distribution function* $F(x)$ *satisfies* (199). *Then*

$$(234) \qquad \int_{-\infty}^{\infty} |x|^{\delta} dF(x) < \infty$$

for $0 \leq \delta < 2$.

Proof. Since

$$(235) \qquad \int_{|x|\geq a} |x|^{\delta} dF(x) = a^{\delta}[1-F(a) + F(-a)] + \delta \int_{a}^{\infty} x^{\delta-1}[1-F(x) + F(-x)]dx$$

for any $a > 0$ and $\delta > 0$, it is sufficient to prove that the last term in (235) is finitefor some $a > 0$ and for $0 < \delta < 2$. If $h(x)$ is defined by (219), then by (199) we have

$$(236) \qquad 0 \leq x^2[1-F(x) + F(-x)] < h(x)$$

for sufficiently large x values. Accordingly (234) is satisfied if

$$(237) \qquad \int_{a}^{\infty} x^{\delta-3} h(x)dx < \infty$$

for some $a > 0$. Let $\rho > 1$. If (199) holds, then by (214) we obtain that for any $\varepsilon > 0$

$$(238) \qquad \frac{h(\rho x)}{h(x)} < 1+\varepsilon$$

if $x \geq a$ and a is sufficiently large. If $0 < \delta < 2$ and if we choose ρ such that $\rho^{2-\delta} > 1+\varepsilon$ and if we choose a such that (238) holds for $x \geq a$, then

$$(239) \qquad \int_{a}^{\infty} x^{\delta-3} h(x)dx = \sum_{k=1}^{\infty} \int_{a\rho^{k-1}}^{a\rho^{k}} x^{\delta-3} h(x)dx < \sum_{k=1}^{\infty} h(a\rho^{k}) \int_{a\rho^{k-1}}^{\infty} x^{\delta-3}dx \leq$$

$$\leq \frac{h(a)}{(2-\delta)} \left(\frac{\rho}{a}\right)^{2-\delta} \sum_{k=1}^{\infty} \left(\frac{1+\varepsilon}{\rho^{2-\delta}}\right)^{k} < \infty .$$

This proves (234).

Our next aim is to give necessary and sufficient conditions for $F(x)$ to belong to the domain of attraction of a nondegenerate and non-normal stable distribution function $R(x)$.

If $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,m)$ defined by (42.97), then $R(x)$ is nondegenerate $(c > 0)$ and nonnormal $(0 < \alpha < 2)$ if and only if $0 < \alpha < 2$, $-1 \leqq \beta \leqq 1$, $c > 0$ and $-\infty < m < \infty$.

If in this case

$$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} dR(x) \tag{240}$$

for $\mathrm{Re}(s) = 0$, then

$$\log \psi(s) = -ms - c|s|^{\alpha}\left(1 + \beta \frac{s}{|s|} \tan \frac{\alpha\pi}{2}\right) \tag{241}$$

for $\mathrm{Re}(s) = 0$, $0 < \alpha < 1$ or $1 < \alpha < 2$, $-1 \leqq \beta \leqq 1$, $c > 0$ and $-\infty < m < \infty$, and

$$\log \psi(s) = -ms - c|s|\left(1 - \frac{2\beta s}{\pi|s|} \log|s|\right) \tag{242}$$

for $\mathrm{Re}(s) = 0$, $\alpha = 1$, $-1 \leqq \beta \leqq 1$, $c > 0$ and $-\infty < m < \infty$.

For any $\tau > 0$ let us write

$$\mu(\tau) = \begin{cases} m + \dfrac{2\beta c\alpha\tau^{1-\alpha}}{\pi(1-\alpha)} \Gamma(\alpha) \sin \dfrac{\alpha\pi}{2} & \text{if } 0 < \alpha < 1 \text{ or } 1 < \alpha < 2, \\[2ex] m + \dfrac{2\beta c}{\pi}[\log \tau - (1-C)] & \text{if } \alpha = 1 \end{cases} \tag{243}$$

where $C = 0.577215 \ldots$ is Euler's constant.

Let $F(x)$ be a distribution function and denote by $F_n(x)$ the n-th iterated convolution of $F(x)$ with itself.

The following theorem was found in 1938 by W. Doeblin [508], [510] and in 1939 B. V. Gnedenko [529], [530] deduced it as a particular case of his more general theorem ([527], [528]).

Theorem 8. *Let $R(x)$ be a stable distribution function of type $S(\alpha,\beta,c,m)$ where $0 < \alpha < 2$, $-1 \leq \beta \leq 1$, $c > 0$ and $-\infty < m < \infty$. The distribution function $F(x)$ belongs to the domain of attraction of $R(x)$ if and only if $1-F(x) + F(-x) > 0$ for all $x \geq 0$,*

$$\lim_{x \to \infty} \frac{F(-x)}{1-F(x)} = \frac{1-\beta}{1+\beta}, \tag{244}$$

and

$$\lim_{x \to \infty} \frac{1-F(x) + F(-x)}{1-F(\rho x) + F(-\rho x)} = \rho^{\alpha} \tag{245}$$

for $0 < \rho < \infty$.

Let A_n $(n = 1,2,\ldots)$ and $B_n > 0$ $(n = 1,2,\ldots)$ be real constants. We have

$$\lim_{n \to \infty} F_n(A_n + B_n x) = R(x) \tag{246}$$

if and only if in addition to (244) and (245) the following conditions are satisfied too

$$(247)\quad \lim_{n\to\infty} n[1-F(B_n)+F(-B_n)] = \begin{cases} \frac{2c}{\pi}\Gamma(\alpha)\sin\frac{\alpha\pi}{2} & \underline{\text{when}}\ \ \alpha \neq 1\ , \\ \frac{2c}{\pi} & \underline{\text{when}}\ \ \alpha = 1\ , \end{cases}$$

and

$$(248)\qquad A_n = n\int_{|x|<\tau B_n} x\,dF(x) - \mu(\tau)B_n - \varepsilon_n B_n$$

where τ is some positive number, $\mu(\tau)$ is defined by (243) and $\lim_{n\to\infty}\varepsilon_n = 0$.

Proof. First we observe that if $F(x)$ is nondegenerate and $1-F(x) + F(-x) = 0$ for some $x > 0$, then $F(x)$ is the distribution function of a bounded random variable. In this case the second moment of $F(x)$ is finite and by Theorem 7 $F(x)$ belongs to the domain of attraction of a nondegenerate normal distribution function. Thus $1-F(x) + F(-x) > 0$ for all $x > 0$ is a necessary condition in the theorem.

If $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,m)$ where $0 < \alpha < 2$, $-1 \leq \beta \leq 1$, $c > 0$ and $-\infty < m < \infty$, then by Theorem 5 we can conclude that (246) holds if and only if the following conditions are satisfied:

$$(249)\qquad \lim_{n\to\infty} nF(-B_n x) = \frac{c_1}{x^\alpha}$$

for $x > 0$ and

$$(250)\qquad \lim_{n\to\infty} n\,[1-F(B_n x)] = \frac{c_2}{x^\alpha}$$

for $x > 0$ where c_1 and c_2 are determined by (142) and (143), and furthermore

$$(251) \qquad \lim_{n\to\infty} \frac{1}{B_n} [n \int_{|x|<\tau B_n} x dF(x) - A_n] = \mu(\tau)$$

where τ is some given positive number and $\mu(\tau)$ is defined by (243), and

$$(252) \qquad \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{n}{B_n^2} [\int_{|x|<\varepsilon B_n} x^2 dF(x) - (\int_{|x|<\varepsilon B_n} x dF(x))^2] = 0 .$$

In (249) and (250) $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 > 0$ and the constants c_1 and c_2 can be expressed by α and β by the relations (142) and (143) which are as follows:

$$(253) \qquad \beta = \frac{c_2 - c_1}{c_2 + c_1} \quad \text{for } 0 < \alpha < 2$$

and

$$(254) \qquad c = \frac{(c_1 + c_2)}{2\Gamma(\alpha)\sin\frac{\alpha\pi}{2}} \quad \text{for } 0 < \alpha < 2,$$

where, in particular, $c = \frac{(c_1 + c_2)\pi}{2}$ for $\alpha = 1$.

First we shall prove that the conditions (244) and (245) are necessary and that the constants B_n $(n = 1,2,\ldots)$ and A_n $(n = 1,2,\ldots)$ should be chosen according to (247) and (248) respectively.

Now we suppose that (249), (250), (251) and (252) are satisfied. Since $B_n \to \infty$, for every sufficiently large x there is an n such that $B_n \leqq x < B_{n+1}$. If $x \to \infty$, then $n \to \infty$. If $B_n \leqq x < B_{n+1}$, then

$$(255) \qquad F(-\rho B_{n+1}) \leqq F(-\rho x) \leqq F(-\rho B_n)$$

and

$$(256) \qquad 1-F(\rho B_{n+1}) \leqq 1-F(\rho x) \leqq 1-F(\rho B_n)$$

for any $\rho > 0$, and consequently we have also

$$(257) \qquad \frac{F(-B_{n+1})}{1-F(B_n)} \leqq \frac{F(-x)}{1-F(x)} \leqq \frac{F(-B_n)}{1-F(B_{n+1})}$$

and

$$(258) \qquad \frac{1-F(B_{n+1})+F(-B_{n+1})}{1-F(\rho B_n)+F(-\rho B_n)} \leqq \frac{1-F(x)+F(-x)}{1-F(\rho x)+F(-\rho x)} \leqq \frac{1-F(B_n)+F(-B_n)}{1-F(\rho B_{n+1})+F(-\rho B_{n+1})} .$$

If we let $x \to \infty$ in (257), then by (249) and (250) we obtain that

$$(259) \qquad \lim_{x \to \infty} \frac{F(-x)}{1-F(x)} = \frac{c_1}{c_2}$$

where the right-hand side of (259) is ∞ if $c_2 = 0$. If $c_2 > 0$, then by (253) we obtain (244) from (259). If $c_2 = 0$, then $\beta = -1$, and thus (244) follows in this case too. If we let $x \to \infty$ in (258) and if we take into consideration that $c_1 + c_2 > 0$, then by (249) and (250) we obtain that (245) holds for all $\rho > 0$.

If we put $x = 1$ in (249) and (250) and add the two equations, then we obtain that

$$(260) \qquad \lim_{n \to \infty} n[1-F(B_n) + F(-B_n)] = c_1 + c_2$$

where $c_1 + c_2$ can expressed by (254). This proves (247).

The condition (248) is exactly the same as (251).

This completes the proof of the necessity of the conditions (244), (245), (247) and (248).

Next we shall prove that the conditions (244), (245), (247) and (248) are sufficient too, that is, they imply (249), (250), (251), and (252).

Let

$$G(x) = 1-F(x) + F(-x) \tag{261}$$

defined for $x \geqq 0$. Then $G(x) > 0$ for all $x \geqq 0$, and $G(x)$ is a non-increasing function of x for which $G(0) = 1$ and $\lim_{x \to \infty} G(x) = 0$. For every $n = 1,2,\ldots$ let us choose a $B_n > 0$ such that

$$\lim_{n \to \infty} n\, G(B_n) = \lim_{n \to \infty} n\, [1-F(B_n) + F(-B_n)] = c_1 + c_2 , \tag{262}$$

that is, such that (247) be satisfied. Then $\lim_{n \to \infty} B_n = \infty$.

Now by (245) it follows that

$$\lim_{n \to \infty} n\, [1-F(B_n x) + F(-B_n x)] = \frac{c_1 + c_2}{x^{\alpha}} \tag{263}$$

for $x > 0$. From (244) it follows that

$$\lim_{n \to \infty} \frac{nF(-B_n x)}{n[1-F(B_n x)]} = \frac{c_1}{c_2} \tag{264}$$

for $x > 0$. By (263) and (264) we obtain that both (249) and (250) hold.

If we choose the constants A_n $(n = 1,2,\ldots)$ according to (248), then (251) is satisfied too.

It remains to prove that (252) is satisfied too. First we shall prove that (244) and (245) imply that

$$\int_0^\infty x\,[1-F(x)+F(-x)]dx = \infty \tag{265}$$

and then we shall show that if we choose B_n $(n = 1,2,\ldots)$ according to (247), then

$$\lim_{\varepsilon\to 0}\ \limsup_{n\to\infty} \frac{n}{B_n^2}\int_0^{\varepsilon B_n} x\,[1-F(x)+F(-x)]dx = 0\ . \tag{266}$$

This implies (252) because

$$\frac{n}{B_n^2}\Big[\int_{|x|<\varepsilon B_n} x^2 dF(x) - \Big(\int_{|x|<\varepsilon B_n} x\,dF(x)\Big)^2\Big] \leqq \frac{n}{B_n^2}\int_{|x|<\varepsilon B_n} x^2 dF(x) \leqq \frac{2n}{B_n^2}\int_0^{\varepsilon B_n} x[1-F(x)+F(-x)]dx \tag{267}$$

The last inequality follows from the fact that

$$\begin{aligned}\int_{-a}^{a} x^2 dF(x) &= 2\int_0^a x[F(a)-F(x)+F(-x)-F(-a)]dx = \\ &= 2\int_0^a x[1-F(x)+F(-x)]dx - a^2[1-F(a)+F(-a)]\end{aligned} \tag{268}$$

holds for all $a \geqq 0$.

We shall use the notation (261) and prove that

$$\int_0^\infty x\,G(x)dx = \infty\ . \tag{269}$$

Let $\varepsilon > 0$ and choose ρ so that $\rho^{2-\alpha} > 1+\varepsilon$. By (245) we obtain that

$$(270)\qquad \frac{G(x)}{G(\rho x)} < \rho^{\alpha}(1+\varepsilon)$$

if $x \geqq a > 0$ and a is sufficiently large. Thus if $x \geqq a$, then

$$(271)\qquad \int_x^{\rho x} uG(u)du < \rho^{\alpha}(1+\varepsilon)\int_x^{\rho x} uG(\rho u)du = \frac{1+\varepsilon}{\rho^{2-\alpha}}\int_{\rho x}^{\rho^2 x} vG(v)dv .$$

By applying this inequality repeatedly we get that

$$(272)\qquad \int_0^{\infty} xG(x)dx \geqq \sum_{k=0}^{\infty}\int_{a\rho^k}^{a\rho^{k+1}} xG(x)dx \geqq \int_a^{a\rho} xG(x)dx \sum_{k=0}^{\infty}\left(\frac{\rho^{2-\alpha}}{1+\varepsilon}\right)^k = \infty$$

which proves (265).

Finally, let us prove (266). Let $\varepsilon > 0$, $\rho^{2-\alpha} > 1+\varepsilon$ and $a > 0$ be so large that (270) is satisfied for $x \geqq a$.

Since (269) holds, we have

$$(273)\qquad \int_0^{a} xG(x)dx \leqq \int_a^{\varepsilon B_n} xG(x)dx$$

if n is sufficiently large. Then

$$(274)\qquad \int_0^{\varepsilon B_n} xG(x)dx \leqq 2\int_a^{\varepsilon B_n} xG(x)dx .$$

For each n let us choose an r such that $a\rho^{r-1} \leqq \varepsilon B_n < a\rho^r$. Then by (271) we obtain that

$$(275)\qquad \int_a^{\varepsilon B_n} xG(x)dx \leqq \sum_{s=1}^{r}\int_{a\rho^{s-1}}^{a\rho^s} xG(x)dx \leqq \int_{a\rho^r}^{a\rho^{r+1}} xG(x)dx \sum_{s=1}^{r}\left(\frac{1+\varepsilon}{\rho^{2-\alpha}}\right)^s \leqq$$

$$\leqq \varepsilon B_n\rho^2\, G(\varepsilon B_n)\varepsilon B_n\rho(\rho-1)\left(\frac{1+\varepsilon}{\rho^{2-\alpha}-1-\varepsilon}\right) .$$

Thus by (274) we obtain that if n is sufficiently large, then

$$(276) \qquad \frac{n}{B_n^2}\int_0^{\varepsilon B_n} xG(x)dx \leqq \frac{2\varepsilon^2(1+\varepsilon)\rho^4\, nG(\varepsilon B_n)}{\rho^{2-\alpha}-1-\varepsilon} .$$

Since by (263) $\lim_{n\to\infty} nG(\varepsilon B_n) = (c_1+c_2)\varepsilon^{-\alpha}$, and $0<\alpha<2$, it follows from (276) that

$$(277) \qquad \lim_{\varepsilon\to 0}\ \lim_{n\to\infty}\frac{n}{B_n^2}\int_0^{\varepsilon B_n} xG(x)dx = 0$$

which proves (266). Finally, (252) follows from the inequalities (267). This completes the proof of the theorem.

The following theorem was found by B. V. Gnedenko [775].

Theorem 9. If the distribution function $F(x)$ *satisfies* (245) *for* $0<\rho<\infty$ *and if* $0<\alpha<2$, *then*

$$(278) \qquad \int_{-\infty}^{\infty}|x|^{\delta}dF(x) < \infty$$

for $0 \leqq \delta < \alpha$.

Proof. By (245) for any $\varepsilon>0$ and for any $\rho>1$ there exists a sufficiently large $a>0$ such that

$$(279) \qquad \frac{1-F(x)+F(-x)}{1-F(\rho x)+F(-\rho x)} > \frac{\rho^{\alpha}}{1+\varepsilon}$$

if $x>a$. If $0<\delta<\alpha$ and $\rho^{\alpha-\delta} > 1+\varepsilon$, then we obtain that

$$(280)\quad \begin{aligned}\int_{-\infty}^{\infty}|x|^{\delta}dF(x) &= \int_{|x|\leq a}|x|^{\delta}dF(x) + \sum_{k=0}^{\infty}\int_{a\rho^k<|x|\leq a\rho^{k+1}}|x|^{\delta}dF(x) \leq \\ &\leq \int_{|x|\leq a}|x|^{\delta}dF(x) + \sum_{k=0}^{\infty} a^{\delta}\rho^{(k+1)\delta}[1-F(a\rho^k) + F(-a\rho^k)] \leq \\ &\leq \int_{|x|\leq a}|x|^{\delta}dF(x) + [1-F(a)+F(-a)]a^{\delta}\rho^{\delta}\sum_{k=0}^{\infty}\left(\frac{(1+\varepsilon)\rho^{\delta}}{\rho^{\alpha}}\right)^k < \infty\end{aligned}$$

which proves the theorem.

We observe that if $F(x)$ is a stable distribution function with characteristic exponent α where $0 < \alpha < 2$, then (278) is satisfied, because $F(x)$ belongs to the domain of attraction of itself and thus (245) holds. This proves that (42.199) is indeed true.

<u>Note.</u> If $1 < \alpha < 2$ in Theorem 8, then

$$(281)\qquad \int_{-\infty}^{\infty}|x|dF(x) < \infty$$

and in (248) we can choose

$$(282)\qquad A_n = n\int_{-\infty}^{\infty} x dF(x) - mB_n .$$

For if $\tau \to \infty$, then by (243) $\lim\limits_{\tau\to\infty}\mu(\tau) = m$.

If $0 < \alpha < 1$ in Theorem 8, then in (248) we can choose

$$(283)\qquad A_n = -m\, B_n .$$

For if $\tau \to 0$, then by (243) $\lim\limits_{\tau\to 0}\mu(\tau) = m$.

If $\alpha = 1$ in Theorem 8, then in (248) τ can be chosen as any finite positive number. However, if we suppose that $\tau = e^{1-C}$ where $C = 0.577215...$, then by (243) we have $\mu(\tau) = m$ and by (248) we can choose

$$A_n = n \int_{|x|<(e^{1-C})B_n} x\,dF(x) - m\,B_n \,. \tag{284}$$

Second we observe, that if $F(-x) = 0$ for some $x > 0$, then by Theorem 8 it follows that $F(x)$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(\alpha,\beta,c,m)$ where $0 < \alpha < 2$, $\beta = 1$, $c > 0$ and $-\infty < m < \infty$ if and only if

$$\lim_{x \to \infty} \frac{1-F(x)}{1-F(\rho x)} = \rho^{\alpha} \tag{285}$$

for $0 < \rho < \infty$. For in this case necessarily $\beta = 1$.

Third, we observe that if the limits

$$\lim_{x \to \infty} x^{\alpha} F(-x) = a_1 \tag{286}$$

and

$$\lim_{x \to \infty} x^{\alpha}[1-F(x)] = a_2 \tag{287}$$

exist where $a_1 + a_2 > 0$, and $0 < \alpha < 2$, then the conditions of Theorem 8 are satisfied and $F(x)$ belongs to the domain of attraction of a stable distribution function $R(x)$ of type $S(\alpha,\beta,c,m)$ where now $\beta = (a_2 - a_1)/(a_2 + a_1)$. In this case we can choose

$$B_n = (bn)^{1/\alpha} \tag{288}$$

for $n = 1,2,\ldots$ where $b = (a_1 + a_2)/(c_1 + c_2)$ and $(c_1 + c_2)$ can be obtained by (254).

Fourth, we note that in Theorem 8 the conditions (244), (245), (247), and (248) can also be expressed with the aid of $\phi(s)$, the Laplace-Stieltjes transform of $F(x)$. In this respect we refer to B. V. Gnedenko and V. S. Korolyuk [535], and B. V. Gnedenko [776].

If $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,m)$ and $\psi(s)$ denotes the Laplace-Stieltjes transform of $R(x)$, then $\log\ \psi(s)$ is given by (241) and (242) for $\mathrm{Re}(s) = 0$.

We have already stated in the proof of Theorem 5 that

$$F_n(A_n + B_n x) \Rightarrow R(x) \tag{289}$$

holds if and only if

$$\lim_{n\to\infty} [\phi(\frac{s}{B_n})]^n e^{sA_n/B_n} = \psi(s) \tag{290}$$

for $\mathrm{Re}(s) = 0$, or equivalently,

$$\lim_{n\to\infty} [n \log\phi(\frac{s}{B_n}) + \frac{sA_n}{B_n}] = \log\psi(s) \tag{291}$$

for $\mathrm{Re}(s) = 0$. Now let us put $s = iu$, where u is real, in (291) and form the real part and the imaginary part of (291). Then we obtain that

$$\lim_{n\to\infty} n \log|\phi(\frac{iu}{B})| = -c|u|^\alpha \tag{292}$$

and

$$\lim_{n\to\infty} [n\ \mathrm{Im}(\log\phi(\frac{iu}{B_n})) + \frac{uA_n}{B_n}] = -um + \begin{cases} -u|u|^{\alpha-1}\beta c\tan\frac{\alpha\pi}{2} & \text{for } \alpha \neq 1 \\ \frac{2\beta c}{\pi} u \log|u| & \text{for } \alpha = 1 \end{cases} \tag{293}$$

for real u are the necessary and sufficient conditions for (289).

Now let us suppose that $R(x)$ is nondegenerate, that is, $c > 0$. By (292) and (293) we can easily deduce the following necessary conditions for (289).

First, the constants $B_n > 0$ $(n = 1,2,\ldots)$ should be chosen in such a way that $\lim_{n\to\infty} B_n = \infty$ and

(294) $$\lim_{n\to\infty} n \log|\phi(\frac{1}{B_n})| = -c .$$

This follows from (292) if we put $u = 1$ in it.

By (293) it follows that the constants A_n $(n = 1,2,\ldots)$ should be chosen in the following way:

(295) $$A_n = -(m+\varepsilon_n)B_n + \begin{cases} n\, a \text{ for } 1 < \alpha < 2 , \\ -nB_n \operatorname{Im}(\log\phi(\frac{1}{B_n})) \text{ for } \alpha = 1 , \\ 0 \quad \text{for } 0 < \alpha < 1 \end{cases}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$ and

(296) $$a = \int_{-\infty}^{\infty} x dF(x) .$$

If we divide (293) by u and if we let $u \to 0$, then we obtain A_n for $1 < \alpha < 2$, and if we let $u \to \infty$, then we obtain A_n for $0 < \alpha < 1$. If we put $u = 1$ in (293) then we obtain (295) for $\alpha = 1$. By (292) and (294) we obtain that

(297) $$\lim_{u\to 0} \frac{\log|\phi(i\rho u)|}{\log|\phi(iu)|} = \rho^{\alpha}$$

for $0 < \rho < \infty$. If $u > 0$ in (293), then by (295) we obtain that

$$(298)\quad \lim_{u \to 0} \frac{\operatorname{Im}(\log\phi(i\rho u)) + a\rho u}{\log|\phi(iu)|} = (\beta \tan \frac{\alpha\pi}{2})\rho^{\alpha} \quad \text{for} \quad 1 < \alpha < 2 \, ,$$

$$(299)\quad \lim_{u \to 0} \frac{\operatorname{Im}[\log\phi(i\rho u) - \rho\log\phi(iu)]}{\log|\phi(iu)|} = \frac{2\beta}{\pi} \rho\log\rho \quad \text{for} \quad \alpha = 1$$

and

$$(300)\quad \lim_{u \to 0} \frac{\operatorname{Im}(\log\phi(i\rho u))}{\log|\phi(iu)|} = (\beta \tan \frac{\alpha\pi}{2})\rho^{\alpha} \quad \text{for} \quad 0 < \alpha < 1$$

and for any $0 < \rho < \infty$.

It can be proved that these conditions are not only necessary but sufficient too for $F_n(A_n + B_n x)$ to converge weakly to a nondegenerate stable distribution function $R(x)$ of type $S(\alpha,\beta,c,m)$.

To close this section we shall give a brief account of some results concerning the limiting distributions of suitably normalized sums of mutually independent real random variables whose distributions are not necessarily identical. Most of the results mentioned here are concerned with the solutions of two main problems.

First, let us assume that $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent real random variables and write $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$. Furthermore, let $R(x)$ be a nondegenerate distribution fucntion.

The first problem is as follows: What conditions should be imposed on $\underset{\sim}{P}\{\xi_k \leq x\}$ $(k = 1,2,\ldots)$ in order that there exist constants A_n and $B_n > 0$ such that

$$(301) \qquad \lim_{n \to \infty} P\{ \frac{\zeta_n - A_n}{B_n} \leq x\} = R(x)$$

in every continuity point of $R(x)$ and what kind of distribution functions $R(x)$ can appear in (301)?

Second, let us assume that $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nk_n}$ are a finite number of mutually independent real random variables for each $n = 1,2,\ldots$ and write $\zeta_n = \xi_{n1} + \xi_{n2} + \ldots + \xi_{nk_n}$ for $n = 1,2,\ldots$. Furthermore, let $R(x)$ be a nondegenerate distribution function.

The second problem is as follows: What conditions should be imposed on $P\{\xi_{nk} \leq x\}$ $(1 \leq k \leq k_n$ and $1 \leq n < \infty)$ in order that there exist constants A_n $(n = 1,2,\ldots)$ such that

$$(302) \qquad \lim_{n \to \infty} P\{\zeta_n - A_n \leq x\} = R(x)$$

in every continuity point of $R(x)$ and what kind of distribution functions $R(x)$ can appear in (302)?

The first result concerning the first problem was found in 1733 by A. De Moivre [36]. He found that if ν_n denotes the number of successes in n Bernoulli trials with probability p for success and if $0 < p < 1$, then

$$(303) \qquad \lim_{n \to \infty} P\{\alpha \leq \frac{\nu_n - np}{\sqrt{np(1-p)}} \leq \beta\} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-u^2/2}\, du$$

for $\alpha < \beta$. Actually, A. De Moivre demonstrated that

$$(304)\qquad \underset{\sim}{P}\{\nu_n = k\} \sim \frac{1}{\sqrt{2\pi np(1-p)}}\, e^{-\frac{(k-np)^2}{2np(1-p)}}$$

as $n \to \infty$ and $|k-np| < C\sqrt{n}$ and obtained (303) for $\beta = -\alpha = 1,2,3$ by numerical integration.

Let us associate a sequence of random variables $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ with the sequence of Bernoulli trials in the following way: $\xi_k = 1$ if the k-th trial results in success and $\xi_k = 0$ if the k-th trial results in failure. Then $\{\xi_k\}$ is a sequence of mutually independent and identically distributed random variables for which $\underset{\sim}{P}\{\xi_k = 1\} = p$ and $\underset{\sim}{P}\{\xi_k = 0\} = 1-p$. By (303) we obtain that (301) holds for the sequence $\{\xi_k\}$ if we choose $A_n = np$, $B_n = \sqrt{np(1-p)}$ and $R(x)=\Phi(x)$ where

$$(305)\qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du$$

is the normal distribution function.

By using the method of Fourier transforms (characteristic functions) in 1812 P. S. Laplace [107], ([178] pp. 588–604.) demonstrated that if $\{\xi_k\}$ is a sequence of mutually independent and identically distributed symmetric random variables for which $\underset{\sim}{E}\{\xi_k\} = 0$ and $\underset{\sim}{\mathrm{Var}}\{\xi_k\} = b^2 > 0$ exist, then (301) holds whenever $A_n = 0$, $B_n = b\sqrt{n}$ and $R(x) = \Phi(x)$ defined by (305). It should be noted that although Laplace's proof is ingenious, it is not rigorous by present standards. A rigorous proof for this result was given only in 1925 by P. Lévy [111 p. 233] by using a continuity theorem for characteristic functions.

Now let us follow first the historical development of the solution of the first problem mentioned above in the particular case when $R(x) = \Phi(x)$ is given by (305).

In 1887 P. L. Chebyshev [616] considered the case where $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent random variables for which $E\{|\xi_k|^r\} < C_r < \infty$ for $r = 1,2,\ldots$ and $k = 1,2,\ldots$. By writing $E\{\xi_k\} = a_k$, $Var\{\xi_k\} = b_k^2$ for $k = 1,2,\ldots$ and $A_n = a_1+\ldots+ a_n$ and $B_n^2 = b_1^2+\ldots + b_n^2$ for $n = 1,2,\ldots,$ Chebyshev proved that if $B_n \to \infty$ as $n \to \infty$, then

$$\lim_{n \to \infty} E\{(\frac{\zeta_n - A_n}{B_n})^r\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^r e^{-x^2/2} dx \tag{306}$$

for $r = 0,1,2,\ldots$ and hence he concluded that (301) holds with these A_n, B_n and $R(x) = \Phi(x)$. Chebyshev's conclusion is based on two auxiliary theorems. First, that the normal distribution function $\Phi(x)$ is uniquely determined by its moments. This follows from some extremely useful inequalities of P. L. Chebyshev [197] which he announced in 1874 without proof. These inequalities were proved first in 1884 by A. A. Markov [130], [131]. See also P. L. Chebyshev [198], [199], T. J. Stieltjes [189], [190], [192] and J. V. Uspensky [204 pp. 356-395]. From these results of Chebyshev it follows that if $F(x)$ is any distribution function for which

$$\int_{-\infty}^{\infty} x^r dF(x) = \int_{-\infty}^{\infty} x^r d\Phi(x) \tag{307}$$

holds whenever $r = 0,1,\ldots,2m$, then

$$|F(x)-\Phi(x)| < \sqrt{\frac{\pi}{2m}} . \tag{308}$$

Since the right-hand side of (308) tends to 0 as $m \to \infty$, it follows that the normal distribution function is uniquely determined by its moments. The second auxiliary theorem which is needed for Chebyshev's conclusion is Theorem 11 in Section 41. Chebyshev did not prove this theorem. He accepted it as an obvious fact. The proof of this auxiliary theorem was given in 1898 by A. A. Markov [580] .

It should be noted that in his proof Chebyshev proved the convergence of the semiinvariants instead of the moments; however, the equivalence of the two procedures is obvious. Indeed in 1899 A. A. Markov [579] provided a direct proof for (306).

In later years A. A. Markov [584 pp. 77-81]proved that Chebyshev's conditions for the validity of (306) can be weakened. He showed that if $E\{|\xi_k|^r\} < \infty$ for $r = 1,2,\ldots$ and $k = 1,2,\ldots$, if $B_n \to \infty$ and if

$$(309) \qquad \lim_{n \to \infty} \frac{1}{B_n^r} \sum_{k=1}^{n} E\{|\xi_k - a_k|^r\} = 0$$

for $r = 3,4,5,\ldots$, then (306) is satisfied and therefore (301) holds with $R(x) = \Phi(x)$.

In 1900 a significant step was made by A. Liapounoff [564] concerning the solution of the first problem in the case of a limiting normal distribution. He supposed that $\{\xi_k\}$ is a sequence of mutually independent random variables for which $E\{\xi_k\} = a_k$ and $Var\{\xi_k\} = b_k^2$ exist. Write $A_n = a_1 + \ldots + a_n$ and $B_n^2 = b_1^2 + \ldots + b_n^2$ for $n = 1,2,\ldots$. Liapounoff proved that if $B_n \to \infty$ as $n \to \infty$ and if

$$(310) \qquad \lim_{n\to\infty} \frac{1}{B_n^{2+\delta}} \sum_{k=1}^{n} E\{|\xi_k - a_k|^{2+\delta}\} = 0$$

for $\delta = 1$, then (301) holds with $R(x) = \Phi(x)$. Let us observe that Liapounoff's conditions do not require the existence of the moments $E\{|\xi_k|^r\}$ for $r > 3$. In 1901 A. Liapounoff [565] showed that the same result holds unchangeably if we require only that (310) hold for some $\delta > 0$. In his proof Liapounoff made use of Dirichlet's discontinuity factor

$$(311) \qquad J(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin hu}{u} \cos xu \, du = \begin{cases} 1 & \text{for } |x| < h , \\ \frac{1}{2} & \text{for } |x| = h , \\ 0 & \text{for } |x| > h . \end{cases}$$

(See also A. A. Markov [584 pp. 67-76].) It should be mentioned that the factor (311) was already used in 1872 by J. W. L. Glaisher [525] in his study on the generalization of Laplace's result mentioned earlier. It is interesting to mention that in 1913 A. A. Markov [585 pp.319-338] demonstrated that Liapounoff's result can also be proved by the method of moments by introducing an ingenious artifice, the truncation of random variables.

If $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent random variables for which $E\{\xi_k\} = a_k$ and $E\{(\xi_k - a_k)^2\} = b_k^2$ exist and if we write $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$, $A_n = a_1 + a_2 + \ldots + a_n$ and $B_n^2 = b_1^2 + b_2^2 + \ldots + b_n^2$, then following the terminology which was introduced in 1920 by G. Pólya [596] we say that the central limit theorem is valid for the sequence

$\{\xi_k\}$ whenever

$$(312) \qquad \lim_{n\to\infty} P\{\frac{\zeta_n - A_n}{B_n} \leq x\} = \Phi(x)$$

where $\Phi(x)$ is defined by (305).

The investigations of P. L. Chebyshev, A. A. Markov and A. Liapounoff yielded weaker and weaker sufficient conditions for the validity of (312). The ultimate condition was found in 1922 by J. W. Lindeberg [566]. (See also [567].) Lindeberg proved that if

$$(313) \qquad \lim_{n\to\infty} \frac{1}{B_n^2} \sum_{k=1}^{n} \int_{|x-a_k|<\varepsilon B_n} (x-a_k)^2 dP\{\xi_k \leq x\} = 0$$

for all $\varepsilon > 0$, then the central limit theorem is valid for the sequence $\{\xi_k\}$, that is (312) holds. Actually, Lindeberg's condition is somewhat different from (313), but it can easily be seen that it can be replaced by (313). Lindeberg's method is entirely different from the previous methods. It is based on the estimation of the difference $P\{\xi_n \leq A_n + B_n x\} - \Phi(x)$ for large n values. Similar methods were used in 1919 by R. v. Mises [588].

In 1922 P. Lévy [110] found powerful theorems which proved to be the stepping stones for the solutions of the problems mentioned at the beginning of this historical review. P. Lévy proved that a distribution function $F(x)$ is uniquely determined by its characteristic function $\psi(\omega)$. His inversion formula is given by Theorem 41.5. P. Lévy also proved that if $F_1(x), F_2(x), \ldots, F_n(x), \ldots$ and $F(x)$ are distribution functions and their characteristic functions are $\psi_1(\omega), \psi_2(\omega), \ldots, \psi_n(\omega), \ldots$ and $\psi(\omega)$ respectively, then $F_n(x) \Longrightarrow F(x)$ as $n \to \infty$ if and only if $\lim_{n \to \infty} \psi_n(\omega) = \psi(\omega)$ for all ω. (See also P. Lévy [111] pp. 195-200.) In 1923 G. Pólya [598] showed that the latter theorem of P. Lévy can be proved in a similar way as a continuity theorem found in 1919 by himself. (See G. Pólya [596].)

For another approach of the proof of the central limit theorem we refer to A. Ya. Khintchine [97].

In 1935 W. Feller [517] proved that Lindeberg's condition (313) is not only sufficient but necessary too for the validity of (312). W. Feller proved that if (312) holds and if

$$\lim_{\varepsilon \to 0} \max_{1 \leq k \leq n} \mathbf{P}\left\{\left|\frac{\xi_k - a_k}{B_n}\right| > \varepsilon\right\} = 0 \tag{314}$$

for all $\varepsilon > 0$, then Lindeberg's condition (313) is satisfied. However, it should be noted that it may happen that $(\zeta_n - A_n)/B_n$ has a limiting normal distribution as $n \to \infty$ and Lindeberg's condition (313) fails. In this case, however, the limiting normal distribution has variance < 1.

In 1926 S. Bernstein [491] gave sufficient conditions and in 1935 and also in 1937 W. Feller [517], [518] gave necessary and sufficient conditions for the existence of constants A_n and $B_n > 0$ such that (301) holds with $R(x) = \Phi(x)$ defined by (305). These results of Bernstein and Feller show that even if the random variables have infinite second moments it may happen that there exist normalizing constants A_n and $B_n > 0$ such that $(\zeta_n - A_n)/B_n$ has a limiting normal distribution as $n \to \infty$.

In the particular case where the random variables $\{\xi_k\}$ have an identical distribution and $R(x) = \Phi(x)$, the solution of the first problem was given in 1935 by W. Feller [517], A. Ya. Khintchine [545] and P. Lévy [560]. (See Theorem 44.6.)

In the case where $R(x) = \Phi(x)$, a necessary and sufficient condition for the validity of (302) was given in 1939 by B. V. Gnedenko [528], as a particular case of a more general result.

Now let us consider the solutions of the first problem in the case where $R(x)$ is not necessarily a normal distribution. Already in 1827 S. D. Poisson [154] and in 1853 A. Cauchy [231] demonstrated that if

$$\text{(315)} \qquad P\{\xi_k \leq x\} = \frac{1}{2} + \frac{1}{\pi} \arctan x ,$$

then $R(x)$ is not a normal distribution. Since in this case

$$\text{(316)} \qquad P\left\{\frac{\zeta_n}{n} \leq x\right\} = \frac{1}{2} + \frac{1}{\pi} \arctan x$$

for all $n = 1,2,\ldots$, it follows that if $A_n = 0$ and $B_n = n$, then

$$\text{(317)} \qquad R(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x .$$

In 1853 A. Cauchy [231] proved that if the random variables have a symmetric distribution, then necessarily

$$\text{(318)} \qquad \psi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x}\, dR(x) = e^{-c|\omega|^{\alpha}}$$

where $\alpha > 0$ and $c > 0$. Cauchy, however, did not recognize that $\psi(\omega)$ is not a characteristic function if $\alpha > 2$. This fact was pointed out only in 1923 by G. Pólya [598]. In 1925 P. Lévy [111 pp. 254-257] showed that $R(x)$ is necessarily a stable distribution function and found the general form of $\psi(\omega)$. (See Definition 42.2 and Theorem 42.4.) Actually P. Lévy used a somewhat more restrictive definition of a stable distribution function than Definition 42.2 (having excluded the case where $\alpha = 1$ and $\beta \neq 0$). A rigorous proof for the general form of $\psi(\omega)$ was given only in 1936 by A. Ya. Khintchine and P. Lévy [279]. It is easy to see that every stable distribution function $R(x)$ can appear as a limiting distribution in (301).

In 1938 W. Doeblin [508],[510] gave necessary and sufficient conditions for the validity of (301) in the case of identically distributed random variables. In 1939 B. V. Gnedenko [529] gave another proof of this result as a particular case of a more general result. See also B. V. Gnedenko [530].

If the random variables $\{\xi_k\}$ are not necessarily identically distributed, the solution of the first problem has interest only if we impose the following conditions on $\{\xi_k\}$

$$(319) \qquad \lim_{n\to\infty} \max_{1\leq k\leq n} P\left\{\left|\frac{\xi_k - m_k}{B_n}\right| > \varepsilon\right\} = 0$$

for all $\varepsilon > 0$ where m_k is a median of ξ_k.

Following A. Ya. Khintchine [278] we say that $R(x)$ belongs to the class L if $R(x)$ can appear as a limiting distribution in (301). In solving a problem of A. Ya. Khintchine (stated in a letter to P. Lévy) in 1936 P. Lévy [561],[563 pp. 195-197] gave necessary and sufficient conditions for $R(x)$ to belong to the class L.

In 1939 B. V. Gnedenko and A. V. Groshev [534] gave necessary and sufficient conditions for the existence of constants A_n and $B_n > 0$ such that (301) holds with an $R(x)$ belonging to the class L.

Now let us consider the solution of the second problem. To exclude obvious cases we assume that the random variables $\{\xi_{nk}\}$ satisfy the following condition:

(320) $$\lim_{n\to\infty} \max_{1\le k\le k_n} P\{|\xi_{nk} - m_{nk}| > \varepsilon\} = 0$$

for any $\varepsilon > 0$ where m_{nk} is a median of ξ_{nk}.

The first result concerning the solution of the second problem was obtained in 1837 by S. D. Poisson [156],[157] (§ 66 – § 93). For every $n = 1,2,\ldots$ he considered a sequence of n Bernoulli trials with probability p_n for success. Let us define $\chi_{nk} = 1$ if the k-th trial results in success and $\chi_{nk} = 0$ if the k-th trials results in failure in the n-th sequence. Write $A_n = p_1 + p_2 + \ldots + p_n$ and $B_n^2 = p_1(1-p_1)+p_2(1-p_2)+\ldots+p_n(1-p_n)$. By the results of Poisson we can conclude that if $B_n \to \infty$, then

(321) $$\lim_{n\to\infty} P\left\{\frac{\chi_{n1}+\ldots+\chi_{nn}-A_n}{B_n} \le x\right\} = \Phi(x)$$

defined by (305) and if $\lim_{n\to\infty} np_n = a$ where a is a positive number, then

(322) $$\lim_{n\to\infty} P\{\chi_{n1}+\ldots+\chi_{nn} = k\} = e^{-a}\frac{a^k}{k!}$$

for $k = 0,1,2,\ldots$.

In 1936 G. M. Bawly [482] considered the case where for each $n = 1,2,\ldots$ the random variables $\xi_{n1}, \xi_{n2}, \ldots, \xi_{nn}$ are independent and $E\{\xi_{nk}\} = a_{nk}$ and $Var\{\xi_{nk}\} = b_{nk}^2$ exist. He supposed that

$$\lim_{n \to \infty} \max_{1 \leq k \leq n} b_{nk}^2 = 0 \tag{323}$$

and

$$B_n^2 = \sum_{k=1}^{n} b_{nk}^2 < C \tag{324}$$

where C is a finite constant independent of n. Under these conditions Bawly showed that $R(x)$ is necessarily an infinitely divisible distribution function with a finite variance. The most general form of the characteristic function of such an $R(x)$ was determined in 1932 by A. N. Kolmogorov [280],[281]. By using a continuity theorem for infinitely divisible distribution functions with finite variances (a particular case of Theorem 44.1) G. M. Bawly [482] gave necessary and sufficient conditions for the validity of

$$\lim_{n \to \infty} P\{\sum_{k=1}^{n} (\xi_{nk} - a_{nk}) \leq x\} = R(x) . \tag{325}$$

In 1937 A. Khintchine [277] proved that if the condition (320) is satisfied, then in (302) $R(x)$ is necessarily an infinitely divisible distribution function. The converse is obvious. Every infinitely divisible distribution function $R(x)$ can appear as a limiting distribution in (302).

In 1938 B. V. Gnedenko [526] gave necessary and sufficient conditions for the validity of (302) in the case where $R(x)$ is an arbitrary infinitely divisible distribution function. This result of B. V. Gnedenko is based on a continuity theorem for infinitely divisible distribution functions which uses Khintchine's representation of the characteristic function of R(x) . (Theorem 41.1.)

In 1939 B. V. Gnedenko [527] published another paper which contains the same fundamental theorem as the previous paper except that in this paper he used Lévy's representation of the characteristic function of R(x) . (Theorem 41.2.)

In 1939 B. V. Gnedenko [528] published in detail the results announced in the previous two papers. In 1944 B. V. Gnedenko [531] published an expository article which contains most of the results concerning the solution of the second problem formulated at the beginning of this historical discussion. Finally, let us call attention to the excellent book of B. V. Gnedenko and A. N. Kolmogorov [260] which was first published in 1949.

The problem of finding the limiting distribution of suitably normalized sums of real random variables have been considered also for various types of dependent random variables. In 1908 and in 1910 A. A. Markov [582],[583] extended the central limit theorem for a sequence of random variables depending on each other like the links of a chain (Markov chains). Markov's results have been extended further by P. Lévy [560], W. Doeblin [640], N. A. Sapogov [670],[671] Yu. V. Linnik [654],[655], R. L. Dobrushin [639], S. V. Nagaev [659] and others. In 1922 and in 1926 S. N. Bernstein [490],[491] extended the central limit theorem for weakly dependent random variables.

Further extensions of the central limit theorem have been given by M. Loève [572],[575], W. Hoeffding and H. Robbins [647], P. H. Dienanda [636],[637], [638], A. Rényi [662],[663] A. N. Kolmogorov [653], M. Rosenblatt [666] and others. Limit theorems for sums of interchangeable random variables have been obtained by H. Chernoff and H. Teicher [633] and H. Bühlmann [771],[772].

Limit distributions for suitably normalized sums of random vectors have been studied by R. v. Mises [588], S. Bernstein [491], A. Ya. Khintchine [97], P. Lévy [113], H. Cramér [503], E. L. Rvacheva [605] and others.

45. Limit Distributions of Various Functionals

In the previous section we considered a sequence of mutually independent and identically distributed real random variables $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ and demonstrated that if $P\{\xi_k \leq x\}$ satisfies certain conditions, then the partial sums $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ have a nondegenerate asymptotic distribution as $n \to \infty$, that is, there exists a nondegenerate distribution function $R(x)$ and suitable normalizing constants A_n and $B_n > 0$ such that

$$\text{(1)} \qquad \lim_{n \to \infty} P\left\{ \frac{\zeta_n - A_n}{B_n} \leq x \right\} = R(x)$$

in every continuity point of $R(x)$.

In this section we shall consider two extensions of the limiting distribution (1).

First, instead of considering the asymptotic distribution of ζ_n as $n \to \infty$, we shall be interested in studying the asymptotic distribution of $\zeta_{\nu(t)}$ as $t \to \infty$ where $\nu(t)$ $(0 \leq t < \infty)$ is a random variable which takes on only nonnegative integers and which converges in probability to ∞ as $t \to \infty$, that is, $\lim_{t \to \infty} P\{\nu(t) \geq m\} = 1$ for all $m = 0,1,2,\ldots$. In general, we shall assume that $\{\zeta_n, n = 0,1,2,\ldots\}$ and $\{\nu(t), 0 \leq t < \infty\}$ are independent. However, some of the results can easily be extended to the case where $\{\nu(t)\}$ may depend on $\{\zeta_n\}$. We note that if $\nu(t) = [t]$ for $0 \leq t < \infty$, where $[t]$ is the greatest integer less than or equal to t, then the general results reduce to (1).

Second, we shall be interested in studying the asymptotic distribution of $\eta_n = h_n(\zeta_0, \zeta_1, \ldots, \zeta_n)$ as $n \to \infty$ where $h_n(\zeta_0, \zeta_1, \ldots, \zeta_n)$ is a Borel measurable function (Baire function) of the random variables $\zeta_0, \zeta_1, \ldots, \zeta_n$. If, in particular, $\eta_n = \zeta_n$ for $n = 0,1,2,\ldots$, then this more general case reduces to the case investigated in the previous section. In this section we shall consider variables such as $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$, $\eta_n = \max(|\zeta_0|, |\zeta_1|, \ldots, |\zeta_n|)$, and $\eta_n = |\zeta_0| + |\zeta_1| + \ldots + |\zeta_n|$.

<u>Sums of a random number of random variables.</u>

In 1948 <u>H. Robbins</u> [165],[696] extended the central limit theorem for sums of a random number of random variables in the following way.

<u>Theorem 1.</u> <u>Let</u> $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ <u>be a sequence of mutually independent and identically distributed random variables for which</u> $E\{\xi_k\} = a$ <u>and</u> $Var\{\xi_k\} = b^2 > 0$ <u>exist.</u> <u>Let</u> $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$

for $n = 1,2,\ldots,$ and $\zeta_0 = 0$. Let $\{\nu(t)\ ,\ 0 \leq t < \infty\}$ be a family of random variables taking on nonnegative integers only. Let us assume that

(2) $$\lim_{t \to \infty} \underset{\sim}{P}\{ \frac{\nu(t)-\mu t}{\sigma\sqrt{t}} \leq x\} = \Phi(x)$$

where $\Phi(x)$ is the normal distribution function and μ and $\sigma > 0$ are constants. If $\{\xi_k\}$ and $\{\nu(t)\}$ are independent, then

(3) $$\lim_{t \to \infty} \underset{\sim}{P}\{ \frac{\zeta_{\nu(t)} - a\mu t}{\sqrt{(a^2\sigma^2 + b^2\mu)t}} \leq x\} = \Phi(x) .$$

Proof. By Theorem 44.6 we have

(4) $$\lim_{n \to \infty} \underset{\sim}{P}\{ \frac{\zeta_n - na}{b\sqrt{n}} \leq x\} = \Phi(x) .$$

By using (2) and (4) we can show that the characteristic function of $(\zeta_{\nu(t)} - a\mu t)/\sqrt{(a^2\sigma^2 + b^2\mu)t}$ tends to the characteristic function of $\Phi(x)$ as $t \to \infty$. Hence by Theorem 41.10 we can conclude that (3) holds. It should be noted that while this proof is conceptually simple, it is quite involved in technical details.

In what follows we shall prove a more general theorem which contains Theorem 1 as a particular case. This more general theorem is based on some simple properties of the convergence of real random variables. Let us summarize briefly these properties.

Let us consider a probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$ and a sequence of real random variables $\xi_n(\omega)$ $(n = 0,1,2,\ldots)$. Let $\xi(\omega)$ be also a real random variable. Then $A = \{\omega: \lim_{n \to \infty} \xi_n(\omega) = \xi(\omega)\}$ is a random event, that is, $A \in \mathcal{B}$, and therefore $\underset{\sim}{P}\{A\}$ is defined. If $\underset{\sim}{P}\{A\} = 1$, then we say that the sequence of random variables $\{\xi_n\}$ converges to ξ with probability one, that is,

(5) $$\underset{\sim}{P}\{\lim_{n \to \infty} \xi_n = \xi\} = 1 .$$

If

(6) $$\lim_{n \to \infty} \underset{\sim}{P}\{|\xi_n - \xi| < \varepsilon\} = 1$$

for all $\varepsilon > 0$, then we say that the sequence of random variables $\{\xi_n\}$ converges to ξ in probability.

Obviously (5) implies (6), whereas the converse is not true in general.

Both (5) and (6) imply that

(7) $$\lim_{n \to \infty} \underset{\sim}{P}\{\xi_n \leq x\} = \underset{\sim}{P}\{\xi \leq x\}$$

in every continuity point of $\underset{\sim}{P}\{\xi \leq x\}$.

Conversely, if $F_n(x)$ $(n = 0,1,2,\ldots)$ and $F(x)$ are distribution functions and

(8) $$\lim_{n \to \infty} F_n(x) = F(x)$$

in every continuity point of $F(x)$, then we can define a probability space $(\Omega,\mathcal{B},\underset{\sim}{P})$ and real random variables ξ_n $(n = 0,1,2,\ldots)$ and ξ in such a way that $\underset{\sim}{P}\{\xi_n \leq x\} = F_n(x)$ $(n = 0,1,2,\ldots)$, $\underset{\sim}{P}\{\xi \leq x\} = F(x)$ and $\underset{\sim}{P}\{\lim_{n \to \infty} \xi_n = \xi\} = 1$.

To see this let us suppose that $\Omega = \{\omega: 0 < \omega < 1\}$, $\mathcal{B}$ is the class of Borel subsets of Ω, and $\underset{\sim}{P}$ is the Lebesgue measure. If $\xi_n(\omega) = F_n^{-1}(\omega)$ for $0 < \omega < 1$ and $\xi(\omega) = F^{-1}(\omega)$ for $0 < \omega < 1$, then all the requirements are satisfied. We define the inverse of a distribution function $F(x)$ as

$$F^{-1}(x) = \inf\{u : F(u) \geq x\} \tag{9}$$

for $0 < x < 1$.

If $\underset{\sim}{P}\{\lim_{n\to\infty} \xi_n = \xi\} = 1$ and $\underset{\sim}{P}\{\lim_{n\to\infty} \eta_n = \eta\} = 1$, then obviously

$$\underset{\sim}{P}\{\lim_{n\to\infty} (\xi_n + \eta_n) = \xi + \eta\} = 1 \quad \text{and} \quad \underset{\sim}{P}\{\lim_{n\to\infty} \xi_n\eta_n = \xi\eta\} = 1 . \tag{10}$$

If $\underset{\sim}{P}\{\lim_{n\to\infty} \xi_n = \xi\}$ and $h(x)$ is a Borel-measurable function (Baire function) of x, then

$$\underset{\sim}{P}\{\lim_{n\to\infty} h(\xi_n) = h(\xi)\} = 1 . \tag{11}$$

We can define the notion of convergence not only for a sequence of random variables, but for a family of random variables too. In waht follows we assume that $\{\xi_t\}$ is a family of random variables defined for $0 \leq t < \infty$ and we shall prove a few properties of the convergence of such random variables.

We say that $\{\xi_t ; 0 \leq t < \infty\}$ converges in probability to the random variables ξ if

$$\lim_{t\to\infty} \underset{\sim}{P}\{|\xi_t - \xi| < \varepsilon\} = 1 \tag{12}$$

for all $\varepsilon > 0$. In this case we use the notation $\xi_t \Rightarrow \xi$ as $t \to \infty$. Obviously, $\xi_t \Rightarrow \xi$ if and only if $(\xi_t - \xi) \Rightarrow 0$ as $t \to \infty$.

We note that if $\lim_{t \to \infty} c_t = c$ where c_t and c are real numbers, then $c_n \Rightarrow c$.

Lemma 1. *If $\xi_t \Rightarrow \xi$ and $\eta_t \Rightarrow \eta$ as $t \to \infty$, then*

(13) $$\xi_t + \eta_t \Rightarrow \xi + \eta \quad \text{and} \quad \xi_t \eta_t \Rightarrow \xi\eta .$$

Proof. Let $\xi_t^* = \xi_t - \xi$ and $\eta_t^* = \eta_t - \eta$. By assumption we have $\xi_t^* \Rightarrow 0$ and $\eta_t^* \Rightarrow 0$. For any $\varepsilon > 0$

(14) $$\mathbf{P}\{|\xi_t^* + \eta_t^*| \geqq \varepsilon\} \leqq \mathbf{P}\{|\xi_t^*| + |\eta_t^*| \geqq \varepsilon\} \leqq \mathbf{P}\{|\xi_t^*| \geqq \frac{\varepsilon}{2}\} + $$
$$+ \mathbf{P}\{|\eta_t^*| \geqq \frac{\varepsilon}{2}\} \to 0 \quad \text{as} \quad t \to \infty .$$

Thus $\xi_t^* + \eta_t^* \Rightarrow 0$, which proves the first half of (13).

Next we shall prove that if $\xi_t \Rightarrow \xi$, then $\xi_t^2 \Rightarrow \xi^2$. First, we observe that $\xi_t^* \Rightarrow 0$ implies that $(\xi_t^*)^2 \Rightarrow 0$. Indeed for any $\varepsilon > 0$ we have

(15) $$\mathbf{P}\{(\xi_t^*)^2 \geqq \varepsilon\} = \mathbf{P}\{|\xi_t^*| \geqq \varepsilon^{\frac{1}{2}}\} \to 0 \quad \text{as} \quad t \to \infty .$$

Second, we observe that $\xi_t^* \Rightarrow 0$ implies that $\xi\xi_t^* \Rightarrow 0$. For any $\varepsilon > 0$ and $m > 0$ we have

(16) $$\mathbf{P}\{|\xi\xi_t^*| \geqq \varepsilon\} \leqq \mathbf{P}\{|\xi| \geqq m\} + \mathbf{P}\{|\xi_t^*| \geqq \frac{\varepsilon}{m}\}.$$

If m is sufficiently large, then the first term on the right-hand side of (16) is arbitrarily close to 0. If $t \to \infty$, then the second term on the right-hand side of (16) tends to 0. This proves that $\xi\xi_t^* \Rightarrow 0$. Since $\xi_t^2 = \xi^2 + 2\xi\xi_t^* + (\xi_t^*)^2$, it follows that $\xi_t^2 \Rightarrow \xi^2$ as $t \to \infty$.

If we write

(17) $$\xi_t\eta_t = \frac{(\xi_t + \eta_t)^2 - (\xi_t - \eta_t)^2}{4}$$

and apply the relations proved previously, then we obtain that

(18) $$\xi_t\eta_t \Rightarrow \frac{(\xi + \eta)^2 - (\xi - \eta)^2}{4} = \xi\eta$$

which completes the proof of (13).

<u>Lemma 2.</u> <u>If</u> $\xi_t \Rightarrow \xi$ <u>and if</u> $h(x)$ <u>is a continuous function of</u> x, <u>then</u>

(19) $$h(\xi_t) \Rightarrow h(\xi) .$$

<u>Proof.</u> Since $h(x)$ is uniformly continuous in any finite closed interval, for any $\varepsilon > 0$ and $m > 0$ there is a $\delta > 0$ such that $|h(x) - h(y)| < \varepsilon$ whenever $|x-y| < \delta$ and $|x| \leq m$ and $|y| \leq m$. On the other hand for any $\varepsilon > 0$, $m > 0$ and $\delta^* > 0$ we have the inequality

(20) $$\underset{\sim}{P}\{|h(\xi_t)-h(\xi)| \geq \varepsilon\} \leq \underset{\sim}{P}\{|\xi| \geq m-\delta^*\} + \underset{\sim}{P}\{|\xi_t - \xi| \geq \delta^*\} + $$
$$+ \underset{\sim}{P}\{|h(\xi_t)-h(\xi)| \geq \varepsilon, |\xi| \leq m-\delta^*, |\xi_t - \xi| < \delta^*\} .$$

If we choose $0 < \delta^* \leq \delta$, then the last term on the right-hand side of (20) is 0. The first term on the right-hand side is arbitrarily close to 0 if m is sufficiently large and δ^* is sufficiently small. The second term on the right-hand side of (20) tends to 0 as $t \to \infty$ for any $\delta^* > 0$. This proves (19). We note that (19) is not true in general for measurable functions $h(x)$.

Lemma 3. *Let us suppose that* $\lim_{t \to \infty} h_t(x) = h(x)$ *for every* x, $h_t(x)$ *and* $h(x)$ *are continuous functions of* x *and the convergence is uniform in every finite interval. If* $\xi_t \Rightarrow \xi$, *then*

$$h_t(\xi_t) \Rightarrow h(\xi) . \tag{21}$$

Proof. By Lemma 2 we have $h(\xi_t) \Rightarrow h(\xi)$. Thus it is sufficient to prove that

$$h_t(\xi_t) - h(\xi_t) \Rightarrow 0 . \tag{22}$$

By assumption for any $\varepsilon > 0$ and $m > 0$ there is a $\tau > 0$ such that $|h_t(x) - h(x)| < \varepsilon$ if $|x| \leq m$ and $t \geq \tau$. On the other hand for any $\varepsilon > 0$, $\delta > 0$ and $m > 0$ we have the inequality

$$\begin{aligned} &\mathbf{P}\{|h_t(\xi_t) - h(\xi_t)| \geq \varepsilon\} \leq \mathbf{P}\{|\xi| \geq m-\delta\} + \mathbf{P}\{|\xi_t - \xi| \geq \delta\} + \\ &\quad + \mathbf{P}\{|h_t(\xi_t) - h(\xi_t)| \geq \varepsilon, |\xi| \leq m-\delta, |\xi_t - \xi| < \delta\} . \end{aligned} \tag{23}$$

Let us choose $0 < \delta < m$. If m is sufficiently large and δ is sufficiently small, then the first term on the right-hand side of (23) is arbitrarily close to zero. For any $\delta > 0$, the second term on the right-hand side of (23) tends to 0 as $t \to \infty$. If $t > \tau$, then the last term on the right-hand side of (23) is zero. This proves (21).

Lemma 4. Let $\{\eta(n), n = 0,1,\ldots\}$ be real random variables for which $P\{\lim_{n\to\infty} \eta(n) = 0\} = 1$. Let $\{\nu(t), 0 \le t < \infty\}$ be discrete random variables taking on nonnegative integers only and let $\nu(t) \Rightarrow \infty$ as $t \to \infty$, that is

$$\lim_{t\to\infty} P\{\nu(t) \ge m\} = 1 \tag{24}$$

for all $m = 0,1,2,\ldots$. Then we have

$$\eta(\nu(t)) \Rightarrow 0 \quad \text{as} \quad t \to \infty . \tag{25}$$

Proof. For any $\varepsilon > 0$ and $m \ge 0$ we have

$$\begin{aligned} P\{|\eta(\nu(t))| \ge \varepsilon\} &= \sum_{n=0}^{\infty} P\{|\eta(n)| \ge \varepsilon \text{ and } \nu(t) = n\} \\ &\le P\{\nu(t) < m\} + P\{\sup_{m\le n<\infty} |\eta(n)| \ge \varepsilon\} . \end{aligned} \tag{26}$$

Since $P\{\lim_{n\to\infty} \eta(n) = 0\} = 1$ if and only if $\sup_{m\le n<\infty} |\eta(n)| \Rightarrow 0$ as $m \to \infty$, it follows that the second term on the right-hand side of (26) is arbitrarily close to 0 when m is large enough. For any $m \ge 0$ the first term on the right-hand side of (26) tends to 0 as $t \to \infty$. This proves (25).

Now we are in a position to prove a fundamental theorem which was found in 1955 by R. L. Dobrushin [678].

Theorem 2. Let $\zeta(n)$ $(n = 0,1,2,\ldots)$ be real random variables and let $\nu(t)$ $(0 \le t < \infty)$ be discrete random variables taking on non-negative integers only. Suppose that $0 < \beta < \alpha$, $b > 0$ and

(27) $$\lim_{n \to \infty} P\{ \frac{\zeta(n) - an^{\alpha}}{bn^{\beta}} \leq x\} = F(x)$$

in every continuity point of the distribution function $F(x)$. Furthermore, let us suppose that $0 < \delta < \gamma$, $d > 0$ and

(28) $$\lim_{t \to \infty} P\{ \frac{\nu(t) - ct^{\gamma}}{dt^{\delta}} \leq x\} = G(x)$$

in every continuity point of the distribution function $G(x)$. If $\{\zeta(n)\}$ and $\{\nu(t)\}$ are independent, then

(29) $$\lim_{t \to \infty} P\{ \frac{\zeta(\nu(t)) - gt^{\lambda}}{ht^{\mu}} \leq x\} = H(x)$$

in every continuity point of the distribution function $H(x)$. The constants g, h, λ, μ and the distribution function $H(x)$ are given in Table I where ζ and ν are independent random variables for which $P\{\zeta \leq x\} = F(x)$ and $P\{\nu \leq x\} = G(x)$.

TABLE I.

$a,c,\alpha,\beta,\gamma,\delta$	g	λ	h	μ	$H(x)$
$(\alpha-1)\gamma+\delta<\beta\gamma$ $a \neq 0,\ c \neq 0$	ac^{α}	$\alpha\gamma$	bc^{β}	$\beta\gamma$	$P\{\zeta \leq x\}$
$(\alpha-1)\gamma+\delta=\beta\gamma$ $\alpha \neq 0,\ c \neq 0$	ac^{α}	$\alpha\gamma$	1	$\beta\gamma$	$P\{bc^{\beta}\zeta + \alpha ac^{\alpha-1}d\nu \leq x\}$
$(\alpha-1)\gamma+\delta>\beta\gamma$ $a \neq 0,\ c \neq 0$	ac^{α}	$\alpha\gamma$	$\alpha ac^{\alpha-1}d$	$(\alpha-1)\gamma+\delta$	$P\{\nu \leq x\}$
$a = 0,\ c \neq 0$	0	–	bc^{β}	$\beta\gamma$	$P\{\zeta \leq x\}$
$a = 0,\ c = 0$	0	–	bd^{β}	$\beta\delta$	$P\{\zeta\nu^{\beta} \leq x\}$
$a \neq 0,\ c = 0$	0	–	ad^{α}	$\alpha\delta$	$P\{\nu^{\alpha} \leq x\}$

Proof. Let us consider the probability space $(\Omega_1, \mathcal{B}_1, \underset{\sim}{P}_1)$ where $\Omega_1 = \{\omega_1 : 0 < \omega_1 < 1\}$, $\mathcal{B}_1$ is the class of Borel subsets of Ω_1 and $\underset{\sim}{P}_1$ is the Lebesgue measure, and define a sequence of random variables $\zeta^*(n) = \zeta^*(n;\omega_1)$ $(n = 0,1,2,\ldots)$ satisfying the following requirements: $\underset{\sim}{P}\{\zeta^*(n) \leq x\} = \underset{\sim}{P}\{\zeta(n) \leq x\}$ for $n = 0,1,2,\ldots$ and all x and

$$\underset{\sim}{P}\{\lim_{n \to \infty} \frac{\zeta^*(n) - an^\alpha}{bn^\beta} = \zeta\} = 1 \tag{30}$$

where $\zeta = \zeta(\omega_1)$ is a random variable with the distribution function $\underset{\sim}{P}\{\zeta \leq x\} = F(x)$.

Let us consider also the probability space $(\Omega_2, \mathcal{B}_2, \underset{\sim}{P}_2)$ where $\Omega_2 = \{\omega_2 : 0 < \omega_2 < 1\}$, $\mathcal{B}_2$ is the class of Borel subsets of Ω_2 and $\underset{\sim}{P}_2$ is the Lebesgue measure, and define a family of random variables $\nu^*(t) = \nu^*(t;\omega_2)$ $(0 \leq t < \infty)$ satisfying the following requirements: $\underset{\sim}{P}\{\nu^*(t) = k\} = \underset{\sim}{P}\{\nu(t) = k\}$ for $k = 0,1,2,\ldots$ and all $t \geq 0$ and

$$\frac{\nu^*(t) - ct^\gamma}{dt^\delta} \Rightarrow \nu \text{ as } t \to \infty \tag{31}$$

where $\nu = \nu(\omega_2)$ is a random variable with the distribution function $\underset{\sim}{P}\{\nu \leq x\} = G(x)$.

Now let us denote by $(\Omega, \mathcal{B}, \underset{\sim}{P})$ the product probability space of $(\Omega_1, \mathcal{B}_1, \underset{\sim}{P}_1)$ and $(\Omega_2, \mathcal{B}_2, \underset{\sim}{P}_2)$, that is, $\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2): 0 < \omega_1 < 1, 0 < \omega_2 < 1\}$, $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$ is the class of Borel subsets of Ω, and $P = P_1 \times P_2$ the two dimensional Lebesgue measure. On this space let us define $\zeta^*(n) = \zeta^*(n; \omega_1, \omega_2) = \zeta^*(n; \omega_1)$ for all ω_2, $\zeta = \zeta(\omega_1, \omega_2) = \zeta(\omega_1)$ for all ω_2, $\nu^*(t) = \nu^*(t; \omega_1, \omega_2) = \nu^*(t; \omega_2)$ for all ω_1, and $\nu = \nu(\omega_1, \omega_2) = \nu(\omega_2)$ for all ω_1.

By this definition $\{\zeta^*(n)\}$ and $\{\nu^*(t)\}$ are independent and $\zeta^*(\nu^*(t))$ has the same distribution as $\zeta(\nu(t))$ for all $t \geqq 0$. Furthermore, ζ and ν are also independent random variables.

If we write

$$\eta(n) = \frac{\zeta^*(n) - an^\alpha}{bn^\beta} - \zeta \tag{32}$$

and

$$\omega(t) = \frac{\nu^*(t) - ct^\gamma}{dt^\delta} - \nu , \tag{33}$$

then $P\{\lim_{n \to \infty} \eta(n) = 0\} = 1$ and $\omega(t) \Rightarrow 0$ as $t \to \infty$.

Accordingly, we have

$$\zeta^*(n) = an^\alpha + bn^\beta(\zeta + \eta(n)) \tag{34}$$

for $n = 0,1,2,\ldots$, and

$$\nu^*(t) = ct^\gamma + dt^\delta(\nu + \omega(t)) \tag{35}$$

for $t \geqq 0$. Hence we obtain that

$$\begin{aligned} \zeta^*(\nu^*(t)) &= a[ct^\gamma + dt^\delta(\nu+\omega(t))]^\alpha + \\ &+ b[\zeta+\eta(\nu^*(t))][ct^\gamma + dt^\delta(\nu+\omega(t))]^\beta \end{aligned} \tag{36}$$

for $t \geqq 0$ where ζ and ν are independent random variables for which $P\{\zeta \leqq x\} = F(x)$ and $P\{\nu \leqq x\} = G(x)$.

If we choose g, h, λ, μ according to Theorem 2, then we can prove that in each case

$$\frac{\zeta^*(\nu^*(t)) - gt^\lambda}{ht^\mu} \Longrightarrow \chi \quad \text{as} \quad t \to \infty \tag{37}$$

where χ is a random variable which depends on ζ and ν and the parameters. The proof is based on the four auxiliary theorems. We note that in each case $\nu^*(t) \Longrightarrow \infty$ as $t \to \infty$ and therefore by Lemma 4 we have $\eta(\nu^*(t)) \Longrightarrow 0$ as $t \to \infty$.

Since $\zeta^*(\nu^*(t))$ and $\zeta(\nu(t))$ have identical distributions for all $t \geqq 0$, it follows from (37) that

$$\lim_{t \to \infty} \underset{\sim}{P}\left\{\frac{\zeta(\nu(t)) - gt^\lambda}{ht^\mu} \leqq x\right\} = \underset{\sim}{P}\{\chi \leqq x\} \tag{38}$$

in every continuity point of $\underset{\sim}{P}\{\chi \leqq x\}$. Thus $H(x) = \underset{\sim}{P}\{\chi \leqq x\}$ and this completes the proof of the theorem.

We shall mention in detail the proof of the second statement in Theorem 2. If we suppose that $a \neq 0$, $c \neq 0$ and $(\alpha-1)\gamma+\delta = \beta\gamma$, then by (36) we obtain that

$$\begin{aligned} \frac{\zeta^*(\nu^*(t)) - ac^\alpha t^{\alpha\gamma}}{t^{\beta\gamma}} &= \frac{ac^\alpha t^{\alpha\gamma}}{t^{\beta\gamma}}\left\{\left[1 + \frac{dt^\delta(\nu+\omega(t))}{ct^\gamma}\right]^\alpha - 1\right\} + \\ &+ bc^\beta[\zeta+\eta(\nu^*(t))]\left[1+ \frac{dt^\delta}{ct^\gamma}(\nu+\omega(t))\right]^\beta \end{aligned} \tag{39}$$

for $t \geqq 0$. Since $\omega(t) \Longrightarrow 0$ as $t \to \infty$, by Lemma 3 we obtain that the first term on the right-hand side of (39) converges in probability to $\alpha ac^\alpha d\nu$. Since $\nu^*(t) \Longrightarrow \infty$ as $t \to \infty$ and $\underset{\sim}{P}\{\lim_{n \to \infty} \eta(n) = 0\} = 1$, by Lemma 4 we obtain that $\eta(\nu^*(t)) \Longrightarrow 0$ as $t \to \infty$, and therefore the second term on the right-hand side of (39) converges in probability to $bc^\beta\zeta$.

Thus we have

$$\chi = \alpha a c^{\alpha} d\nu + b c^{\beta} \zeta \tag{40}$$

which proves the second statement of Theorem 2. The remaining five statements can be proved in a similar way.

Theorem 1 can be obtained as a particular case of the second statement of Theorem 2.

Theorem 2 is in fact an invariance theorem. According to this theorem the asymptotic distribution of $\zeta(\nu(t))$ depends only on the asymptotic distributions of $\zeta(n)$ and $\nu(t)$ as $n \to \infty$ and $t \to \infty$ respectively. If we replace $\{\zeta(n)\}$ by $\{\zeta^*(n)\}$ and $\{\nu(t)\}$ by $\{\nu^*(t)\}$ where $\zeta^*(n)$ has the same asymptotic distribution as $\zeta(n)$ and $\nu^*(t)$ has the asymptotic distribution as $\nu(t)$, then $\zeta(\nu(t)$ has the same asymptotic distribution as $\zeta^*(\nu^*(t))$. We can choose $\{\zeta^*(n)\}$ and $\{\nu^*(t)\}$ in the simplest way as follows. Let

$$\zeta^*(n) = an^{\alpha} + bn^{\beta}\zeta \tag{41}$$

for $n = 0,1,2,\ldots,$ and

$$\nu^*(t) = ct^{\gamma} + dt^{\delta}\nu \tag{42}$$

for $t \geqq 0$ where ζ and ν are independent random variables with distribution functions $\underset{\sim}{P}\{\zeta \leq x\} = F(x)$ and $\underset{\sim}{P}\{\nu \leq x\} = G(x)$. In this particular case we can determine the asymptotic distribution of $\zeta^*(\nu^*(t))$ as $t \to \infty$ without difficulty.

In some cases we can generalize Theorem 2 by removing the assumption of independence. For example, if instead of (27) and (28) we assume that

$$\lim_{\substack{n \to \infty \\ t \to \infty}} \underset{\sim}{P}\left\{ \frac{\zeta(n) - an^\alpha}{bn^\beta} \leq x \ , \ \frac{\nu(t) - ct^\gamma}{dt^\delta} \leq y \right\} = F(x, y) \tag{43}$$

in every continuity point of the distribution function $F(x, y)$, and if we can prove that the limiting distribution (29) exists and depends only on (43) , then $H(x)$ can be obtained in exactly the same way as in Theorem 2 except that now

$$\underset{\sim}{P}\{\zeta \leq x \ , \ \nu \leq y\} = F(x, y) \tag{44}$$

where $F(x, y)$ is given by (43) .

We note that Theorem 2 can easily be extended to more general normalizing functions than power functions without changing the method of proof. Let us consider the following example of this nature.

Let us assume that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of mutually independent and identically distributed real random variables which belong to the domain of attraction of a nondegenerate stable distribution function $R(x)$ of type $S(\alpha,\beta,c,0)$ where $\alpha \neq 1$ or $\alpha = 1$ and $\beta = 0$. Let
(45) $\zeta(n) = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta(0) = 0$. By Theorem 44.6 and Theorem 44.8 we can conclude that

$$\lim_{n \to \infty} \mathbf{P}\left\{\frac{\zeta(n) - na}{n^{1/\alpha}\rho(n)} \leq x\right\} = R(x) \tag{46}$$

where $a = 0$ if $0 < \alpha \leq 1$ and $a = \mathbf{E}\{\xi_n\}$ if $1 < \alpha < 2$, and $\rho(t)$ defined for $t \geq 0$ is a nondecreasing function of t for which $\lim_{t \to \infty} \rho(t) = \infty$ and

$$\lim_{t \to \infty} \frac{\rho(\omega t)}{\rho(t)} = 1 \tag{47}$$

for all $\omega > 0$. (See Problem 46.12.)

Theorem 3. Let $\nu(t)$ $(0 \leq t < \infty)$ be discrete random variables taking on nonnegative integers only and suppose that

$$\lim_{t \to \infty} \mathbf{P}\left\{\frac{\nu(t)}{t} \leq x\right\} = G(x) \tag{48}$$

in every continuity point of the distribution function $G(x)$. Let us suppose that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are mutually independent and identically distributed random variables for which (46) holds. If $\{\xi_n\}$ and $\{\nu(t)\}$ are independent, and $G(0) = 0$, then

(49) $$\lim_{t\to\infty} \underset{\sim}{P}\{ \frac{\zeta(\nu(t) - a\nu(t)}{t^{1/\alpha}\rho(t)} \leq x\} = \underset{\sim}{P}\{\zeta\nu^{\frac{1}{\alpha}} \leq x\}$$

<u>in every continuity point of the distribution function</u> $\underset{\sim}{P}\{\zeta\nu^{\frac{1}{\alpha}} \leq x\}$. <u>In</u> (49) ζ <u>and</u> ν <u>are independent random variables for which</u> $\underset{\sim}{P}\{\zeta \leq x\} = R(x)$ <u>and</u> $\underset{\sim}{P}\{\nu \leq x\} = G(x)$.

<u>Proof</u>. Let us define the random variables $\zeta^*(n)$ $(n = 0,1,\dots)$ and $\nu^*(t)$ $(0 \leq t < \infty)$ in such a way that $\{\zeta^*(n), \nu^*(t)\}$ and $\{\zeta(n), \nu(t)\}$ have the same **joint** distribution function for all $n = 0,1,2,\dots$ and $t \geq 0$, and furthermore

(50) $$\underset{\sim}{P}\{\lim_{n\to\infty} \frac{\zeta^*(n) - an}{n^{1/\alpha}\rho(n)} = \zeta\} = 1$$

where $\underset{\sim}{P}\{\zeta \leq x\} = R(x)$ and

(51) $$\frac{\nu^*(t)}{t} \Rightarrow \nu \quad \text{as} \quad t \to \infty$$

where $\underset{\sim}{P}\{\nu \leq x\} = G(x)$ and ζ and ν are independent random variables. By (50) we can write that

(52) $$\zeta^*(n) - an = n^{1/\alpha}\rho(n)(\zeta+\eta(n))$$

where $\underset{\sim}{P}\{\lim_{n\to\infty} \eta(n) = 0\} = 1$ and by (51) we can write that

(53) $$\nu^*(t) = t(\nu+\omega(t))$$

where $\omega(t) \Rightarrow 0$ as $t \to \infty$. Thus we obtain that

(54) $$\frac{\zeta^*(\nu^*(t)) - a\nu^*(t)}{t^{1/\alpha}\,\rho(t)} = \frac{\rho(\nu^*(t))}{\rho(t)} [\zeta + \eta(\nu^*(t))][\nu + \omega(t)]^{\frac{1}{\alpha}} .$$

In (54) $\omega(t) \Longrightarrow 0$ as $t \to \infty$. By Lemma 4 $\eta(\nu^*(t)) \Longrightarrow 0$ as $t \to \infty$. We shall prove that

$$\frac{\rho(\nu^*(t))}{\rho(t)} \Rightarrow 1 \quad \text{as} \quad t \to \infty, \tag{55}$$

and thus it follows from (54) that

$$\frac{\zeta^*(\nu^*(t)) - a\nu^*(t)}{t^{1/\alpha}\rho(t)} \Rightarrow \zeta\nu^{\frac{1}{\alpha}} \tag{56}$$

as $t \to \infty$. This implies (49) which was to be proved.

It remains to prove (55). For any $\varepsilon > 0$ and $m > 0$ we can write that

$$\mathbf{P}\left\{\left|\frac{\rho(\nu^*(t))}{\rho(t)} - 1\right| > \varepsilon\right\} \leqq \mathbf{P}\left\{\frac{\nu^*(t)}{t} < \frac{1}{m}\right\} + \mathbf{P}\left\{\frac{\nu^*(t)}{t} > m\right\} + \sum_{n \in A_t} \mathbf{P}\{\nu^*(t) = n\} \tag{57}$$

where

$$A_t = \left\{n : \left|\frac{\rho(n)}{\rho(t)} - 1\right| > \varepsilon \text{ and } \frac{t}{m} \leqq n \leqq mt\right\}. \tag{58}$$

Let us assume that $x = m$ and $x = 1/m$ are continuity points of $G(x)$. If $t \to \infty$, then by (51) the sum of the first two terms on the right-hand side of (57) tends to $\mathbf{P}\{\nu < \frac{1}{m}\} + \mathbf{P}\{\nu > m\}$ which is arbitrarily close to zero for sufficiently large m values. By (47) we can conclude that for any m the set A_t is empty if t is sufficiently large. This proves (55).

We note that in the particular case where $\rho(n) = 1$ for $n = 1, 2, \ldots$ Theorem 3 reduces to the fifth statement of Theorem 2.

We **note also** that if we do not assume in Theorem 3 that $\{\xi_k\}$

and $\{\nu(t)\}$ are independent, and if

(59) $$\lim_{\substack{n \to \infty \\ t \to \infty}} \underset{\sim}{P}\left\{ \frac{\zeta(n) - na}{n^{1/\alpha} \rho(n)} \leqq x \text{ , } \frac{\nu(t)}{t} \leqq y \right\} = F(x, y)$$

in every continuity point of the distribution function $F(x, y)$, then **in some cases** (49) remains valid except that $\underset{\sim}{P}\{\zeta \leqq x \text{ , } \nu \leqq y\} = F(x, y)$.

If in particular

(60) $$\frac{\nu(t)}{t} \Rightarrow q \text{ as } t \to \infty$$

where q is a positive constant, then in the result mentioned above $\nu \equiv q$ (constant), and consequently Theorem 3 is valid without the assumption of independence. In fact this particular case can be proved **directly as follows.**

<u>Theorem 4.</u> <u>If</u> $\nu(t)$ $(0 \leqq t < \infty)$ <u>are discrete random variables taking on nonnegative integers only, if</u>

(61) $$\frac{\nu(t)}{t} \Rightarrow q \text{ as } t \to \infty$$

<u>where</u> q <u>is a positive constant, and if</u> $\xi_1, \xi_2, \dots, \xi_k, \dots$ <u>are mutually independent and identically distributed random variables for which</u> (46) <u>holds, then</u>

(62) $$\lim_{t \to \infty} \underset{\sim}{P}\left\{ \frac{\zeta(\nu(t)) - a\nu(t)}{(qt)^{1/\alpha} \rho(t)} \leqq x \right\} = R(x)$$

<u>regardlesss of whether</u> $\{\nu(t)\}$ <u>depends on</u> $\{\xi_k\}$ <u>or not.</u>

<u>Proof.</u> In proving (62) we may assume without loss of generality that $a = 0$. Let us define $\zeta(t) = \zeta(n)$ for $n \leqq t < n+1$ and introduce the following events:

$$A_t(x) = \{ \frac{\zeta(\nu(t))}{(qt)^{1/\alpha}\rho(t)} \leqq x\} ,$$

$$A_t^*(x) = \{ \frac{\zeta(qt)}{(qt)^{1/\alpha}\rho(t)} \leqq x\} ,$$

$$B_t = \{|\nu(t) - qt| < \varepsilon^{\alpha+1} t\} ,$$

and

$$C_t = \{ \max_{|n-qt|<\varepsilon^{\alpha+1}t} |\zeta(n) - \zeta(qt)| < \varepsilon(qt)^{1/\alpha}\rho(t)\} \tag{63}$$

for $\varepsilon > 0$ and $t > 0$. We can easily see that

$$\underset{\sim}{P}\{A_t(x)B_tC_t\} \leqq \underset{\sim}{P}\{A_t(x)\} \leqq \underset{\sim}{P}\{A_t(x)B_tC_t\} + \underset{\sim}{P}\{\overline{B}_t\} + \underset{\sim}{P}\{\overline{C}_t\}$$

and

$$\underset{\sim}{P}\{A_t^*(x-\varepsilon)\} - \underset{\sim}{P}\{\overline{B}_t\} - \underset{\sim}{P}\{\overline{C}_t\} \leqq \underset{\sim}{P}\{A_t(x)B_tC_t\} \leqq \underset{\sim}{P}\{A_t^*(x+\varepsilon)\}$$

hold for $\varepsilon > 0$ and $t > 0$.

Since $\lim_{t\to\infty} \rho(qt)/\rho(t) = 1$, it follows from (46) that $\lim_{t\to\infty} \underset{\sim}{P}\{A_t^*(x)\} = R(x)$ for every x. By (61) we have $\lim_{t\to\infty} \underset{\sim}{P}\{\overline{B}_t\} = 0$, and we shall prove presently that

$$\lim_{\varepsilon\to 0} \limsup_{t\to\infty} \underset{\sim}{P}\{\overline{C}_t\} = 0 . \tag{64}$$

Thus it follows that $\lim_{t\to\infty} \underset{\sim}{P}\{A_t(x)\} = R(x)$ for every x which proves (62).

Let

$$D_m(\varepsilon) = \{|\zeta(n)| > \varepsilon m^{1/\alpha}\rho(m) \text{ for some } n = 1,2,\ldots, [2m\,\varepsilon^{\alpha+1}]\} .$$

Then we have $\underset{\sim}{P}\{\overline{C}_t\} \leq 2\underset{\sim}{P}\{D_m(\varepsilon)\}$ if $m \leq t < m+1$ and m is sufficiently large. We shall prove that

$$\lim_{\varepsilon \to 0} \limsup_{m \to \infty} \underset{\sim}{P}\{D_m(\varepsilon)\} = 0 \tag{65}$$

and this implies (64). If in (46), $R(0) = 0$ or $R(0) = 1$, then (65) is trivially true. If $0 < R(0) < 1$, then let

$$r = \inf\{\underset{\sim}{P}\{\zeta(n) \geq 0\} \text{ and } \underset{\sim}{P}\{\zeta(n) \leq 0\} \text{ for } n = 1,2,\dots\}.$$

Let us prove that $r > 0$. Since $\lim_{n \to \infty} \underset{\sim}{P}\{\zeta(n) \geq 0\} = 1-R(0) > 0$ and $\lim_{n \to \infty} \underset{\sim}{P}\{\zeta(n) \leq 0\} = R(0) > 0$, therefore $r = 0$ would imply that $\underset{\sim}{P}\{\zeta(n) \geq 0\} = 0$ or $\underset{\sim}{P}\{\zeta(n) \leq 0\} = 0$ for some $n = 1,2,\dots$. In the first case necessarily $R(0) = 0$ and in the second case $R(0) = 1$. This contradiction proves that $r > 0$ if $0 < R(0) < 1$.

If $0 < R(0) < 1$, $\varepsilon > 0$ and $m = 1,2,\dots,$ then we have

$$\underset{\sim}{P}\{D_m(\varepsilon)\} \leq \frac{1}{r} \underset{\sim}{P}\{|\zeta(2m\, \varepsilon^{\alpha+1})| > \varepsilon\, m^{1/\alpha} \rho(m)\}. \tag{66}$$

To prove (66) let us write $z = \varepsilon\, m^{1/\alpha} \rho(m)$ and $N = [2m\, \varepsilon^{\alpha+1}]$, and denote by τ the smallest $n = 1,2,\dots$ for which $|\zeta(n)| > z$. Then we have

$$\underset{\sim}{P}\{|\zeta(n)| > z \text{ for some } n = 1,2,\dots,N\} = \sum_{k=1}^{N} [\underset{\sim}{P}\{\tau=k,\ \zeta(k) > z\} + \underset{\sim}{P}\{\tau =$$

$$= k,\ \zeta(k) < -z\}] \leq \frac{1}{r} \sum_{k=1}^{N} [\underset{\sim}{P}\{\tau = k,\ \zeta(k) > z\}\, \underset{\sim}{P}\{\zeta(N)-\zeta(k) \geq 0\} + \underset{\sim}{P}\{\tau =$$

$$= k,\ \zeta(k) < -z\} \underset{\sim}{P}\{\zeta(N)-\zeta(k) \leq 0\}] \leq \frac{1}{r} \sum_{k=1}^{N} \underset{\sim}{P}\{\tau = k,\ |\zeta(N)| > z\} =$$

$$= \frac{1}{r} \underset{\sim}{P}\{|\zeta(N)| > z\}$$

which proves (66) . By (66)

$$\limsup_{m\to\infty} \mathbf{P}\{D_m(\varepsilon)\} \leqq \frac{1}{r}\left[1 - R\left(\frac{1}{(2\varepsilon)^{1/\alpha}}\right) + R\left(-\frac{1}{(2\varepsilon)^{1/\alpha}}\right)\right]$$

and if $\varepsilon \to 0$, the right-hand side tends to 0 . This proves (65).

We note that Theorem 4 can be generalized in the following way.

Theorem 5. *If* $\nu(t)$ $(0 \leqq t < \infty)$ *are discrete random variables taking on nonnegative integers only, if*

$$\frac{\nu(t)}{t} \Rightarrow \nu \tag{67}$$

as $t \to \infty$ *where* ν *is a positive random variable, and if,* $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ *are mutually independent and identically distributed random variables for which* (46) *holds, then*

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\zeta(\nu(t)) - a\nu(t)}{t^{1/\alpha}\rho(t)} \leqq x\right\} = \int_0^\infty R\left(\frac{x}{y^{1/\alpha}}\right) d\mathbf{P}\{\nu \leqq y\} \tag{68}$$

regardless of whether $\{\nu(t)\}$ *depends on* $\{\xi_k\}$ *or not.*

Finally, we shall give a brief historical review of the problem of finding the asymptotic distribution of a sum of a random number of random variables.

In 1938 W. Doeblin [679] proved Theorem 4 in the case where $R(x) = \Phi(x)$, the normal distribution function, and $\rho(n) \equiv 1$. In 1948 H. Robbins [165], [696] proved Theorem 1. In 1952 F. J. Anscombe [676] proved

Theorem 4. In 1955 R. L. Dobrushin [678] proved Theorem 2. It should be noted that while Dobrushin's results are correct in his proof in one place weak convergence should be replaced by convergence with probability 1. In 1957 the author [698],[699],[700],[701],[702] found the asymptotic distribution of sums of a random number of random variables where the number of variables depends on the variables themselves. In the papers [698],[699] direct methods are used, and in the papers [700],[701],[702] Theorem 2 is used. In 1957 A. Rényi [692] showed that a result of the author [698] can be obtained by a theorem which, as it turned out, was found first by W. Doeblin [679] and which is a particular case of a theorem of F. J. Anscombe [676].

In 1960 A. Rényi [693] proved that if $\zeta(n) = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ where $\{\xi_n\}$ is a sequence of mutually independent and identically distributed random variables for which $E\{\xi_n\} = 0$ and $E\{\xi_n^2\} = 1$, if $\{\nu_n, n = 1,2,\ldots\}$ is a sequence of positive random variables taking on integers only and if

$$\frac{\nu_n}{n} \Rightarrow \nu \text{ as } n \to \infty \tag{69}$$

where ν is a positive discrete random variable, then

$$\lim_{n\to\infty} P\left\{\frac{\zeta(\nu_n)}{\sqrt{\nu_n}} \leq x\right\} = \Phi(x) , \tag{70}$$

where $\Phi(x)$ is the normal distribution function. In 1962 J. Mogyoródi [689] and in 1963 J. R. Blum, D. L. Hanson and J. I. Rosenblatt [677] proved that if in (69) ν is an arbitrary positive random variable, then (70) holds unchangeably. In 1964 H. Wittenberg [706] extended the above result to the case when the random variables $\{\xi_n\}$ belong to the domain of attraction of a stable distribution function and (69) holds with a positive random variable ν. (Theorem 5.) For other extensions of the results mentioned above we refer to M. Csörgő and R. Fischler [681].

Theorem 4 has been extended by J. Mogyoródi [688] for non-identically distributed random variables, and Theorem 3 has been extended by H. Teicher [703] for vector random variables.

The Maximum of Sums of Independent Random Variables. Our main interest is to find the asymptotic distribution of the maximum of partial sums of mutually independent and identically distributed real random variables. We shall assume that $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ is a sequence of mutually independent and identically distributed random variables for which $P\{\xi_k \leq x\} = F(x)$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. We shall consider the random variable

$$\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n) \tag{71}$$

for $n = 0,1,2,\ldots$ and our aim is to find the asymptotic distribution of η_n as $n \to \infty$. In Chapter II we gave methods for finding the distribution of η_n for $n = 1,2,\ldots$. In Section 44 of this chapter we found the asymptotic distribution of ζ_n as $n \to \infty$. In fact we proved that if $F(x)$ belongs to the domain of attraction of a nondegenerate stable distribution function $R(x)$ of type $S(\alpha,\beta,c,m)$ then there exist constants A_n and $B_n > 0$ such that

$$\lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\zeta_n - A_n}{B_n} \leq x \right\} = R(x) \ . \tag{72}$$

We can choose B_n $(n = 0,1,\ldots)$ in such a way that $B_0 \leq B_1 \leq B_2 \leq \cdots \leq B_n \leq \cdots$, $\lim_{n \to \infty} B_n = \infty$ and

$$B_n = n^{1/\alpha} \rho(n) \tag{73}$$

where

$$\lim_{n \to \infty} \frac{\rho(\omega n)}{\omega(n)} = 1 \tag{74}$$

for all $\omega > 0$. (See Problem 46.12.) In this section we shall show that if the above conditions are satisfied then η_n has an asymptotic distribution as $n \to \infty$.

Beside (71) we shall also consider other functionals defined on the sequence of random variables $\{\zeta_n\}$.

First, let us consider the case where $\{\xi_k\}$ is a sequence of mutually independent and identically distributed random variables for which $\underset{\sim}{E}\{\xi_k\} = 0$ and $\underset{\sim}{E}\{\xi_k^2\} = 1$. Then we have

$$\lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\zeta_n}{\sqrt{n}} \leq x \right\} = \Phi(x) \tag{75}$$

for all x where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \tag{76}$$

is the normal distribution function.

Now let us suppose that $\{\xi(u),\ 0 \leq u < \infty\}$ is a Brownian motion process, that is, a family of random variables for which $\underset{\sim}{P}\{\xi(0) = 0\} = 1$,

$P\{\xi(u+t) - \xi(t) \leq x\} = \Phi(x/\sqrt{u})$ if $u > 0$, and $t \geq 0$ and $\xi(u_1)-\xi(u_0)$, $\xi(u_2) - \xi(u_1), \ldots, \xi(u_k) - \xi(u_{k-1})$ are mutually independent random variables for any $0 \leq u_0 < u_1 < u_2 < \ldots < u_k$ and $k = 2,3,\ldots$.

For any $u \geq 0$ and $n = 1,2,\ldots$ let us define

$$\xi_n(u) = \frac{\zeta_{[nu]}}{\sqrt{n}} . \tag{77}$$

Then $\{\xi_n(u), 0 \leq u < \infty\}$ is a stochastic process for which $P\{\xi_n(0) = 0\} = 1$,

$$\lim_{n \to \infty} P\{\xi_n(t+u) - \xi_n(t) \leq x\} = \Phi\left(\frac{x}{\sqrt{u}}\right) \tag{78}$$

if $u > 0$ and $t \geq 0$ and $\xi_n(u_1) - \xi_n(u_0), \xi_n(u_2) - \xi_n(u_1), \ldots, \xi_n(u_k)-\xi_n(u_{k-1})$ are mutually independent random variables for any $0 \leq u_0 < u_1 < u_2 < \ldots < u_k$ and $k = 2,3,\ldots$ and $n = 1,2,\ldots$.

By (78) it follows immediately that

$$\begin{aligned} &\lim_{n \to \infty} P\{\xi_n(t_1) \leq x_1, \xi_n(t_2) \leq x_2, \ldots, \xi_n(t_k) \leq x_k\} = \\ &= P\{\xi(t_1) \leq x_1, \xi(t_2) \leq x_2, \ldots, \xi(t_k) \leq x_k\} \end{aligned} \tag{79}$$

for $0 \leq t_1 < t_2 < \ldots < t_k$ and $k = 1,2,\ldots$, that is, the finite dimensional distributions of the process $\{\xi_n(u), 0 \leq u < \infty\}$ converge to the corresponding finite dimensional distributions of the process $\{\xi(u), 0 \leq u < \infty\}$.

By (79) it follows that if $a(u)$ and $b(u)$ are two real functions defined for $0 \leq u \leq t$ and $0 \leq t_1 < t_2 < \ldots < t_k \leq t$ where $k = 1,2,\ldots,$ then

$$\lim_{n\to\infty} \underset{\sim}{P}\{a(t_i) \leqq \xi_n(t_i) \leqq b(t_i) \quad \text{for} \quad i = 1,2,\ldots,k\} =$$

(80)

$$= \underset{\sim}{P}\{a(t_i) \leqq \xi(t_i) \leqq b(t_i) \quad \text{for} \quad i = 1,2,\ldots,k\} .$$

By (80) we would expect that if $\{a(u) \leqq \xi(u) \leqq b(u) \ \text{for} \ 0 \leqq u \leqq t\}$ is a random event concerning the process $\{\xi(u) , 0 \leqq u < \infty\}$, then we have

$$\lim_{n\to\infty} \underset{\sim}{P}\{a(u) \leqq \xi_n(u) \leqq b(u) \quad \text{for} \quad 0 \leqq u \leqq t\} =$$

(81)

$$= \underset{\sim}{P}\{a(u) \leqq \xi(u) \leqq b(u) \quad \text{for} \quad 0 \leqq u \leqq t\} .$$

If we suppose that the functions $a(u)$ and $b(u)$ $(0 \leqq u \leqq t)$ behave reasonably well and if we suppose, for example, that the process $\{\xi(u), 0 \leqq u < \infty\}$ is separable (see **Section 47**), then $\{a(u) \leqq \xi(u) \leqq b(u) \ \text{for} \ 0 \leqq u \leqq t\}$ is a random event and therefore the probability on the right-hand side of (81) is defined. Even if the right-hand side of (81) is defined, we are still left with the problem of whether (81) is true?

This problem was solved for the first time in 1931 by A. N. Kolmogorov [744],[745]. Actually Kolmogorov considered a somewhat different case. He did not assume that the random variables $\{\xi_k\}$ are identically distributed, but assumed that Liapounoff's conditions are satisfied for $\{\xi_k\}$. In this case too ζ_n has an asymptotic normal distribution as $n \to \infty$. Under these conditions Kolmogorov proved that if $a(u)$ and $b(u)$ $(0 \leqq u \leqq t)$ satisfy some differentiability conditions, then (81) holds and the probability on the right-hand side of (81) can be obtained by solving the heat-equation

$$\frac{\partial f(u,x)}{\partial u} = \frac{1}{2}\,\frac{\partial^2 f(u,x)}{\partial x^2} \tag{82}$$

in the domain $\{(u,x) : 0 < u < t\ ,\ a(u) < x < b(u)\}$ with the boundary conditions:

$$\begin{aligned} f(u,a(u)) &= 0 \quad \text{for} \quad 0 \leq u < t\ , \\ f(u,b(u)) &= 0 \quad \text{for} \quad 0 \leq u < t\ , \\ f(0,x) &= 0 \quad \text{for} \quad x \neq 0\ . \end{aligned} \tag{83}$$

The probability (81) can be obtained as the integral of $f(t,x)$ from $x = a(t)$ to $x = b(t)$.

In 1946 P. Erdős and M. Kac [730] proved the following result.

Theorem 6. *Let* $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ *be mutually independent and identically distributed random variables for which* $E\{\xi_n\} = 0$ *and* $E\{\xi_n^2\} = 1$. *Let* $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ *for* $n = 1,2,\ldots,$ $\zeta_0 = 0$ *and* $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$ *for* $n = 1,2,\ldots$. *Then we have*

$$\lim_{n\to\infty} P\left\{ \frac{\eta_n}{\sqrt{n}} \leq x \right\} = G(x) \tag{84}$$

where

$$G(x) = \begin{cases} 2\Phi(x) - 1 & \text{for} \quad x \geq 0\ , \\ 0 & \text{for} \quad x < 0\ . \end{cases} \tag{85}$$

Proof. If $x < 0$, then $G(x) = 0$. Let us suppose that $x \geq 0$. First we shall prove (84) in the particular case when

$$P\{\xi_k = 1\} = P\{\xi_k = -1\} = \frac{1}{2} \tag{86}$$

for $k = 1,2,\ldots$. In this case by the theory of random walks we obtain easily that

(87) $$P\{\eta_n < a\} = P\{\zeta_n < a\} - P\{\zeta_n < -a\}$$

for $a = 1,2,\ldots$. If we put $a = a_n = [x\sqrt{n}]$ in (87) and let $n \to \infty$, then by (75) we obtain that

(88) $$\lim_{n\to\infty} P\{\frac{\eta_n}{\sqrt{n}} \leqq x\} = \lim_{n\to\infty} P\{-x \leqq \frac{\zeta_n}{\sqrt{n}} \leqq x\} = \Phi(x) - \Phi(-x)$$

for $x \geqq 0$ which proves (85) in this particular case.

Next, let us suppose that $\xi_1^*, \xi_2^*,\ldots, \xi_n^*,\ldots$ are mutually independent random variables for which $P\{\xi_n^* \leqq x\} = \Phi(x)$. Let $\zeta_n^* = \xi_1^* + \xi_2^* + \ldots + \xi_n^*$ for $n = 1,2,\ldots$, $\zeta_0^* = 0$, and $\eta_n^* = \max(\zeta_0^*, \zeta_1^*,\ldots, \zeta_n^*)$ for $n = 0,1,2,\ldots$.

Define

(89) $$G_n(x) = P\{\eta_n \leqq x\sqrt{n}\} .$$

Now we shall prove that for every $k = 1,2,\ldots$ and $\varepsilon > 0$ we have the inequality

(90) $$P\{\eta_k^* \leqq (x-\varepsilon)\sqrt{k}\} - \frac{1}{\varepsilon^2 k} \leqq \liminf_{n\to\infty} G_n(x) \leqq \limsup_{n\to\infty} G_n(x) \leqq P\{\eta_k^* \leqq x\sqrt{k}\} .$$

For every $n = 1,2,\ldots$ and every $k = 1,2,\ldots$ let us define

(91) $$n_j = [\frac{nj}{k}] \quad (j = 0,1,\ldots,k)$$

and write

(92) $$G_{nk}(x) = P\{\max(\zeta_{n_0}, \zeta_{n_1},\ldots, \zeta_{n_k}) \leqq x\sqrt{n}\} .$$

By (75) we obtain that

(93) $$\lim_{n \to \infty} G_{nk}(x) = \underset{\sim}{P}\{\max(\zeta_0^*, \zeta_1^*, \ldots, \zeta_k^*) \leq x\sqrt{k}\}.$$

For by (75) we have

(94) $$\lim_{n \to \infty} \underset{\sim}{P}\{\zeta_{n_1} \leq x_1\sqrt{n}, \zeta_{n_2} - \zeta_{n_1} \leq x_2\sqrt{n}, \ldots, \zeta_{n_k} - \zeta_{n_{k-1}} \leq x_k\sqrt{n}\} =$$
$$= \Phi(x_1\sqrt{k})\Phi(x_2\sqrt{k})\ldots\Phi(x_k\sqrt{k}) = \underset{\sim}{P}\{\xi_1^* \leq x_1\sqrt{k}, \xi_2^* \leq x_2\sqrt{k}, \ldots, \xi_k^* \leq x_k\sqrt{k}\}$$
$$= \underset{\sim}{P}\{\zeta_1^* \leq x_1\sqrt{k}, \zeta_2^* - \zeta_1^* \leq x_2\sqrt{k}, \ldots, \zeta_k^* - \zeta_{k-1}^* \leq x_k\sqrt{k}\}$$

for any $x_1, x_2, \ldots, x_k$. This implies (93).

Let

(95) $$Q_r(x) = \underset{\sim}{P}\{\zeta_0 \leq x\sqrt{n}, \zeta_1 \leq x\sqrt{n}, \ldots, \zeta_{r-1} \leq x\sqrt{n}, \zeta_r > x\sqrt{n}\}.$$

Then

(96) $$\sum_{r=1}^{n} Q_r(x) = 1 - G_n(x) \leq 1.$$

For $n_i < r \leq n_{i+1}$ $(i = 0,1,\ldots,k-1)$ we have

(97) $$Q_r(x) = \underset{\sim}{P}\{\zeta_0 \leq x\sqrt{n}, \ldots, \zeta_{r-1} \leq x\sqrt{n}, \zeta_r > x\sqrt{n}, |\zeta_{n_{i+1}} - \zeta_r| \geq \varepsilon\sqrt{n}\} +$$
$$+ \underset{\sim}{P}\{\zeta_0 \leq x\sqrt{n}, \ldots, \zeta_{r-1} \leq x\sqrt{n}, \zeta_r > x\sqrt{n}, |\zeta_{n_{i+1}} - \zeta_r| < \varepsilon\sqrt{n}\}.$$

For any $\varepsilon > 0$ the first term on the right-hand side of (97) is

(98) $$Q_r(x)\,\underset{\sim}{P}\{|\zeta_{n_{i+1}} - \zeta_r| \geq \varepsilon\sqrt{n}\} \leq Q_r(x)\,\frac{1}{k\varepsilon^2}$$

which follows from Theorem 41.3 being $\underset{\sim}{E}\{(\zeta_{n_{i+1}} - \zeta_r)^2\} = n_{i+1}-r \leq \frac{n}{k}$.

Thus from (97) and (98) it follows that

$$1 - G_n(x) = \sum_{r=1}^{n} Q_r(x) \leq \frac{1}{k\varepsilon^2} +$$

$$(99) \qquad + \sum_{i=0}^{n-1} \sum_{n_i < r \leq n_{i+1}} \underset{\sim}{P}\{\zeta_0 \leq x\sqrt{n}, \ldots, \zeta_{r-1} \leq x\sqrt{n},\ \zeta_r > x\sqrt{n},\ |\zeta_{n_{i+1}} - \zeta_r| < \varepsilon\sqrt{n}\}$$

$$\leq \frac{1}{k\varepsilon^2} + \underset{\sim}{P}\{\max(\zeta_{n_0}, \zeta_{n_1}, \ldots, \zeta_{n_k}) > (x-\varepsilon)\sqrt{n}\} \quad ,$$

that is,

$$(100) \qquad 1 - G_n(x) \leq \frac{1}{k\varepsilon^2} + 1 - G_{nk}(x-\varepsilon)$$

for any x and $\varepsilon > 0$. Since evidently $G_n(x) < G_{nk}(x)$, it follows that

$$(101) \qquad G_{nk}(x-\varepsilon) - \frac{1}{k\varepsilon^2} < G_n(x) < G_{nk}(x)$$

for all x and $\varepsilon > 0$. If we let $n \to \infty$ in (101), then we obtain (90).

If we apply (90) to the random variables (86), then we obtain that

$$(102) \qquad \underset{\sim}{P}\{\eta_k^* \leq (x-\varepsilon)\sqrt{k}\} - \frac{1}{k\varepsilon^2} \leq G(x) \leq \underset{\sim}{P}\{\eta_k^* \leq x\sqrt{k}\}$$

where $G(x)$ is given by (85). If we replace x by $x+\varepsilon$ in (102) then we get

$$(103) \qquad G(x) \leq \underset{\sim}{P}\{\eta_k^* \leq x\sqrt{k}\} \leq G(x+\varepsilon) + \frac{1}{k\varepsilon^2}$$

and hence by (90), (102) and (103)

$$(104)\qquad G(x-\varepsilon)-\frac{1}{k\varepsilon^2} \leq \liminf_{n\to\infty} G_n(x) \leq \limsup_{n\to\infty} G_n(x) \leq G(x+\varepsilon) + \frac{1}{k\varepsilon^2}$$

for any x and $\varepsilon > 0$ and $k = 1,2,\ldots$. Let $k \to \infty$ and $\varepsilon \to 0$ in (104). Since $G(x)$ is continuous, we obtain that

$$(105)\qquad \lim_{n\to\infty} G_n(x) = G(x)$$

for any x where $G(x)$ is given by (85). This completes the proof of the theorem.

In the above proof it has been demonstrated that if $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are mutually independent and identically distributed random variables for which $\mathbf{E}\{\xi_n\} = 0$ and $\mathbf{E}\{\xi_n^2\} = 1$, then the limiting distribution (84) exists and $G(x)$ does not depend on the distribution function $\mathbf{P}\{\xi_n \leq x\}$. In the particular case where ξ_n has the distribution (86) it is easy to show that $G(x)$ is given by (85) and consequently (84) holds with the same $G(x)$ for all sequences $\{\xi_n\}$ which satisfy the requirements stated above.

From the above result it follows immediately that if $\{\xi(u) , 0 \leq u < \infty\}$ is a Brownian motion process for which $\mathbf{E}\{\xi(u)\} = 0$ and $\mathbf{E}\{[\xi(u)]^2\} = u$ for $u \geq 0$, then

$$(106)\qquad \lim_{n\to\infty} \mathbf{P}\{\max_{0\leq k\leq n} \xi(\tfrac{kt}{n}) \leq x\sqrt{t}\} = \lim_{n\to\infty} \mathbf{P}\{\max_{0\leq k\leq n} \xi(k) \leq x\sqrt{n}\} = G(x)$$

for all x and $t > 0$ where $G(x)$ is given by (85). If $\{\xi(u) , 0 \leq u < \infty\}$ is a separable Brownian motion process, then it follows from (106) that

(107) $$\underset{\sim}{P}\{\sup_{0 \leq u \leq t} \xi(u) \leq x\sqrt{t}\} = G(x)$$

for all x and $t > 0$.

Similar invariance properties can be proved for other functionals of the sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$.

If we suppose again that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of mutually independent and identically distributed random variables for which $\underset{\sim}{E}\{\xi_n\} = 0$ and $\underset{\sim}{E}\{\xi_n^2\} = 1$ and we write $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1, 2, \ldots$ and $\zeta_0 = 0$, then we have

(108) $$\lim_{n\to\infty} \underset{\sim}{P}\{\max(|\zeta_0|, |\zeta_1|, \ldots, |\zeta_n|) \leq x\sqrt{n}\} = \sum_{k=-\infty}^{\infty} (-1)^k \Phi(2kx+x) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} e^{-\frac{(2j+1)^2\pi^2}{8x^2}}$$

for $x > 0$. P. Erdős and M. Kac [730] proved that the limiting distribution (108) does not depend on the distribution of ξ_n and the particular case (86) yields (108). See also A. Wald [766] and Theorem 37.2.

In the above case P. Erdős and M. Kac [730] also found the limiting distributions

(109) $$\lim_{n\to\infty} \underset{\sim}{P}\{\zeta_0^2 + \zeta_1^2 + \ldots + \zeta_n^2 \leq n^2 x\}$$

and

(110) $$\lim_{n\to\infty} \underset{\sim}{P}\{|\zeta_0| + |\zeta_1| + \ldots + |\zeta_n| \leq n^{3/2} x\}.$$

In 1947 P. Erdős and M. Kac [731] proved that if $\Delta(n)$ denotes the number of positive partial sums in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$, then

$$(111) \qquad \lim_{n \to \infty} \underset{\sim}{P}\{ \frac{\Delta(n)}{n} \leq x\} = \frac{2}{\pi} \text{arc} \sin\sqrt{x}$$

for $0 \leq x \leq 1$. This limiting distribution can be deduced from a result found in 1940 by P. Lévy [292].

Further examples for invariant results have been given by A. M. Mark [748] and R. Fortet [734] .

In 1951 M. D. Donsker [728]extended the above results for a large class of functionals defined on the sequence of random variables $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$. Donsker's result can be formulated in the following way:

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed real random variables for which $\underset{\sim}{E}\{\xi_n\} = 0$ and $\underset{\sim}{E}\{\xi_n^2\} = 1$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$ and define

$$(112) \qquad \xi_n^*(u) = \frac{\zeta_{[nu]} + (nu - [nu])\xi_{[nu+1]}}{\sqrt{n}}$$

for $u \geq 0$. Then the stochastic process $\{\xi_n^*(u) , 0 \leq u < \infty\}$ has continuous sample functions and the finite dimensional distributions of the process $\{\xi_n^*(u) , 0 \leq u < \infty \}$ converge to the corresponding finite dimensional distributions of a Brownian motion process $\{\xi(u) , 0 \leq u < \infty\}$, that is,

$$\lim_{n\to\infty} \underset{\sim}{P}\{\xi_n^*(t_1) \leqq x_1,\ \xi_n^*(t_2) \leqq x_2, \dots,\ \xi_n^*(t_k) \leqq x_k\} = \tag{113}$$
$$\underset{\sim}{P}\{\xi(t_1) \leqq x_1,\ \xi(t_2) \leqq x_2, \dots,\ \xi(t_k) \leqq x_k\}$$

for any $0 \leqq t_1 < t_2 < \dots < t_k$ and $k = 1,2,\dots$. This follows immediately from (79) because $\xi_{[nu+1]}/\sqrt{n} \Rightarrow 0$ as $n \to \infty$ for any $u \geqq 0$.

To present Donsker's theorem we shall first define a probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$ which was introduced in 1923 by N. Wiener [767].

Let Ω , the sample space, be the set of continuous functions defined on the interval $[0, t]$. We shall use the notation $C[0, t]$ for denoting this set of functions.

Let $\mathcal{B}$, the class of random events, be the smallest σ - algebra which contains the sets

$$A(u,x) = \{f : f(u) \leqq x \text{ and } f \in C[0,t]\} \tag{114}$$

for all $u \in [0,t]$ and $x \in (-\infty, \infty)$.

Let us assume that

$$\underset{\sim}{P}\{A(t_1, x_1)A(t_2, x_2)\dots A(t_k, x_k)\} = \tag{115}$$
$$\frac{1}{\sqrt{(2\pi)^k(t_1-t_0)(t_2-t_1)\dots(t_k-t_{k-1})}} \underset{\substack{y_1+\dots+y_i \leqq x_i \\ (i=1,2,\dots,k)}}{\iint\dots\int} e^{-\frac{1}{2}\sum_{i=1}^{k}\frac{y_i^2}{t_i-t_{i-1}}} dy_1 dy_2 \dots dy_k$$

for $0 = t_0 < t_1 < t_2 < \dots < t_k$ $(k = 1,2,\dots)$ and all real $x_1, x_2, \dots, x_k$.

By Carathéodory's extension theorem (see Theorem 1.2 in the Appendix) we can prove that there is a unique probability $\underset{\sim}{P}\{A\}$ defined for $A \in \mathcal{B}$ which satisfies (115). Let us choose this probability as $\underset{\sim}{P}$ in the probability space $(\Omega, \mathcal{B}, \underset{\sim}{P})$.

Let us define a family of random variables $\{\xi(u), 0 \leqq u \leqq t\}$ in the following way:

$$(116) \qquad \xi(u) = \xi(u\,;\,\omega) = f(u)$$

for $0 \leqq u \leqq t$ whenever $\omega = f(u) \in \Omega = C[0, t]$. In this case $\{\xi(u), 0 \leqq u \leqq t\}$ is a Brownian motion process for which the sample functions are continuous functions of u for every $\omega \in \Omega$.

In the space $C[0, t]$ let us define the norm of a function $f(u)$ $(0 \leqq u \leqq t)$ by

$$(117) \qquad \|f\| = \sup_{0 \leqq u \leqq t} |f(u)| \; .$$

We define the distance between two functions $f(u)$ $(0 \leqq u \leqq t)$ and $g(u)$ $(0 \leqq u \leqq t)$ by

$$(118) \qquad d(f,g) = \|f-g\| = \sup_{0 \leqq u \leqq t} |f(u)-g(u)| \; .$$

With this distance function the space $C[0,t]$ becomes a metric space and we can define open sets, closed sets, compact sets, separability, completeness and so on in the same way as in Euclidean spaces.

A functional Q on $C[0,t]$ is a mapping, that is, a function, from

$C[0, t]$ to the set of real or complex numbers. The value of Q for $f \in C[0, t]$ will be denoted by $Q(f)$. The functional Q is said to be bounded if there exists a real constant $M \geqq 0$ such that $|Q(f)| \leqq M$ for all $f \in C[0, t]$. The functional Q is said to be continuous at $f \in C[0, t]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

(119) $$|Q(f) - Q(g)| < \varepsilon$$

whenever $||f-g|| < \delta$.

Now we can formulate Donsker's theorem in the following way:

<u>Theorem 7.</u> <u>Let</u> $(\Omega, \mathcal{B}, \underset{\sim}{P})$ <u>the Wiener probability space defined above and let</u> $\underset{\sim}{\xi} = \{\xi(u),\ 0 \leqq u \leqq t\}$ <u>be a Brownian motion process defined by</u> (116). <u>For each</u> $n = 1,2,\ldots$ <u>let</u> $\underset{\sim}{\xi}_n^* = \{\xi_n^*(u),\ 0 \leqq u \leqq t\}$ <u>be the stochastic process defined by</u> (112). <u>If</u> Q <u>is a real functional on</u> $C[0, t]$ <u>and if</u> Q <u>is almost everywhere continuous on</u> $C[0, t]$ <u>with respect to the probability</u> $\underset{\sim}{P}$, <u>then</u>

(120) $$\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n^*) \leqq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leqq x\}$$

<u>in every continuity point of the limiting distribution function.</u>

<u>Proof.</u> For any $A \in \mathcal{B}$ let us define

(121) $$\mu_n(A) = \underset{\sim}{P}\{\underset{\sim}{\xi}_n^* \in A\},$$

that is, $\mu_n(A)$ is the probability that $\underset{\sim}{\xi}_n^* = \{\xi_n^*(u),\ 0 \leqq u \leqq t\}$ belongs to A, and

(122) $$\mu(A) = \underset{\sim}{P}\{\underset{\sim}{\xi} \in A\}$$

that is, $\mu(A)$ is the probability that $\underset{\sim}{\xi} = \{\xi(u), 0 \leq u \leq t\}$ belongs to A. Probability (121) is determined by the distribution function of the random variables $\{\xi_k\}$, and probability (122) is determined by $\underset{\sim}{P}$.

We say that the sequence of measures μ_n $(n = 1,2,\ldots)$ converges weakly to the measure μ and write $\mu_n \Rightarrow \mu$ if

(123) $$\lim_{n\to\infty} \int_{\Omega} Q(f)d\mu_n = \int_{\Omega} Q(f)d\mu$$

for all continuous bounded functionals Q on Ω.

We say that the sequence of measures μ_n $(n = 1,2,\ldots)$ is weakly compact if every subsequence of $\{\mu_n\}$ contains a weakly convergent subsequence.

We shall prove the theorem in several steps. First, we shall prove that for any $\varepsilon > 0$

(124) $$\lim_{h\to 0} \limsup_{n\to\infty} \underset{\sim}{P}\{\sup_{|u-v|\leq h} |\xi_n^*(u) - \xi_n^*(v)| > \varepsilon\} = 0 .$$

Second, we shall show that for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \in B$ such that

(125) $$\mu_n(K_\varepsilon) \geq 1 - \varepsilon$$

for all $n = 1,2,\ldots$. A set K_ε is said to be compact if every class of open sets which covers K_ε contains a finite subclass which is also a covering of K_ε.

Third, we shall show that the sequence $\{\mu_n\}$ is weakly compact.

Fourth, by (113) we conclude that $\mu_n \Longrightarrow \mu$ and this easily implies (120).

In what follows we shall need the following inequality: Let $\xi_1, \xi_2, \ldots, \xi_n$ be mutually independent random variables for which $E\{\xi_k\} = 0$ and $E\{\xi_k^2\} = 1$. Write $\zeta_k = \xi_1 + \xi_2 + \ldots + \xi_k$ for $k = 1,2,\ldots, n$. Then

(126) $$P\{\max_{1\leq k\leq n} |\zeta_k| > \varepsilon\} \leq \frac{P\{|\zeta_n| > \frac{\varepsilon}{2}\}}{1 - \frac{4n}{\varepsilon^2}}$$

for $\varepsilon > 2\sqrt{n}$. This follows from the following inequality which holds for all $\varepsilon > 0$.

(127) $$P\{|\zeta_n| > \frac{\varepsilon}{2}\} \geq \sum_{k=1}^{n} P\{|\zeta_1| \leq \varepsilon, \ldots, |\zeta_{k-1}| \leq \varepsilon, |\zeta_k| > \varepsilon, |\zeta_n - \zeta_k| \leq \frac{\varepsilon}{2}\} =$$
$$= \sum_{k=1}^{n} P\{|\zeta_1| \leq \varepsilon, \ldots, |\zeta_{k-1}| \leq \varepsilon, |\zeta_k| > \varepsilon\} P\{|\zeta_n - \zeta_k| \leq \frac{\varepsilon}{2}\} .$$

Since

(128) $$P\{|\zeta_n - \zeta_k| \leq \frac{\varepsilon}{2}\} \geq 1 - \frac{4E\{(\zeta_n - \zeta_k)^2\}}{\varepsilon^2} = 1 - \frac{4(n-k)}{\varepsilon^2} \geq 1 - \frac{4n}{\varepsilon^2}$$

for $k = 1,2,\ldots,n$, it follows from (127) that

(129) $$P\{|\zeta_n| > \frac{\varepsilon}{2}\} \geq (1 - \frac{4n}{\varepsilon^2}) P\{\max_{1\leq k\leq n} |\zeta_k| > \varepsilon\}$$

for $\varepsilon > 0$ which proves (126).

Now for each $n = 1,2,\ldots$ and each $h > 0$ let us define $a_j = [nhj]/n$ $(j = 0,1,2,\ldots)$. If $n > 1/h$, then $a_{j+2} - a_j \geq h$. If $\sup|\xi_n^*(u) - \xi_n^*(v)| > \varepsilon$

whenever $|u-v| \leq h$, then, obviously, there is a $j = 0,1,\ldots, [t/h]$ such that $|\xi_n^*(u) - \xi_n^*(a_j)| > \frac{\varepsilon}{2}$ or $|\xi_n^*(v)-\xi_n^*(a_j)| > \frac{\varepsilon}{2}$ where $a_j \leqq u \leqq a_{j+3}$ and $a_j \leqq v \leqq a_{j+3}$. Thus by (126) it follows that

$$\begin{aligned} &\underset{\sim}{P}\{ \sup_{|u-v|\leqq h} |\xi_n^*(u)-\xi_n^*(v)|>\varepsilon\} \leqq 2 \sum_{j=0}^{[t/h]} \underset{\sim}{P}\{ \sup_{a_j\leqq u\leqq a_{j+3}} |\xi_n^*(u)-\xi_n^*(a_j)|>\frac{\varepsilon}{2}\} \\ (130)\quad & \\ &\leqq 2(1+\frac{t}{h})\underset{\sim}{P}\{ \sup_{0\leqq k<4nh} |\xi_n^*(\frac{k}{n})|>\frac{\varepsilon}{2}\} \leqq 2(1+\frac{t}{h}) \frac{\underset{\sim}{P}\{|\xi_n^*(\frac{[4nh]}{n})|>\frac{\varepsilon}{4}\}}{1-\frac{64h}{\varepsilon^2}} \end{aligned}$$

for $n > 1/h$ and $\varepsilon > 8\sqrt{h}$. If $n \to \infty$, then by (75) the extreme right member in (130) tends to

$$(131)\qquad \frac{4(1+\frac{t}{h})}{1-\frac{64h}{\varepsilon^2}} \frac{1}{\sqrt{2\pi}} \int_{\varepsilon/8\sqrt{h}}^{\infty} e^{-x^2/2}\, dx .$$

Since

$$(132)\qquad \int_{\varepsilon/8\sqrt{h}}^{\infty} e^{-x^2/2} dx \leqq \frac{64h}{\varepsilon^2} \int_{\varepsilon/8\sqrt{h}}^{\infty} x^2 e^{-x^2/2}\, dx ,$$

therefore (131) tends to 0 as $h \to 0$. This proves (124).

From (124) we can conclude that

$$(133)\qquad \lim_{h\to 0} \sup_{1\leqq n\leqq\infty} \underset{\sim}{P}\{ \sup_{|u-v|\leqq h} |\xi_n^*(u) - \xi_n^*(v)| > \varepsilon\} = 0$$

for all $\varepsilon > 0$. For we have

$$(134)\qquad \lim_{h\to 0} \max_{1\leqq n\leqq N} \underset{\sim}{P}\{ \sup_{|u-v|\leqq h} |\xi_n^*(u) - \xi_n^*(v)| > \varepsilon\}= 0$$

for any $\varepsilon > 0$ and $N = 1,2,\ldots$. On the other hand by (124)

$$(135) \qquad \sup_{N \leq n < \infty} \underset{\sim}{P}\{\sup_{|u-v| \leq h} |\xi_n^*(u) - \xi_n^*(v)| > \varepsilon\}$$

is arbitrarily close to 0 if N is sufficiently large and $h > 0$ is sufficiently small. This proves (133).

We can prove (125) by (133). For each $\varepsilon > 0$ and $r = 1,2,\ldots$ let us choose an $h_r > 0$ such that

$$(136) \qquad \sup_{1 \leq n < \infty} |\underset{\sim}{P}\{\sup_{|u-v| \leq h_r} |\xi_n^*(u) - \xi_n^*(v)| > \frac{1}{r}\} \leq \frac{\varepsilon}{2^r} .$$

Let us define a sequence of closed sets $F_0, F_1, F_2, \ldots$ in $\mathcal{B}$ in the following way: $F_0 = \{f : f(0) = 0\}$ and

$$(137) \qquad F_r = \{f : \sup_{|u-v| \leq h_r} |f(u) - f(v)| \leq \frac{1}{r}\}$$

and let

$$(138) \qquad K_\varepsilon = \bigcap_{r=0}^{\infty} F_r .$$

If $f \varepsilon K_\varepsilon$, then $f \varepsilon F_r$ for all $r = 0,1,\ldots,$ and therefore $f(0) = 0$ and

$$(139) \qquad \sup_{f \varepsilon K_\varepsilon} \sup_{|u-v| \leq h} |f(u) - f(v)| \to 0 \quad \text{as} \quad h \to 0 .$$

Furthermore, $\|f\| = \sup_{0 \leq u \leq t} |f(u)| < M < \infty$ for all $f \varepsilon K_\varepsilon$. This last statement follows from the inequality

$$(140) \qquad |f(u)| \leq |f(0)| + \sum_{i=1}^{m} |f(\frac{iu}{m}) - f(\frac{(i-1)u}{m})|$$

which holds for all $m = 1,2,\ldots$. If $f \varepsilon K_\varepsilon$ and $m > 1/h_r$, then by (140)

$||f|| \leq \frac{m}{r}$.

Accordingly, K_ε is closed and K_ε is a family of uniformly bounded and equicontinuous functions on $C[0,t]$. Thus by a theorem of C. Arzelà (see A. N. Kolmogorov and S. V. Fomin [102 p. 54]) K_ε is compact.

Since $\mu_n(F_0) = 1$ for $n = 1,2,\ldots,$ and

(141) $$\mu_n(F_r) > 1 - \frac{\varepsilon}{2^r}$$

for $r = 1,2,\ldots$ and $n = 1,2,\ldots,$ it follows that

(142) $$\mu_n(K_\varepsilon) \geq 1 - \sum_{r=0}^{\infty} [1-\mu_n(F_r)] \geq 1 - \sum_{r=1}^{\infty} \frac{\varepsilon}{2^r} = 1 - \varepsilon$$

for all $n = 1,2,\ldots$. This completes the proof of the second statement.

The third statement follows from a general theorem of Yu. V. Prochorov [756] (see Theorem 3.2 in the Appendix). According to this theorem the conditions $\mu_n(\Omega) = 1$ and (125) imply that $\{\mu_n\}$ is weakly compact.

If we assume that (113) is satisfied, then the weak compactness of $\{\mu_n\}$ implies that $\{\mu_n\} \Rightarrow \mu$. Now we are going to prove this statement.

For any set $A \in \mathcal{B}$ denote by $A^{(i)}$ the interior of A and by $A^{(c)}$ the closure of A , that is, $A^{(i)}$ contains the interior points of A and $A^{(c)}$ contains the limit points and isolated points of A .

By the third statement, every infinite subsequence of $\{\mu_n\}$ contains a weakly convergent subsequence $\{\mu_{n_k}\}$. That is there exists a measure $\overline{\mu}$

such that $\mu_{n_k} \Rightarrow \bar{\mu}$ as $k \to \infty$. We shall show that $\bar{\mu} = \mu$ for every weakly convergent sequence $\{\mu_{n_k}\}$. Hence it follows that the whole sequence $\{\mu_n\}$ is weakly convergent and $\mu_n \Rightarrow \mu$ as $n \to \infty$.

If $\mu_{n_k} \Rightarrow \bar{\mu}$ then for every $A \varepsilon \mathcal{B}$ and for every $\varepsilon > 0$ we can find a continuous nonnegative functional Q for which $Q(f) = 1$ whenever $f \varepsilon A^{(c)}$ and

$$(143) \qquad \bar{\mu}(A^{(c)}) \geqq \int_{\Omega} Q(f) d\bar{\mu} - \varepsilon .$$

Hence we have

$$(144) \qquad \bar{\mu}(A^{(c)}) \geqq \int_{\Omega} Q(f) d\bar{\mu} - \varepsilon = \lim_{k\to\infty} \int_{\Omega} Q(f) d\mu_{n_k} - \varepsilon \geqq \limsup_{k\to\infty} \mu_{n_k}(A) - \varepsilon$$

for any $\varepsilon > 0$. This implies that

$$(145) \qquad \limsup_{k\to\infty} \mu_{n_k}(A) \leqq \bar{\mu}(A^{(c)})$$

for any $A \varepsilon \mathcal{B}$. By (145) we can conclude that

$$(146) \qquad \bar{\mu}(A^{(i)}) \leqq \liminf_{k\to\infty} \mu_{n_k}(A) \leqq \limsup_{k\to\infty} \mu_{n_k}(A) \leqq \bar{\mu}(A^{(c)})$$

holds for every $A \varepsilon \mathcal{B}$ .If we replace A by $\Omega - A$ in (145), then we obtain the first half of (146). The second half is precisely (145).

Now denote by $\mathcal{M}$ the class of sets $A \varepsilon \mathcal{B}$ for which

$$(147) \qquad \bar{\mu}(A^{(i)}) \leqq \mu(A) \leqq \bar{\mu}(A^{(c)})$$

holds, that is,

$$(148) \qquad \mathcal{M} = \{A : \bar{\mu}(A^{(i)}) \leqq \mu(A) \leqq \bar{\mu}(A^{(c)}) \text{ and } A \varepsilon \mathcal{B}\} .$$

Since for any sequence of sets $\{A_r\}$ we have

(149) $$\left(\sum_{r=1}^{\infty} A_r\right)^{(c)} = \sum_{r=1}^{\infty} A_r^{(c)} \quad \text{and} \quad \left(\sum_{r=1}^{\infty} A_r\right)^{(i)} \subset \sum_{r=1}^{\infty} A_r^{(i)}$$

and

(150) $$\left(\prod_{r=1}^{\infty} A_r\right)^{(c)} \subset \prod_{r=1}^{\infty} A_r^{(c)} \quad \text{and} \quad \left(\prod_{r=i}^{\infty} A_r\right)^{(i)} = \prod_{r=1}^{\infty} A_r^{(i)} ,$$

it follows from (148) that if $A_r \varepsilon \mathcal{B}$ and $\{A_r\}$ is a monotone sequence, then $\lim_{r \to \infty} A_r \varepsilon \mathcal{M}$. Consequently, $\mathcal{M}$ is a monotone class.

Let $\mathcal{A}$ be the minimal algebra which contains the sets

(151) $$A(u_1, u_2, \ldots, u_k; S) = \{f : (f(u_1), f(u_2), \ldots, f(u_k)) \varepsilon S \text{ and } f \varepsilon C[0,t]\}$$

for all $u \varepsilon [0,t]$ and Borel sets S in the k-dimensional Euclidean space where $k = 1,2,\ldots$.

Furthermore, let

(152) $$\mathcal{A}_0 = \{A : \mu(A^{(i)}) = \mu(A^{(c)}) \text{ and } A \varepsilon \mathcal{A}\} .$$

If $A \varepsilon \mathcal{A}_0$ then by (113) it follows that

(153) $$\lim_{n \to \infty} \mu_n(A) = \mu(A)$$

and $\mathcal{B}$ can be characterized as the minimal σ-algebra which contains $\mathcal{A}_0$.

By (146) it follows that

(154) $$\overline{\mu}(A^{(i)}) \leqq \mu(A) \leqq \overline{\mu}(A^{(c)})$$

if $A \varepsilon \mathcal{A}_0$.

We can easily see that A_0 is an algebra which is not empty. Furthermore, $A_0 \subset M$. Thus M is a σ-algebra. (See Theorem **1.1** in the Appendix). By definition $M \subset B$. Since B is the minimal σ-algebra which contains A_0, it follows that necessarily $M = B$.

Thus we proved that

(155) $$\bar{\mu}(A^{(i)}) \leq \mu(A) \leq \bar{\mu}(A^{(c)})$$

holds for all $A \in B$.

Let

(156) $$B_0 = \{A : \bar{\mu}(A^{(i)}) = \bar{\mu}(A^{(c)}) \text{ and } A \in B\} .$$

Obviously B_0 is an algebra and it is easy to see that the minimal σ-algebra which contains B_0 is B. Since by (155) $\bar{\mu}(A) = \mu(A)$ if $A \in B_0$, it follows by Carathéodory's extension theorem (Theorem **1.2** in the Appendix) that

(157) $$\bar{\mu}(A) = \mu(A)$$

for all $A \in B$. Since $\bar{\mu}$ does not depend on the particular sequence $\{\mu_{n_k}\}$, consequently $\mu_n \Rightarrow \mu$ also holds.

If $A \in B$ and $\mu(A^{(i)}) = \mu(A^{(c)})$, then we say that A is a continuity set of μ. By (146) and (157) we can conclude that if $\{\mu_n\}$ is weakly compact, then

(158) $$\lim_{n \to \infty} \mu_n(A) = \mu(A)$$

for every continuity set $A \in B$ of μ.

Finally, it remains to prove that if Q is a functional on $C[0,t]$, if Q is measurable with respect to $\mathcal{B}$ and if Q is almost everywhere continuous with respect to the measure μ, then

$$\underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n^*) \leqq x\} \Rightarrow \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leqq x\} \tag{159}$$

as $n \to \infty$.

Let us denote by D the set of discontinuity points of Q. By assumption $\mu(D) = 0$.

For every real x let

$$E_x = \{f : Q(f) \leqq x\} \tag{160}$$

and

$$G_x = E_x^{(c)} \cap (\Omega - E_x)^{(c)}, \tag{161}$$

that is, G_x is the boundary of E_x. For $x < y$ we have

$$G_x \cap G_y \subset E_x^{(c)} \cap (\Omega - E_y)^{(c)}. \tag{162}$$

Therefore $f \varepsilon G_x \cap G_y$ implies that

$$\liminf_{g \to f} Q(g) \leqq x \quad \text{and} \quad \limsup_{g \to f} Q(g) \geqq y, \tag{163}$$

that is $G_x \cap G_y \subset D$. Consequently $\mu(G_x \cap G_y) = 0$ for $x < y$. Hence it follows that for an arbitrary sequence of distinct real numbers $\{x_r\}$ we have

$$\mu(\sum_r G_{x_r}) = \sum_r \mu(G_{x_r}). \tag{164}$$

By (164) we can conclude that the set of real numbers x for which $\mu(G_x) > 0$ is at most countable. Thus E_x is a continuity set of μ for every x except possibly for countably many x values. That is $\lim_{n\to\infty} \mu_n(E_x) = \mu(E_x)$ or

$$\text{(165)} \qquad \lim_{n\to\infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n^*) \leq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$$

for every x except possibly for countably many x values. This completes the proof of the theorem. For the above proof of this theorem we refer to I. I. Gikhman and A. V. Skorokhod [735]. Furthermore, we refer to M. D. Donsker [728], Yu. V. Prochorov [756], A. V. Skorookhod [767], P. Billingsley [712] and K. H. Parthasarathy [755].

Now let us demonstrate how we can use Theorem 7 in proving the particular results mentioned earlier.

If we suppose that

$$\text{(166)} \qquad Q(f) = \sup_{0\leq u\leq t} f(u) ,$$

then Q is a continuous functional on $C[0,t]$ because

$$\text{(167)} \qquad |Q(f) - Q(g)| \leq \sup_{0\leq u\leq t} |f(u) - g(u)| = \|f-g\|$$

and so $|Q(f) - Q(g)| \to 0$ as $\|f-g\| \to 0$. Now by Theorem 7 it follows that

$$\text{(168)} \qquad \underset{\sim}{P}\{\sup_{0\leq u\leq t} \xi_n^*(u) \leq x\} \Rightarrow \underset{\sim}{P}\{\sup_{0\leq u\leq t} \xi(u) \leq x\} .$$

Since

$$(169) \qquad \sup_{0\leq u\leq t} \xi_n(u) \leq \sup_{0\leq u\leq t} \xi_n^*(u) \leq \sup_{0\leq u\leq t} \xi_n(u) \qquad + [\xi_{[nt+1]}]^+/\sqrt{n}$$

for every n and $\xi_{[nt+1]}/\sqrt{n} \Rightarrow 0$ as $n \to \infty$, it follows that

$$(170) \qquad \underset{\sim}{P}\{\sup_{0\leq u\leq t} \xi_n(u) \leq x\} \Rightarrow \underset{\sim}{P}\{\sup_{0\leq u\leq t} \xi(u) \leq x\} .$$

This proves that in Theorem 6 the limiting distribution exists and is independent of the distribution of ξ_k. By performing the calculations for any particular sequence $\{\xi_k\}$ we can determine the limiting distribution.

The result (108) can be proved in a similar way. If we define

$$(171) \qquad Q(f) = \sup_{0\leq u\leq t} |f(u)| ,$$

then Q is a continuous functional on $C[0,t]$ because

$$(172) \qquad |Q(f) - Q(g)| \leq \sup_{0\leq u\leq t} |f(u) - g(u)| = ||f-g||$$

and so $|Q(f) - Q(g)| \to 0$ as $||f-g|| \to 0$. Now by Theorem 7 it follows that

$$(173) \qquad \underset{\sim}{P}\{ \sup_{0\leq u\leq t} |\xi_n^*(u)| \leq x\} \Longrightarrow \underset{\sim}{P}\{ \sup_{0\leq u\leq t} |\xi(u)| \leq x\}$$

and this implies that

$$(174) \qquad \underset{\sim}{P}\{ \sup_{0\leq u\leq t} |\xi_n(u)| \leq x\} \Rightarrow \underset{\sim}{P}\{ \sup_{0\leq u\leq t} |\xi(u)| \leq x\}$$

also holds. This proves that the limiting distribution (108) exists and is independent of the distribution of ξ_k. The limiting distribution can be determined by considering any particular sequence.

The functionals

$$Q(f) = \int_0^t |f(u)|^r du , \tag{175}$$

where $r = 1,2,\ldots,$ are well defined for every $f \in C[0,t]$ and continuous on $C[0,t]$. Thus by Theorem 7 it follows that the limiting distribution

$$\lim_{n\to\infty} \underset{\sim}{P}\{\int_0^t |\xi_n^*(u)|^r du \leq x\} = \underset{\sim}{P}\{\int_0^t |\xi(u)|^r du \leq x\} \tag{176}$$

exists for $r = 1,2,\ldots$. We can prove that

$$\int_0^t |\xi_n(u)|^r du \quad \text{and} \quad \frac{1}{n^{(r+2)/2}} \sum_{k=1}^n |\zeta_k|^r \tag{177}$$

where $r = 1,2,\ldots$ have also the limiting distribution (176) as $n \to \infty$.

Finally, let us prove (111). Let $Q(f)$ be the Lebesgue measure of the set $\{u : f(u) \geq 0 \text{ for } 0 \leq u \leq 1\}$, that is

$$Q(f) = \int_0^t \delta(f(u))du \tag{178}$$

where

$$\delta(x) = \begin{cases} 1 & \text{for } x > 0 , \\ 0 & \text{for } x \leq 0 . \end{cases} \tag{179}$$

Since $\delta(f(u))$ is bounded and Borel-measurable on the interval $[0,t]$, therefore $Q(f)$ is defined for all $f \in C[0,t]$. Since

$$|Q(f) - Q(g)| \leq \int_0^t |f(u) - g(u)| du \leq t \,||f-g|| , \tag{180}$$

it follows that $Q(f)$ is a continuous functional on $C[0,t]$. Thus by Theorem 7 we get

$$\lim_{n\to\infty} \underset{\sim}{P}\{\int_0^t \delta(\xi_n^*(u))du \leqq x\} = \underset{\sim}{P}\{\int_0^t \delta(\xi(u))du \leqq x\} . \tag{181}$$

We can easily show that if on the left-hand side of (181) we replace the integral by

$$\int_0^t \delta(\xi_n(u))du \quad \text{or} \quad \frac{\Delta([nt])}{n} , \tag{182}$$

then the right-hand side remains unchanged.

Now let us consider the general case when $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of mutually independent and identically distributed real random variables with distribution function $\underset{\sim}{P}\{\xi_n \leqq x\} = F(x)$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. We are interested in studying the asymptotic distribution of the random variable

$$\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n) \tag{183}$$

as $n \to \infty$.

Let us introduce the following notation

$$M = \sum_{n=1}^{\infty} \frac{\underset{\sim}{P}\{\zeta_n > 0\}}{n} \tag{184}$$

and

$$\bar{M} = \sum_{n=1}^{\infty} \frac{\underset{\sim}{P}\{\zeta_n < 0\}}{n} . \tag{185}$$

If $\underset{\sim}{E}\{|\xi_n|\} < \infty$ and $\underset{\sim}{E}\{\xi_n\} > 0$, then $M = \infty$ and $\bar{M} < \infty$. If $\underset{\sim}{E}\{|\xi_n|\} < \infty$ and $\underset{\sim}{E}\{\xi_n\} < 0$, then $M < \infty$ and $\bar{M} = \infty$. If $\underset{\sim}{E}\{\xi_n\}$ exists, $\underset{\sim}{E}\{\xi_n\} = 0$ and $\underset{\sim}{P}\{\xi_n = 0\} < 1$, then both $M = \infty$ and $\bar{M} = \infty$. This follows from Corollary 43.1 .

First we shall prove two particular results, and then we shall consider the solution of the general problem.

Theorem 8. If $M < \infty$, *then the limiting distribution*

(186) $$\lim_{n \to \infty} \mathbf{P}\{\eta_n \leq x\} = W(x)$$

exists and it can be obtained either by Theorem 43.13 *or by Theorem* 43.15.

Proof. By Theorem 43.12 we can state that $\eta_n \Rightarrow \eta$ where η is a nonnegative random variable for which $\mathbf{P}\{\eta < \infty\} = 1$. Thus it follows that $W(x) = \mathbf{P}\{\eta \leq x\}$.

If $M = \infty$, then $\mathbf{P}\{\lim_{n \to \infty} \eta_n = \infty\} = 1$ and so it is of some interest to find the asymptotic distribution of η_n as $n \to \infty$.

Theorem 9. If $\overline{M} < \infty$, *and if there are constants* A_n *and* $B_n > 0$ *such that* $B_n \to \infty$ *and*

(187) $$\lim_{n \to \infty} \mathbf{P}\left\{ \frac{\zeta_n - A_n}{B_n} \leq x\right\} = R(x)$$

in every continuity point of the distribution function $R(x)$, *then*

(188) $$\lim_{n \to \infty} \mathbf{P}\left\{ \frac{\eta_n - A_n}{B_n} \leq x\right\} = R(x)$$

also holds in every continuity point of $R(x)$. *Conversely,* $\overline{M} < \infty$ *and* (188) *imply* (187).

Proof. Let

(189) $$\overline{\eta}_n = \max(-\zeta_0, -\zeta_1, \ldots, -\zeta_n) .$$

If we apply Theorem 8 to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n, \ldots,$ then $\overline{M} < \infty$ implies that $\overline{\eta}_n \Rightarrow \overline{\eta}$ where $\overline{\eta}$ is a nonnegative random variable for which $\mathbf{P}\{\overline{\eta} < \infty\} = 1$.

On the other hand $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$ has the same distribution as

$$\zeta_n + \overline{\eta}_n = \max(\zeta_n - \zeta_0, \zeta_n - \zeta_1, \ldots, \zeta_n - \zeta_n) \tag{190}$$

for $n = 1,2,\ldots$.

If $B_n \to \infty$, then $\overline{\eta}_n/B_n \Rightarrow 0$, and consequently we have

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\eta_n - A_n}{B_n} \leqq x\right\} = \lim_{n\to\infty} \mathbf{P}\left\{\frac{\zeta_n - A_n}{B_n} \leqq x\right\} \tag{191}$$

where the existence of one of the limits implies the existence of the other limit, and the two limiting distributions are equal in every continuity point.

The above method in the proof has previously been used by the author [763],[764] in the context of queuing theory. See also C. C. Heyde [737].

By the results of Section 44 we know that if (187) exists then $R(x)$ is necessarily a stable distribution function (possibly degenerate). In Section 44 we gave necessary and sufficient conditions for $F(x)$ to belong to the domain of attraction of a nondegenerate stable distribution function $R(x)$.

It remains to consider the case when both $M = \infty$ and $\overline{M} = \infty$. In particular, if $\underset{\sim}{E}\{\xi_n\}$ exists, then $M = \infty$ and $\overline{M} = \infty$ if and only if $\underset{\sim}{E}\{\xi_n\} = 0$, and $\underset{\sim}{P}\{\xi_n = 0\} < 1$.

In what follows we shall prove a general theorem which covers the case $M = \overline{M} = \infty$ apart from a single particular case, and which also contains some of the results given in Theorem 9.

In Section 44 we proved that if $F(x)$ belongs to the domain of attraction of a nondegenerate stable distribution function $R(x)$, and only in this case, there exist constant A_n and $B_n > 0$ such that $\lim_{n \to \infty} B_n = \infty$ and

$$(192) \qquad \lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\zeta_n - A_n}{B_n} \leq x \right\} = R(x) .$$

In the following discussion we consider only such cases in which $A_n = 0$ $(n = 1,2,\ldots)$ can be chosen. Let us suppose that $R(x)$ is of type $S(\alpha,\beta,c,0)$ where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$ and $c > 0$. If $1 < \alpha \leq 2$, then $\underset{\sim}{E}\{\xi_n\}$ exists, and the cases $\underset{\sim}{E}\{\xi_n\} > 0$ and $\underset{\sim}{E}\{\xi_n\} < 0$ are covered by Theorem 1 and Theorem 2. If $1 < \alpha < 2$ and $\underset{\sim}{E}\{\xi_n\} = 0$, then we can choose $A_n = 0$. If $0 < \alpha < 1$, then we can always choose $A_n = 0$. If $\alpha = 1$, then we can choose $A_n = 0$ only in the case when $\beta = 0$. Thus we shall exclude the case $\alpha = 1$, $\beta \neq 0$.

Accordingly, if we assume that $\underset{\sim}{E}\{\xi_n\} = 0$ in the case when $1 < \alpha \leq 2$, and that $\beta = 0$ in the case when $\alpha = 1$, then (192) can be reduced to

$$(193) \qquad \lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\zeta_n}{B_n} \leq x \right\} = R(x) .$$

In (193) we may assume without loss of generality that $\{B_n\}$ is a nondecreasing sequence of positive numbers for which $\lim_{n\to\infty} B_n = \infty$ and that

$$(194) \qquad B_n = n^{1/\alpha}\rho(n)$$

where $\rho(x)$ $(0 < x < \infty)$ satisfies the following relation

$$(195) \qquad \lim_{x\to\infty} \frac{\rho(\omega x)}{\rho(x)} = 1$$

for every $\omega > 0$. (See Problem 46.12.)

In the particular case where $E\{\xi_n\} = 0$ and $E\{\xi_n^2\} = 1$ by Theorem 6 we have

$$(196) \qquad \lim_{n\to\infty} P\left\{\frac{\eta_n}{\sqrt{n}} \leq x\right\} = \begin{cases} 2\Phi(x) - 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

where $\Phi(x)$ is the normal distribution function. This result was found in 1946 by P. Erdős and M. Kac [730]. The following theorem is an extension of Theorem 6 and the proof follows on the same lines as in Theorem 6. (See also C. C. Heyde [738].)

Theorem 10. *Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables for which $P\{\xi_k \leq x\} = F(x)$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1, 2, \ldots$ and $\zeta_0 = 0$. If*

$$(197) \qquad \lim_{n\to\infty} P\left\{\frac{\zeta_n}{B_n} \leq x\right\} = R(x)$$

exists where $R(x)$ is a nondegenerate stable distribution function and $\lim_{n\to\infty} B_n = \infty$, then

$$(198) \qquad \lim_{n\to\infty} P\left\{\frac{\eta_n}{B_n} \leq x\right\} = H(x)$$

also exists and the distribution function $H(x)$ does not depend on $F(x)$.

Proof. Let

(199) $$H_n(x) = P\{\max(\zeta_0, \zeta_1, \ldots, \zeta_n) \leq B_n x\}$$

for $n = 0,1,2,\ldots$ and

(200) $$H_{nk}(x) = P\{\max(\zeta_{n_0}, \zeta_{n_1}, \ldots, \zeta_{n_k}) \leq B_n x\}$$

for $n = 0,1,2,\ldots$ and $k = 1,2,\ldots$ where

(201) $$n_j = \left[\frac{nj}{k} \right]$$

for $j = 0,1,\ldots, k$.

First we shall prove that for every $\varepsilon > 0$ there exists a positive constant C such that

(202) $$H_{nk}(x-\varepsilon) - \frac{C}{k\varepsilon^\alpha} \leq H_n(x) \leq H_{nk}(x)$$

holds for all $n = 0,1,\ldots,$ $k = 1,2,\ldots$ and x .

Let us denote by ν the smallest subscript $r = 0,1,2,\ldots$ for which $\zeta_r > B_n x$. If there is no such r , then $\nu = \infty$. Then we can write that for any $\varepsilon > 0$

(203) $$H_{nk}(x-\varepsilon) \leq H_n(x)+ \sum_{i=0}^{k} \sum_{n_i \leq r < n_{i+1}} P\{\nu=r \text{ and } \zeta_{n_j} \leq B_n(x-\varepsilon) \text{ for } 0 \leq j \leq k\} .$$

If $n_i \leq r < n_{i+1}$, then we have

$$\begin{aligned}
&\underset{\sim}{P}\{\nu = r \text{ and } \zeta_{n_j} \leqq B_n(x-\varepsilon) \text{ for } 0 \leqq j \leqq k\} \leqq \\
&\leqq \underset{\sim}{P}\{\nu = r \text{ and } \zeta_{n_{i+1}} - \zeta_r > B_n\varepsilon\} = \\
(204)\qquad &= \underset{\sim}{P}\{\nu = r\}\underset{\sim}{P}\{\zeta_{n_{i+1}} - \zeta_r > B_n\varepsilon\} \leqq \\
&\leqq \underset{\sim}{P}\{\nu = r\} \max_{1 \leqq s \leqq \frac{n}{k}} \underset{\sim}{P}\{\zeta_s > B_n\varepsilon\}.
\end{aligned}$$

For if $\nu = r$ and $n_i \leqq r < n_{i+1}$, then $\zeta_r > B_n x$ and $\zeta_{n_{i+1}} \leqq B_n(x-\varepsilon)$, and the events $\{\nu = r\}$ and $\{\zeta_{n_{i+1}} - \zeta_r > B_n\varepsilon\}$ are independent. Now we shall prove that for any $\varepsilon > 0$ there exists a sufficiently large positive constant C such that

$$(205)\qquad \max_{1 \leqq s \leqq \frac{n}{k}} \underset{\sim}{P}\{\zeta_s > B_n\varepsilon\} < \frac{C}{k\varepsilon^\alpha} .$$

holds for all $n = 0,1,\ldots$ and $k = 1,2,\ldots$.

Since

$$(206)\qquad \lim_{n \to \infty} \max_{0 \leqq s \leqq N} \underset{\sim}{P}\{\zeta_s > B_n\varepsilon\} = 0$$

for any $N = 1,2,\ldots,$ and since

$$\begin{aligned}
&\max_{N \leqq s \leqq \frac{n}{k}} \underset{\sim}{P}\{\zeta_s > B_n\varepsilon\} \leqq \max_{N \leqq s \leqq \frac{n}{k}} \underset{\sim}{P}\{\frac{\zeta_s}{B_s} > \frac{B_{sk}\varepsilon}{B_s}\} \leqq \\
(207)\qquad &\leqq \sup_{N \leqq s < \infty} \underset{\sim}{P}\{\frac{\zeta_s}{B_s} > \frac{B_{sk}\varepsilon}{B_s}\} \to 1 - R(\varepsilon k^{1/\alpha})
\end{aligned}$$

as $N \to \infty$, it follows that

$$(208)\qquad \limsup_{n \to \infty} \max_{0 \leqq s \leqq \frac{n}{k}} \underset{\sim}{P}\{\zeta_s > B_n\varepsilon\} \leqq 1 - R(\varepsilon k^{1/\alpha})$$

for all $\varepsilon > 0$ and $k = 1,2,\ldots$. On the other hand $R(x)$ belongs to the domain of attraction of itself, and therefore by (44. 250)

$$(209) \qquad \lim_{k \to \infty} k[1-R(\varepsilon k^{1/\alpha})] = \frac{c_2}{\varepsilon^\alpha}$$

holds for $0 < \alpha < 2$ and for every $\varepsilon > 0$ where $0 \leqq c_2 \leqq c$. (See also (42.201).) If $R(x) = \Phi(x)$, that is, $\alpha = 2$, then (209) trivially holds with $c_2 = 0$. By (208) and (209) it follows that for every $\varepsilon > 0$ there is a sufficiently large positive constant C so that (205) is satisfied for all $n = 0,1,2,\ldots$ and $k = 1,2,\ldots$.

By (203), (204) and (205) we obtain the first inequality in (202). The second inequality in (202) is obvious.

Now let us suppose that $\xi_1^*, \xi_2^*,\ldots, \xi_n^*,\ldots$ is a sequence of mutually independent and identically distributed random variables for which $\underset{\sim}{P}\{\xi_n^* \leqq x\} = R(x)$ is given by (197). Let $\zeta_n^* = \xi_1^* + \xi_2^* + \ldots + \xi_n^*$ for $n = 1,2,\ldots$ and $\zeta_0^* = 0$. If

$$(210) \qquad \eta_k^* = \max(\zeta_0^*, \zeta_1^*,\ldots, \zeta_k^*)$$

for $k = 0,1,2,\ldots,$ then we have

$$(211) \qquad \lim_{n \to \infty} H_{nk}(x) = \underset{\sim}{P}\{\eta_k^* \leqq k^{1/\alpha}x\} .$$

For the random variables $\zeta_{n_j} - \zeta_{n_{j-1}}$ $(j = 1,2,\ldots,k)$ are independent and by (193), (194) and (195) we have

$$(212) \qquad \lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\zeta_{n_j} - \zeta_{n_{j-1}}}{B_n} \leqq x\right\} = \underset{\sim}{P}\{\zeta_j^* - \zeta_{j-1}^* \leqq k^{\frac{1}{\alpha}} x\}$$

for $j = 1,2,\dots, k$. Hence it follows that

$$\lim_{n\to\infty} \underset{\sim}{P}\{\zeta_{n_j} - \zeta_{n_{j-1}} \leqq B_n x_j \text{ for } 1\leqq j\leqq k\} = \underset{\sim}{P}\{\zeta_j^* - \zeta_{j-1}^* \leqq k^{1/\alpha} x_j \text{ for } 1\leqq j\leqq k\} \tag{213}$$

holds for all $x_1, x_2,\dots, x_k$. This implies (211).

If we let $n \to \infty$ in (202), then by (211) we get

$$\underset{\sim}{P}\{\eta_k^* \leqq k^{\frac{1}{\alpha}}(x-\varepsilon)\} - \frac{C}{k\varepsilon^\alpha} \leqq \liminf_{n\to\infty} H_n(x) \leqq \limsup_{n\to\infty} H_n(x) \leqq \tag{214}$$

$$\leqq \underset{\sim}{P}\{\eta_k^* \leqq k^{\frac{1}{\alpha}} x\} .$$

Finally, we shall show that

$$\lim_{k\to\infty} \underset{\sim}{P}\{\eta_k^* \leqq k^{\frac{1}{\alpha}} x\} = H(x) \tag{215}$$

exist. If in (214) we let $k \to \infty$ and $\varepsilon \to 0$, then we obtain that

$$\lim_{n\to\infty} H_n(x) = H(x)$$

in every continuity point of $H(x)$. Since $H(x)$ is the same for every $F(x)$ which belongs to the domain of attraction of $R(x)$, consequently (198) holds.

To prove (215) let us assume that $\xi_k^* = \xi(k) - \xi(k-1)$ for $k = 1,2,\dots$ where $\{\xi(u) , 0 \leqq u < \infty\}$ is a stochastic process which is homogeneous and has independent increments, and for which

$$\underset{\sim}{P}\{\xi(u) \leqq u^{\frac{1}{\alpha}} x\} = R(x) \tag{216}$$

where $R(x)$ is given by (197). Such a process exists and we shall call it a stable process.

By using the above interpretation of $\{\xi_k^*\}$ we can write that

$$(217)\qquad \underset{\sim}{P}\{\eta_k^* \leqq k^{\frac{1}{\alpha}} x\} = \underset{\sim}{P}\{\max_{0\leqq j\leqq k} \xi(j) \leqq k^{1/\alpha} x\} = \underset{\sim}{P}\{\max_{0\leqq j\leqq k} \xi(\tfrac{j}{k}) \leqq x\}.$$

If we assume that the process $\{\xi(u),\ 0 \leqq u < \infty\}$ is separable, which can be done without loss of generality, then letting $k \to \infty$ in (217) we obtain that

$$(218)\qquad \lim_{k\to\infty} \underset{\sim}{P}\{\eta_k^* \leqq k^{\frac{1}{\alpha}} x\} = \underset{\sim}{P}\{\sup_{0\leqq u\leqq 1} \xi(u) \leqq x\},$$

that is,

$$(219)\qquad H(x) = \underset{\sim}{P}\{\sup_{0\leqq u\leqq 1} \xi(u) \leqq x\}.$$

To provide a complete solution of the problem we need to determine $H(x)$ in the case when $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,0)$ where $0 < \alpha \leqq 2$, $-1 \leqq \beta \leqq 1$ and $c > 0$. In the above discussion the case $\alpha = 1$, $\beta \neq 0$ has been excluded.

First let us consider some particular cases. If $\alpha = 2$, then β is immaterial and for $c = 1/2$ we have

$$(220)\qquad H(x) = \begin{cases} 2\Phi(x) - 1 & \text{when } x \geqq 0, \\ 0 & \text{when } x < 0. \end{cases}$$

If $0 < \alpha < 1$ and $\beta = 1$, then $R(0) = 0$, and consequently $\overline{M} = 0$. In this case by Theorem 9 we obtain that

$$(221)\qquad H(x) = R(x).$$

If $0 < \alpha < 1$ and $\beta = -1$, then $R(0) = 1$, and consequently $M = 0$. In this case by Theorem 8 we obtain that

(222)
$$H(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

If $1 < \alpha < 2$ and $\beta = -1$, then we have

(223)
$$H(x) = \begin{cases} 1 - \dfrac{1-R(x)}{1-R(0)} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0 \end{cases}$$

where by (42.192)

(224)
$$R(0) = \frac{\alpha-1}{\alpha}.$$

This follows from a result of A. V. Skorokhod [761 p. 157]. Skorokhod proved that if $1 < \alpha < 2$ and $\beta = -1$, then

(225)
$$\underset{\sim}{P}\{\sup_{0 \leq u \leq t} \xi(u) > x\} = \frac{\underset{\sim}{P}\{\xi(t) > x\}}{\underset{\sim}{P}\{\xi(1) > 0\}}$$

for $x \geq 0$ and $t > 0$. (See **formula (56.38)**.) Putting $t = 1$ in (225) we get (223).

In 1956 D. A. Darling [726] proved that if $\alpha = 1$, $\beta = 0$ and $c = 1$, that is, if

(226)
$$R(x) = \frac{1}{2} + \frac{1}{\pi} \text{arc tan } x,$$

then

(227)
$$\int_0^\infty x^s dH(x) = \frac{\sin \pi s}{\pi s} \int_0^\infty x^s dG(x)$$

for $-\frac{1}{2} < \text{Re}(s) < 1$ where

$$(228) \qquad G(x) = \exp\{-\frac{1}{\pi}\int_0^\infty \frac{\log(1+\frac{y}{x})}{1+y^2}\,dy\}\,.$$

By (227) we have

$$(229) \qquad \frac{dH(x)}{dx} = \frac{G(xe^{\pi i}) - G(xe^{-\pi i})}{2\pi i x}$$

for $x > 0$ where the definition of $G(s)$ is extended by analytical continuation to the complex plane cut along the negative real axis from the origin to infinity. By (229) it follows that

$$(230) \qquad \frac{dH(x)}{dx} = \frac{1}{\pi x^{1/2}(1+x^2)^{3/4}}\, e^{-\frac{1}{\pi}\int_0^x \frac{\log y}{1+y^2}\,dy}$$

for $x > 0$.

It is easy to extend the above result of Darling to the case where $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,0)$ where either $0 < \alpha < 1$, $-1 < \beta < 1$, $c > 0$ or $1 < \alpha < 2$, $-1 \leq \beta \leq 1$, $c > 0$. See C. C. Heyde [738] and the author [765] .

Theorem 11. *Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables having a stable distribution function $R(x)$ of type $S(\alpha,\beta,c,0)$ where either $0 < \alpha < 1$, $-1 < \beta < 1$, $c > 0$ or $1 < \alpha < 2$, $-1 \leq \beta \leq 1$, $c > 0$. Write $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$, $\zeta_0 = 0$ and $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$. Then we have*

$$(231) \qquad \lim_{n\to\infty} \mathbf{P}\{\frac{\eta_n}{n^{1/\alpha}} \leq x\} = H(x)$$

where $H(x) = 0$ for $x \leq 0$ and

$$(232) \qquad \int_0^\infty x^s dH(x) = \frac{\left(\dfrac{c}{\cos\frac{\gamma\pi}{2\alpha}}\right)^{s/\alpha}}{\Gamma(1-s)\Gamma(1+\frac{s}{\alpha})} \int_0^\infty x^s dG(x)$$

for $|\mathrm{Re}(s)| < \sigma$ where σ is a sufficiently small positive number, furthermore

$$(233) \qquad G(x) = \exp\{-\frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi}\int_0^\infty \frac{\log(1+\frac{y^\alpha}{x^\alpha})}{1-2y\sin\frac{\gamma\pi}{2\alpha}+y^2}dy\}$$

for $x \geq 0$ and

$$(234) \qquad \gamma = \frac{2}{\pi}\arctan(\beta\tan\frac{\alpha\pi}{2})$$

with $-1 < \gamma < 1$.

Proof. In this case we have

$$(235) \qquad \psi(s) = \int_{-\infty}^\infty e^{-sx}dR(x) = e^{-c|s|^\alpha\{1+\beta\frac{s}{|s|}\tan\frac{\alpha\pi}{2}\}}$$

for $\mathrm{Re}(s) = 0$. It is sufficient to prove (232) for some particular $c > 0$, because the general case can be obtained from any particular case by a simple transformation. It will be convenient to assume in the proof that

$$(236) \qquad c = \cos\frac{\gamma\pi}{2}$$

where γ is defined by (234). In this case by (235) we have

$$\psi(iy) = \begin{cases} e^{-y^{\alpha}} e^{\frac{i\gamma\pi}{2}} & \text{for } y \geqq 0 , \\ e^{-(-y)^{\alpha}} e^{-\frac{i\gamma\pi}{2}} & \text{for } y \leqq 0 . \end{cases} \tag{237}$$

Now by Theorem 14.3 we can write that

$$\sum_{n=0}^{\infty} E\{e^{-s\underset{\sim}{\eta}_n}\}\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\psi(s)]\}} \tag{238}$$

for $Re(s) \geqq 0$ and $|\rho| < 1$. Let

$$K(s) = \underset{\sim}{T}\{\log[1-\rho\psi(s)]\} . \tag{239}$$

By Theorem 5.1 we have

$$K(s) = \frac{\log(1-\rho)}{2} + \lim_{\varepsilon \to 0} \frac{s}{2\pi i} \left[\int_{\varepsilon}^{\infty} \frac{\log[1-\rho\psi(iy)]}{y(s-iy)} dy - \int_{\varepsilon}^{\infty} \frac{\log[1-\rho\psi(-iy)]}{y(s+iy)} dy \right] \tag{240}$$

for $Re(s) > 0$ and $|\rho| < 1$. If we make the substitution $y = e^{-\frac{i\gamma\pi}{2\alpha}} sz$ in the first integral and $y = e^{\frac{i\gamma\pi}{2\alpha}} sz$ in the second integral, then we obtain that for $Re(s) > 0$

$$K(s) = \frac{\log(1-\rho)}{2} + \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \left[\int_{L_1(\varepsilon)} \frac{\log(1-\rho e^{-s^{\alpha}z^{\alpha}})}{z(1-iz\, e^{-\frac{i\gamma\pi}{2\alpha}})} dz - \int_{L_2(\varepsilon)} \frac{\log(1-\rho e^{-s^{\alpha}z^{\alpha}})}{z(1+iz\, e^{\frac{i\gamma\pi}{2\alpha}})} dz \right] \tag{241}$$

where $L_1(\varepsilon) = \{z : z = e^{\frac{i\gamma\pi}{2\alpha}} y/s$ and $\varepsilon \leq y < \infty\}$ and $L_2(\varepsilon) = \{ z : z = e^{-\frac{i\gamma\pi}{2\alpha}} y/s$ and $\varepsilon \leq y < \infty\}$. Denote by $C_1(\varepsilon)$ the path which varies from $z = e^{\frac{i\gamma\pi}{2\alpha}} \varepsilon/s$ to $z = \varepsilon/|s|$ along the arc $|z| = \varepsilon/|s|$ and from $z = \varepsilon/|s|$ to ∞ along the real axis. Denote by $C_2(\varepsilon)$ the path which varies from $z = e^{-\frac{i\gamma\pi}{2\alpha}} \varepsilon/s$ to $z = \varepsilon/|s|$ along the arc $|z| = \varepsilon/|s|$ and from $z = \varepsilon/|s|$ to ∞ along the real axis. If we replace $L_1(\varepsilon)$ by $C_1(\varepsilon)$ in the first integral and $L_2(\varepsilon)$ by $C_2(\varepsilon)$ in the second integral, then by Cauchy's integral theorem both integrals remain unchanged. If $\varepsilon \to 0$ the difference of the two integrals taken along the arcs tends to $\pi i\gamma[\log(1-\rho)]/\alpha$, and thus (241) reduces to

$$K(s) = \left(\frac{1}{2} + \frac{\gamma}{2\alpha}\right)\log(1-\rho) + \frac{1}{2\pi i}\int_0^\infty \left[\frac{\log(1-\rho e^{-s^\alpha x^\alpha})}{x(1-ixe^{-\frac{i\gamma\pi}{2\alpha}})} - \right.$$

$$(242)\qquad \left. - \frac{\log(1-\rho e^{-s^\alpha x^\alpha})}{x(1+ixe^{\frac{i\gamma\pi}{2\alpha}})}\right]dx = \left(\frac{1}{2} + \frac{\gamma}{2\alpha}\right)\log(1-\rho) +$$

$$+ \frac{\cos\frac{\gamma\pi}{2\alpha}}{\pi}\int_0^\infty \frac{\log(1-\rho e^{-s^\alpha x^\alpha})}{1-2x\sin\frac{\gamma\pi}{2\alpha} + x^2}\,dx\,.$$

By (238) we have

$$(243)\qquad (1-\rho)\sum_{n=0}^\infty E\{e^{-s(1-\rho)^{1/\alpha}\eta_n}\}\rho^n = e^{K(0)-K((1-\rho)^{1/\alpha}s)}$$

for $\mathrm{Re}(s) \geq 0$ and $|\rho| < 1$, being $K(0) = \log(1-\rho)$.

Since

$$\lim_{\rho \to 1} [K((1-\rho)^{1/\alpha}s) - K(0)] = \lim_{\rho \to 1} \frac{\cos \frac{\gamma\pi}{2\alpha}}{\pi} \int_0^\infty \frac{\log(1-\rho\, e^{-(1-\rho)s^\alpha x^\alpha}) - \log(1-\rho)}{1-2x \sin \frac{\gamma\pi}{2\alpha} + x^2}\, dx$$

(244)

$$= \frac{\cos \frac{\gamma\pi}{2\alpha}}{\pi} \int_0^\infty \frac{\log(1+s^\alpha x^\alpha)}{1-2x \sin \frac{\gamma\pi}{2\alpha} + x^2}\, dx$$

for $\mathrm{Re}(s) > 0$, we can write that

$$\lim_{\rho \to 1} (1-\rho) \sum_{n=0}^\infty \underset{\sim}{E}\{e^{-s(1-\rho)^{1/\alpha}\eta_n}\}\rho^n = e^{-L(s)} \tag{245}$$

for $\mathrm{Re}(s) \geqq 0$ where

$$L(s) = \frac{\cos \frac{\gamma\pi}{2\alpha}}{\pi} \int_0^\infty \frac{\log(1+s^\alpha x^\alpha)}{1-2x \sin \frac{\gamma\pi}{2\alpha} + x^2}\, dx \; . \tag{246}$$

Here we extended the definition of $L(s)$ for $\mathrm{Re}(s) \geqq 0$ by continuity.

The above result can also be interpreted in the following way. Let us define a family of random variables $\{\nu(\rho) \, , \, 0 < \rho < 1\}$ in such a way that $\{\nu(\rho)\}$ is independent of the sequence of random variables $\{\xi_n\}$ and

$$\underset{\sim}{P}\{\nu(\rho) = n\} = (1-\rho)\rho^n \tag{247}$$

for $n = 0,1,\ldots$ and $0 < \rho < 1$. Then by (245) we can write that

$$\lim_{\rho \to 1} \underset{\sim}{E}\{e^{-s(1-\rho)^{1/\alpha}\eta_{\nu(\rho)}}\} = \lim_{\rho \to 1} \underset{\sim}{E}\{e^{-s[(1-\rho)\nu(\rho)]^{1/\alpha} \frac{\eta_{\nu(\rho)}}{[\nu(\rho)]^{1/\alpha}}}\} = e^{-L(s)} \tag{248}$$

for $\mathrm{Re}(s) \geqq 0$. Since

(249)
$$\lim_{\rho \to 1} \underset{\sim}{P}\{(1-\rho)\nu(\rho) \leq x\} = \begin{cases} 1-e^{-x} & \text{for } x \geq 0 , \\ 0 & \text{for } x < 0 , \end{cases}$$

and

(250)
$$\lim_{n \to \infty} \underset{\sim}{P}\{ \frac{\eta_n}{n^{1/\alpha}} \leq x\} = H(x) ,$$

therefore by (248) we have

(251)
$$\int_0^\infty (\int_0^\infty e^{-sx^{1/\alpha}y}\, dH(y))e^{-x}\, dx = e^{-L(s)}$$

for $\operatorname{Re}(s) \geq 0$.

Let us define

(252)
$$I(s) = \int_0^\infty e^{-u-u^{1/\alpha}/s}\, du$$

for $\operatorname{Re}(s) \geq 0$. Then by (251) we have

(253)
$$\int_0^\infty I(\frac{1}{sy})dH(y) = e^{-L(s)}$$

for $\operatorname{Re}(s) \geq 0$.

We observe that for $0 \leq x < \infty$ the function $I(x)$ is a distribution function of a positive random variable. Consequently, for $0 \leq x < \infty$

(254)
$$G(x) = \int_0^\infty I(\frac{x}{y})dH(y)$$

can be interpreted as the distribution function of the product of two independent positive random variables having distribution functions $I(x)$

and $H(x)$ respectively. On the other hand by (253) we have

$$G(x) = e^{-L(\frac{1}{x})} \tag{255}$$

for $x \geqq 0$.

Finally the unknown $H(x)$ can be obtained from (254) by Mellin-Stieltjes transform. Since

$$\int_0^\infty x^s dI(x) = \Gamma(1-s)\Gamma(1+\frac{s}{\alpha}) \tag{256}$$

for $-\alpha < \mathrm{Re}(s) < 1$, we obtain that

$$\int_0^\infty x^s dH(x) = \frac{1}{\Gamma(1-s)\Gamma(1+\frac{s}{\alpha})} \int_0^\infty x^s dG(x) \tag{257}$$

if s satisfies the inequalities $-\alpha < \mathrm{Re}(s) < 1$ and $-\sigma < \mathrm{Re}(s) < \alpha$ where σ is a sufficiently small positive number. This proves (232) in the particular case where c is given by (236). For an arbitrary $c > 0$ the right-hand side of (257) should be multiplied by $[c/\cos(\gamma\pi/2\alpha)]^{s/\alpha}$. Thus we obtain (232). From (232) $H(x)$ can be obtained by inversion by using formulas (41.64) or (41.65).

We note that Theorem 11 is also valid in the case where $\alpha = 1$, $\beta = 0$ and $c > 0$. This can easily be seen if assume that $\gamma = 0$ in the proof. Thus Theorem 11 proves (227) and (228) too.

Theorem 11 yields the distribution $H(x)$ defined by (219) which is identical with the limiting distribution (198).

We note that if $\{\xi(u) , 0 \leq u < \infty\}$ is a separable stable process for which

(258) $$\underset{\sim}{P}\{\xi(u) \leq u^{1/\alpha}x\} = R(x)$$

for $u > 0$ where $R(x)$ is a stable distribution function of type $S(\alpha,\beta,c,0)$ where $0 < \alpha < 1 < \alpha \leq 2$, $-1 \leq \beta \leq 1$ and $c > 0$ or $\alpha = 1$, $\beta = 0$, $c > 0$, and if we define $\eta(t) = \sup_{0 \leq u \leq t} \xi(u)$ for $t \geq 0$, then

(259) $$\underset{\sim}{P}\{\eta(t) \leq x\} = H\left(\frac{x}{t^{1/\alpha}}\right)$$

for $t > 0$.

If in particular $c = \cos(\gamma\pi/2\alpha)$, then by (251) we obtain that

(260) $$\int_0^\infty e^{-t} \underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \int_0^\infty\left(\int_0^\infty e^{-st^{1/\alpha}y}\, dH(y)\right)e^{-t}dt = e^{-L(s)}$$

for $\mathrm{Re}(s) \geq 0$ where $L(s)$ is given by (246). If $c > 0$ is arbitrary and $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(s) \geq 0$, then it follows from (260) that

(261) $$q \int_0^\infty e^{-qt} \underset{\sim}{E}\{e^{-s\eta(t)}\}dt = \exp\left\{-L\left(\frac{c^{1/\alpha}s}{(q \cos \frac{\gamma\pi}{2\alpha})^{1/\alpha}} \right)\right\}$$

where $L(s)$ is given by (246). Formula (261) is obvious for positive real q values. For $\mathrm{Re}(q) > 0$ (261) follows by analytical continuation.

We can also obtain (261) by a result of G. Baxter and M. D. Donsker [711]. By inversion (261) determines $\underset{\sim}{P}\{\eta(t) \leq x\}$ uniquely. Thus we can determine $H(x)$ in this way too.

The problems which we discussed above can be generalized in the following way: Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed real random variables. Write $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$

and $\zeta_0 = 0$. Let us assume that

(262) $$\lim_{n\to\infty} P\{\frac{\zeta_n}{B_n} \leq x\} = R(x)$$

where $R(x)$ is a nondegenerate stable distribution function, $B_n > 0$ for $n = 1,2,\ldots$ and $\lim_{n\to\infty} B_n = \infty$.

Define

(263) $$\xi_n(u) = \frac{\zeta_{[nu]}}{B_n}$$

for $0 \leq u < 1$ and $n = 1,2,\ldots$, and let $\xi_n(1) = \zeta_{n-1}/B_n$ for $n = 1,2,\ldots$.

If we assume that $\{\xi(u)\ ,\ 0 \leq u \leq 1\}$ is a stable stochastic process for which

(264) $$P\{\xi(u) \leq u^{1/\alpha}x\} = R(x)$$

where $0 < u \leq 1$ and $0 < \alpha \leq 2$ is the characteristic exponent of $R(x)$, then we can easily see that the finite dimensional distribution functions of the process $\{\xi_n(u)\ ,\ 0 \leq u \leq 1\}$ converge to the corresponding finite dimensional distributions of the process $\{\xi(u)\ ,\ 0 \leq u \leq 1\}$. Since both $\{\xi_n(u)\ ,\ 0 \leq u \leq 1\}$ and $\{\xi(u)\ ,\ 0 \leq u \leq 1\}$ have independent increments, it is sufficient to show that

(265) $$\lim_{n\to\infty} P\{\xi_n(v)-\xi_n(u) \leq x\} = P\{\xi(v)-\xi(u) \leq x\}$$

for all $0 \leq u < v \leq 1$. This, however, follows easily from (262) and from the relation

$$\lim_{n \to \infty} \frac{B_{[nu]}}{B_{[nv]}} = \left(\frac{u}{v}\right)^{1/\alpha} \tag{266}$$

for $0 \leq u < v \leq 1$.

Let Q be some real functional defined for $\underset{\sim}{\xi}_n = \{\xi_n(u) \, , \, 0 \leq u \leq 1\}$ and $\underset{\sim}{\xi} = \{\xi(u) \, , \, 0 \leq u \leq 1\}$. The problem arises what conditions should we impose on Q in order that

$$\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n) \leq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\} \tag{267}$$

be satisfied in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$?

By Theorem 10 and Theorem 12 we can conclude that if Q is the supremum functional that is $Q(f) = \sup_{0 \leq u \leq 1} f(u)$ and if $\{\xi(u) \, , \, 0 \leq u \leq 1\}$ is a separable stable process, then (267) is satisfied.

The solution of the general problem was provided in 1955 by A. V. Skorokhod [759],[760],[761],[784]. In what follows we shall present Skorokhod's results.

Denote by $D[0,1]$ the space of real functions $f(u)$ defined on the interval $[0,1]$ for which $f(u+0)$ and $f(u-0)$ exist at every point and $f(u+0) = f(u)$, $f(0) = f(+0)$ and $f(1) = f(1-0)$.

Denote by Λ the set of continuous, increasing, real functions $\lambda(u)$ defined on the interval $[0,1]$ for which $\lambda(0) = 0$ and $\lambda(1) = 1$.

Let us introduce a metric in the space $D[0,1]$ in the following way: If $f \in D[0,1]$ and $g \in D[0,1]$, then let us define the distance between f

and g by

(268) $$d(f,g) = \inf_{\lambda\in\Lambda}\{ \sup_{0\le u\le 1} |f(u) - g(\lambda(u))| + \sup_{0\le u\le 1} |u - \lambda(u)|\} .$$

We can easily check that $d(f,g)$ defines a metric on $D[0,1]$, and the space $D[0,1]$ with the metric (268) is a separable metric space.

By definition the sample functions of the process $\{\xi_n(u), 0 \le u \le 1\}$ belong to the space $D[0,1]$.

If we suppose that $\{\xi(u), 0 \le u \le 1\}$ is a separable stable process, then we can prove that with probability one the sample functions of the process $\{\xi(u), 0 \le u \le 1\}$ belong to the space $D[0,1]$. By removing a set of sample functions having probability 0 from the sample space we can achieve that all the sample functions of the process $\{\xi(u), 0 \le u \le 1\}$ belong to $D[0,1]$. This can be done without loss of generality.

Let Q be a real functional defined on the space $D[0,1]$. Write $\underset{\sim}{\xi}_n = \{\xi_n(u), 0 \le u \le 1\}$ and $\underset{\sim}{\xi} = \{\xi(u), 0 \le u \le 1\}$. Skorokhod proved that if Q is a real functional defined on $D[0,1]$ and if Q is continuous in the metric (268), then

(269) $$\lim_{n\to\infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n) \le x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \le x\}$$

in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \le x\}$. This result is based on the following theorem.

<u>Theorem 12.</u> <u>Let us suppose that the sample functions of the processes</u> $\{\xi_n(u), 0 \le u \le 1\}$ <u>and</u> $\{\xi(u), 0 \le u \le 1\}$ <u>belong to the space</u> $D[0,1]$ <u>and that the finite dimensional distributions of the process</u> $\{\xi_n(u),$ $0 \le u \le 1\}$ <u>converge to the corresponding finite dimensional distributions</u>

of the process $\{\xi(u), 0 \leq u \leq 1\}$. For each $f \in D[0,1]$ let us define

(270)
$$\Delta_a(f) = \sup_{0 \leq u-a \leq t \leq u \leq v \leq v+a \leq 1} \{\min(|f(t)-f(u)|, |f(v)-f(u)|)\} + \sup_{0 \leq u \leq a} |f(u)-f(0)| + \sup_{1-a \leq u \leq 1} |f(u)-f(1)| .$$

If for every $\varepsilon > 0$

(271)
$$\lim_{a \to 0} \lim_{n \to \infty} \sup \underset{\sim}{P}\{\Delta_a(\underset{\sim}{\xi}_n) > \varepsilon\} = 0 ,$$

and if Q is a real functional defined on $D[0,1]$ and if Q is continuous in the metric (269), then

(272)
$$\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n) \leq x\} = \underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$$

in every continuity point of $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$.

For the proof of this theorem we refer to I. I. Gikhman and A. V. Skorokhod [735 pp. 469-478].

If the process $\{\xi_n(u), 0 \leq u \leq 1\}$ is defined by (263) and if $\{\xi(u), 0 \leq u \leq 1\}$ is a stable process for which (264) holds and for which the sample functions belong to $D[0,1]$, then (271) is satisfied and consequently (272) holds for any functional Q which is continuous in the metric (268). This follows from a more general result of I. I. Gikhman and A. V. Skorokhod [735 pp. 478-484]. If we want to find the limiting distribution $\underset{\sim}{P}\{Q(\underset{\sim}{\xi}) \leq x\}$, then it is sufficient to determine the limit $\lim_{n \to \infty} \underset{\sim}{P}\{Q(\underset{\sim}{\xi}_n) \leq x\}$ for any particular process $\{\xi_n(u)\ ,\ 0 \leq u \leq 1\}$ which satisfies the requirements. Thus we may assume that, in particular, the random variables $\{\xi_n\}$ are mutually independent and identically distributed and $\underset{\sim}{P}\{\xi_n \leq x\} = R(x)$ where $R(x)$ is the distribution function given by (262). In this particular case the limiting distribution (267) has been found for several functionals Q, and by the above results we can conclude that (267) holds for any sequence $\{\xi_n\}$ for which (262) is satisfied.

Now we shall mention a few results of this nature.

In 1950 M. Kac and H. Pollard [742] gave a method for finding the limiting distribution

$$(273) \qquad \lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\max(|\zeta_0|,\ |\zeta_1|,\dots,|\zeta_n|)}{n} \leq x \right\}$$

in the case when

$$(274) \qquad \underset{\sim}{P}\{\xi_n \leq x\} = \frac{1}{2} + \frac{1}{\pi} \text{arc tan } x\ .$$

Thus we can obtain (267) for $Q(f) = \sup_{0 \leq u \leq 1} |f(u)|$ and $R(x)$ of type $S(1,0,1,0)$.

In 1951 K. L. Chung and M. Kac [724],[725] considered the case when $\underset{\sim}{P}\{\xi_n \leq x\} = R(x)$ is a symmetric stable distribution function of type $S(\alpha,0,1,0)$ where $0 < \alpha \leq 2$. They determined the asymptotic distributions of the following random variables: ν_n the number of changes of sign in the sequence $\zeta_1, \zeta_2, \dots, \zeta_n$ and $\mu_n(a)$ the number of subscripts $k = 1,2,\dots, n$

for which $|\zeta_k| < a$ where a is a positive constant. K. L. Chung and M. Kac [724],[725] proved that if $1 < \alpha \leq 2$, then

(275) $$\lim_{n\to\infty} P\left\{\frac{\nu_n}{n^{1-\frac{1}{\alpha}}} \leq \frac{2\Gamma(1-\frac{1}{\alpha})x}{\pi\alpha \sin\frac{\pi}{\alpha}}\right\} = \lim_{n\to\infty} P\left\{\frac{\mu_n(a)}{n^{1-\frac{1}{\alpha}}} \leq \frac{2ax}{\alpha\sin\frac{\pi}{\alpha}}\right\} = G_{1-\frac{1}{\alpha}}(x)$$

where the distribution function $G_{1-\frac{1}{\alpha}}(x)$ is defined by (42.178). In the particular case of $\alpha = 2$ the asymptotic distribution of ν_n was found in 1950 by K. L. Chung [722],[723].

If $\alpha = 1$, then

(276) $$\lim_{n\to\infty} P\left\{\frac{\nu_n}{2\pi^2\log n} \leq x\right\} = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases}$$

and

(277) $$\lim_{n\to\infty} P\left\{\frac{\mu_n(a)}{\log n} \leq \frac{2ax}{\pi}\right\} = \begin{cases} 1-e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

If $0 < \alpha < 1$, then

(278) $$\lim_{n\to\infty} P\left\{\frac{\nu_n}{\log n} \leq \frac{2x\tan\frac{\alpha\pi}{2}}{\alpha\pi}\right\} = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases}$$

and

(279) $$P\{\lim_{n\to\infty} \mu_n(a) < \infty\} = 1.$$

In 1956 F. Spitzer [181] proved that if Δ_n denotes the number of positive elements in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$ and if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P\{\zeta_k > 0\} = \alpha \tag{280}$$

exists, then

$$\lim_{n \to \infty} P\{ \frac{\Delta_n}{n} \leq x\} = F_\alpha(x) \tag{281}$$

where

$$F_\alpha(x) = \begin{cases} 0 & \text{for } x \leq 0 , \\ \dfrac{\sin\alpha\pi}{\pi} \displaystyle\int_0^x u^{\alpha-1}(1-u)^{-\alpha}du & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geq 1 \end{cases} \tag{282}$$

and for $0 < \alpha < 1$, $F_0(x) = 0$ for $x < 0$, $F_0(x) = 1$ for $x \geq 0$, $F_1(x) = 0$ for $x < 1$ and $F_1(x) = 1$ for $x \geq 1$.

We mention some more results. In 1949 Feller [732] determined the asymptotic distribution of the number of zeros in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$

for the case when the random variables $\{\xi_n\}$ have a lattice distribution which belongs to the domain of attraction of a stable distribution function.

In 1954 G. Kallianpur and H. Robbins [743] studied the asymptotic distribution of

$$\sum_{k=1}^{n} h(\zeta_k) \tag{283}$$

in the case when $h(x)$ is Riemann integrable on some finite interval (a,b) and 0 elsewhere and $P\{\xi_n \leq x\}$ belongs to the domain of attraction of a symmetric stable distribution function.

In 1957 M. Kac [741] demonstrated as a particular case of a somewhat more general result that if $P\{\xi_k \leq x\} = R(x)$ is a symmetric stable distribution function of type $S(\alpha,0,c,0)$ where $1 \leq \alpha \leq 2$ and $c > 0$ and $\mu_n(S)$ denotes the number of partial sums $\zeta_1, \zeta_2, \ldots, \zeta_n$ belonging to the set S where S is a bounded and measurable linear set, then the limit

$$\lim_{n \to \infty} n^{1-\frac{1}{\alpha}} P\{\mu_n(S) = j\} \tag{284}$$

exists for $1 < \alpha \leq 2$ and $j = 0,1,2,\ldots$ and the limit

$$\lim_{n \to \infty} (\log n) P\{\mu_n(S) = j\} \tag{285}$$

exists for $\alpha = 1$ and $j = 0,1,2,\ldots$. M. Kac gave methematical methods for finding these limits.

Finally, we shall mention another result of somewhat different nature. In 1956 D. A. Darling and P. Erdős [727] proved that if $\{\xi_n\}$ is a sequence of mutually independent and identically distributed random variables for which $\underset{\sim}{P}\{\xi_n \leq x\} = \Phi(x)$, the normal distribution function, and $\zeta_n = \xi_1 + \xi_2 + \dots, \xi_n$ for $n = 1,2,\dots,$ then

$$(286)\quad \lim_{n\to\infty} \underset{\sim}{P}\left\{\max_{1\leq k\leq n} \frac{\zeta_k}{\sqrt{k}} < (2\log\log n)^{1/2} + \frac{\log\log\log n}{2(2\log\log n)^{1/2}} + \frac{x}{(2\log\log n)^{1/2}}\right\} = \exp\{-e^{-x}/2\sqrt{\pi}\}$$

and

$$(287)\quad \lim_{n\to\infty} \underset{\sim}{P}\left\{\max_{1\leq k\leq n} \frac{|\zeta_k|}{\sqrt{k}} < (2\log\log n)^{1/2} + \frac{\log\log\log n}{2(2\log\log n)^{1/2}} + \frac{x}{(2\log\log n)^{1/2}}\right\} = \exp\{-e^{-x}/\sqrt{\pi}\}$$

for $-\infty < x < \infty$. Furthermore, they demonstrated that (283) and (284) also hold if we assume only that $\{\xi_n\}$ is a sequence of mutually independent random variables for which $\underset{\sim}{E}\{\xi_n\} = 0$, $\underset{\sim}{E}\{\xi_n^2\} = 1$ and $\underset{\sim}{E}\{|\xi_n|^3\} < C < \infty$ for all $n = 1,2,\dots$.

46. Problems

46.1. Let

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

Determine

$$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} f(x)dx .$$

46.2. Let $f(x)$ be a stable density function of type $S(\frac{1}{2},\beta,1,0)$ where $-1 \leq \beta \leq 1$, that is, $f(x) = f(x ; \frac{1}{2}, \beta,1,0)$. Find $f(x)$ for $x > 0$.

46.3. Let $F(x)$ be a distribution function. Prove that

$$\int_{-\infty}^{\infty} |x|^{\delta} dF(x) = \delta \int_{0}^{\infty} x^{\delta-1} [1-F(x) + F(-x)]dx$$

for $0 < \delta < \infty$.

46.4. Let

$$P\{\xi \leq x\} = \frac{1}{2} + \frac{1}{\pi} \text{arc tan } x .$$

Find $E\{|\xi|^{\delta}\}$ for $-1 < \delta < 1$.

46.5. Evaluate the integral

$$I_{\alpha}(s) = \int_{0}^{\infty} (e^{-sx} - 1 + \frac{sx}{1+x^2}) \frac{\alpha dx}{x^{\alpha+1}} .$$

for $\text{Re}(s) \geq 0$ and $0 < \alpha < 2$.

46.6 The random variable ξ has a stable distribution of type $S(\alpha,\beta,c,0)$ where $\alpha \neq 1$ and $c > 0$. Find $E\{|\xi|^{\delta}\}$ for $-1 < \delta < \alpha$.

46.7. The random variable ξ has a stable distribution of type $S(\alpha,\beta,c,0)$ where $\alpha \neq 1$ and $c > 0$. Find $P\{\xi \geq 0\}$.

46.8. Let $R(x)$ be a stable distribution function of type $S(\alpha,\beta,c,0)$ where either $0 < \alpha < 1$, $-1 \leq \beta \leq 1$, or $\alpha = 1$, $\beta = 0$ or $1 < \alpha < 2$, $-1 \leq \beta \leq 1$ and $c > 0$. In this case

$$\psi(s) = \int_{-\infty}^{\infty} e^{-sx} dR(x) = e^{-c|s|^{\alpha}\{1+\beta \frac{s}{|s|} \tan \frac{\alpha\pi}{2}\}}$$

for $\operatorname{Re}(s) = 0$. Determine $\psi^{+}(s) = \underset{\sim}{T}\{\psi(s)\}$ for $\operatorname{Re}(s) \geq 0$. (See V. M. Zolotarev [341].)

46.9. The random variable η has a stable distribution of type $S(\frac{1}{2}, 1, c, 0)$ where $c > 0$. Prove that η can be represented in the form $\eta = c^2/\xi^2$ where

$$\underset{\sim}{P}\{\xi \leq x\} = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du .$$

46.10. Let $F(x)$ be a stable distribution function of type $S(\alpha,\beta,c,m)$ where $0 < \alpha < 2$, $-1 \leq \beta \leq 1$ and $c > 0$. Prove (42.201) and (42.202), that is,

$$\lim_{x \to \infty} x^{\alpha} F(-x) = c_1 \quad \text{and} \quad \lim_{x \to \infty} x^{\alpha}[1-F(x)] = c_2$$

where c_1 and c_2 are determined by the equations

$$\beta = \frac{c_2 - c_1}{c_2 + c_1} \quad \text{and} \quad c = \frac{(c_1 + c_2)\pi}{2\Gamma(\alpha)\sin \frac{\alpha\pi}{2}} .$$

46.11. Let $F(x)$ be a stable distribution function. Then $F(x)$ is of type $S(\alpha,\beta,c,m)$ where $0 < \alpha < 2$, $-1 \leq \beta \leq 1$, $c \geq 0$ and m is a real number. Give a procedure of finding α,β,c,m.

46.12. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables belonging to the domain of attraction of a non-degenerate stable distribution function $R(x)$ of type $S(\alpha,\beta,c,m)$. Then there exist constant A_n and $B_n > 0$ such that

$$\lim_{n \to \infty} P\{ \frac{\xi_1 + \ldots + \xi_n - A_n}{B_n} \leq x\} = R(x) .$$

Prove that $B_n = n^{1/\alpha}\rho(n)$ and

$$\frac{A_n}{B_n} = \frac{h(n)}{n^{\frac{1}{\alpha}-1}} + \begin{cases} \frac{2\beta c}{\pi} \log n & \text{if } \alpha = 1 , \\ -m & \text{if } \alpha \neq 1 \end{cases}$$

where $\rho(t)$ and $h(t)$ are defined for $0 < t < \infty$ and satisfy the relations

$$\lim_{t \to \infty} \frac{\rho(\omega t)}{\rho(t)} = 1$$

and

$$\lim_{t \to \infty} \frac{h(\omega t) - h(t)}{t^{\frac{1}{\alpha}-1}} = 0 .$$

46.13 Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent positive random variables with a common distribution function $F(x)$. Let us suppose that $[1-F(x)]x^{\alpha} = h(x)$ where $h(\omega x)/h(x) \to 1$ as $x \to \infty$ for any $\omega > 0$ and $0 < \alpha < 1$. Prove that

$$\lim_{n \to \infty} P\{ \frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n^{1/\alpha}\rho(n)} \leq x\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\alpha,1,\Gamma(1-\alpha)\cos \frac{\alpha\pi}{2},0)$ if and only if

$$\lim_{n \to \infty} \frac{h(n^{1/\alpha} \rho(n))}{(\rho(n))^{\alpha}} = 1 .$$

46.14. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent positive random variables with a common distribution function $F(x)$. Let us suppose that $[1-F(x)]x^{\alpha} = h(x)$ where $h(\omega x)/h(x) \to 1$ as $x \to \infty$ for any $\omega > 0$ and $1 < \alpha < 2$. Let $\underset{\sim}{E}\{\xi_n\} = a$. Prove that

$$\lim_{n \to \infty} \underset{\sim}{P}\left\{ \frac{\xi_1 + \xi_2 + \ldots + \xi_n - na}{n^{1/\alpha} \rho(n)} \leq x \right\} = R(x)$$

where $R(x)$ is a stable distribution function of type $S(\alpha, 1, \Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}, 0)$ if and only if

$$\lim_{n \to \infty} \frac{h(n^{1/\alpha}\rho(n))}{(\rho(n))^{\alpha}} = 1 .$$

46.15. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables for which

$$\underset{\sim}{P}\{\xi_n = 2j\} = \frac{1}{2j-1} \binom{2j}{j} \frac{1}{2^{2j}}$$

if $j = 1, 2, \ldots$. Find the limiting distribution of $(\xi_1 + \ldots + \xi_n - A_n)/B_n$ as $n \to \infty$ where A_n and B_n are suitably chosen normalizing constants.

46.16. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables for which

$$\underset{\sim}{P}\{\xi_n > k\} = (-1)^k \binom{q-1}{k}$$

if $k = 1, 2, \ldots$ and $0 < q < 1$. Find the limiting distribution of $(\xi_1 + \ldots + \xi_n - A_n)/B_n$ as $n \to \infty$ where A_n and B_n are suitably chosen normalizing constants.

46.17. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables with distribution function

$$F(x) = \begin{cases} 1 - \dfrac{1}{x(\log x)^2} & \text{for } x \geqq e, \\ 0 & \text{for } x < e. \end{cases}$$

Find the limiting distribution of $(\xi_1 + \ldots + \xi_n - A_n)/B_n$ as $n \to \infty$ where A_n and B_n are suitably chosen normalizing constants.

46.18. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables with distribution function

$$F(x) = \begin{cases} 1 - \dfrac{1}{2x(\log x)^2} & \text{for } x \geqq e, \\ 1/2e & \text{for } -e \leqq x < e, \\ \dfrac{1}{3|x|(\log|x|)^2} & \text{for } x < -e. \end{cases}$$

Find the limiting distribution of $(\xi_1 + \ldots + \xi_n - A_n)/B_n$ as $n \to \infty$ where A_n and B_n are suitably chosen normalizing constants.

46.19. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables with distribution function

$$F(x) = \begin{cases} 1 - \dfrac{1}{x} & \text{for } x \geqq 1, \\ 0 & \text{for } x < 1. \end{cases}$$

Find the limiting distribution of $(\xi_1 + \ldots + \xi_n - A_n)/B_n$ as $n \to \infty$ where A_n and B_n are suitably chosen normalizing constants.

46.20. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables with density function

$$f(x) = \begin{cases} \frac{1}{2} \frac{\log|x|}{x^2} & \text{for } |x| \geq 1 , \\ 0 & \text{for } x \leq 1 , \end{cases}$$

Find the limiting distribution of $(\xi_1 + \ldots + \xi_n - A_n)/B_n$ as $n \to \infty$ where A_n and B_n are suitably chosen normalizing constants. (See G. Kallianpur and H. Robbins [89].)

46.21. Prove (42.181).

46.22. Let ξ and η be independent random variables having the same stable distribution of type $S(\alpha, 1, 1, 0)$ where $0 < \alpha < 1$. Find $H(x) = \underset{\sim}{P}\{\xi\eta^{-1} \leq x\}$.

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Infinitely Divisible and Stable Distributions

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Limit Distributions for Sums of Independent Random Variables

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Limit Distributions for Sums of Dependent Random Variables

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