CHAPTER V.

RANDOM WALKS, BALLOT THEOREMS, AND ORDER STATISTICS

35. Bernoulli Trials. A sequence of independent and identical trials (repeated trials) is called Bernoulli trials if there are two possible results (outcomes) for each trial, namely, either an event A occurs, or it does not occur. Sometimes it is convenient to call A success and $\bar{A}$, failure. Let $\mathbf{P}\{A\} = p$ and $\mathbf{P}\{\bar{A}\} = q$. Then $p+q = 1$.

Denote by ν_n the number of occurrences of A (or, the number of successes) in the first n trials. Then

$$\mathbf{P}\{\nu_n = k\} = \binom{n}{k} p^k q^{n-k} \tag{1}$$

for $k = 0,1,2,\ldots$. In some particular cases this formula had been known by Pierre Fermat (1601-1665), Blaise Pascal (1623-1662) and Christiaan Huygens (1629-1695); however, Jakob Bernoulli (1654-1705) was the first who systematically studied the mathematical laws governing repeated trials.

We say that the random variable ν_n has a Bernoulli distribution with parameters n and p where $n = 1,2,\ldots$ and $0 < p < 1$. In what follows we shall mention a few useful formulas for the Bernoulli distribution.

We have

$$\mathbf{P}\{\nu_n \geq j\} = \sum_{k=j}^{n} \binom{n}{k} p^k q^{n-k} \tag{2}$$

for $j = 0,1,\ldots, n$, which follows from (1). We can write also that

(3) $$P\{\nu_n \geq j\} = \sum_{k=j}^{n} \binom{k-1}{j-1} p^j q^{k-j}$$

for $j = 1,2,\ldots, n$, which can be proved by taking into consideration that the event $\{\nu_n \geq j\}$ can occur in the following mutually exclusive ways: among the first n trials, the j-th success occurs at the j-th, j+1-st,..., n-th trial. We can write also that

(4) $$P\{\nu_n \geq j\} = n \binom{n-1}{j-1} \int_0^p u^{j-1}(1-u)^{n-j} du$$

for $j = 1,2,\ldots, n$. We can prove (4) in several ways. We can show that the integral on the right-hand side of (4) can be expressed either in the form (2) or (3). We can prove (4) also in a probabilistic way by choosing a suitable model for Bernoulli trials.

The r-th binomial moment of ν_n is given by

(5) $$B_r(n) = E\{\binom{\nu_n}{r}\} = \sum_{k=0}^{n} \binom{k}{r} P\{\nu_n = k\} = \binom{n}{r} p^r$$

for $r = 0,1,\ldots, n$. Obviously $B_r(n) = 0$ if $r > n$.

Knowing the binomial moments of ν_n we can easily determine the power moments and the central moments of ν_n . Here are a few particular cases:

(6) $$E\{\nu_n\} = np ,$$

(7) $$Var\{\nu_n\} = E\{(\nu_n - np)^2\} = npq ,$$

(8) $$E\{(\nu_n - np)^3\} = npq(q-p) ,$$

(9) $$E\{(\nu_n - np)^4\} = 3n^2p^2q^2 + npq(1-6pq) ,$$

(10) $$E\{(\nu_n - np)^5\} = npq(q-p)(1-12pq+10npq) ,$$

(11) $$E\{(\nu_n - np)^6\} = npq(1-30pq+120p^2q^2)+5n^2p^2q^2(5-26pq)+15n^3p^3q^3 .$$

By Chebyshev's inequality (Theorem 41.3) we have

(12) $$P\{|\nu_n - np| \geq a\} \leq \frac{E\{(\nu_n - np)^{2s}\}}{a^{2s}}$$

for any $a > 0$ and $s = 1,2,\ldots$. In particular, if $s = 1$, we get

(13) $$P\{|\nu_n - np| \geq a\} \leq \frac{npq}{a^2} \leq \frac{n}{4a^2}$$

for $a > 0$, and if $s = 2$, we get

(14) $$P\{|\nu_n - np| \geq a\} \leq \frac{3(npq)^2 + npq(1-6pq)}{a^4} \leq \frac{3n^2}{16a^4}$$

for $a > 0$.

In 1680 or so Jakob Bernoulli [7] proved the weak law of large numbers which asserts that

(15) $$\lim_{n \to \infty} P\{|\frac{\nu_n}{n} - p| < \varepsilon\} = 1$$

for any $\varepsilon > 0$.

In 1917 F. P. Cantelli [13] proved the strong law of large numbers which asserts that

$$P\{\lim_{n\to\infty} \frac{\nu_n}{n} = p\} = 1 . \tag{16}$$

In 1733 A. De Moivre [18] (see also P. S. Laplace [39]) proved the following limit theorem:

$$\lim_{n\to\infty} P\{\frac{\nu_n - np}{\sqrt{npq}} \leq x\} = \Phi(x) \tag{17}$$

for any x where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \tag{18}$$

is the normal distribution function.

Finally, we would like to mention briefly the development of the notion of binomial coefficients. We can define formally the binomial coefficient $\binom{a}{k}$ for any complex or real a and for any positive integer k as

$$\binom{a}{k} = \frac{a(a-1)\dots(a-k+1)}{k!} \tag{19}$$

where $k! = 1.2\dots k$, and for any a

$$\binom{a}{0} \equiv 1 . \tag{20}$$

Accordingly, $\binom{a}{k}$ is a polynomial of degree k .

The notion of binomial coefficients originates in the notion of figurate numbers, as we call them now. We define F_n^k , the k-th figurate number of order n , for $n \geqq 0$ and $k \geqq 1$ by the following recurrence formula

(21) $$F_{n+1}^{k+1} = F_{n+1}^{k} + F_{n}^{k+1}$$

where $F_0^k = 1$ for $k \geqq 1$ and $F_n^1 = 1$ for $n \geqq 0$.

(See L. E. Dickson [19] II. pp. 1-39.) Here is a table for F_n^k $(0 \leqq n \leqq 6 \ , 1 \leqq k \leqq 7)$.

F_n^k

n \ k	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7
2	1	3	6	10	15	21	28
3	1	4	10	20	35	56	84
4	1	5	15	35	70	126	212
5	1	6	21	56	126	252	464
6	1	7	28	84	212	464	928

Figurate numbers were studied by Nicomachus of Gerasa [46] who lived about the close of the first century. Omar Khayyam of Nishapur (d. 1213) knew them in the eleventh century. (See F. Woepcke [68].) In 1303 Chu Shih-chieh [15] refers to figurate numbers as an old invention and he mentions several surprising relations for figurate numbers. (See Y. Mikami [44].) The figurate numbers arranged in the form of a triangular array

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first appeared in print in 1527 on the title-page of P. Apianus [4]. (See D. E. Smith [58] p. 509.)

In 1544 M. Stifel [60] showed that in the binomial expansion

(22) $$(1+x)^n = \sum_{k=0}^{n} C_n^k x^k$$

the coefficients C_n^k $(0 \leq k \leq n)$ can be obtained by the recurrence relation

(23) $$C_{n+1}^k = C_n^k + C_n^{k-1}$$

where $C_n^0 = C_n^n = 1$ for $n = 0,1,2,\ldots$. He arranged the coefficients C_n^k $(0 \leq k \leq n)$ in the following triangular array which is known now as the Pascal's arithmetic triangle

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & &
\end{array}$$

In 1556 Nicolo Tartaglia [64] (Part 2. pp. 70, 72) gave this triangular array as his own invention. In 1654 Blaise Pascal [47] made many discoveries concerning the numbers C_n^k $(0 \leq k \leq n)$.

The numbers C_n^k $(0 \leq k \leq n)$ appear in the 17-th century in connection with combinations. The number of combinations without repetition of n objects taken k at a time can be expressed as C_n^k. In 1634 P. Hérigone [31] gave the following formula

$$C_n^k = \frac{n(n-1)\dots(n-k+1)}{k!} \tag{24}$$

for $0 < k \leq n$ where $k! = 1.2\dots k$ for $k = 1,2,\dots$. This formula appears also in 1654 in the treatise of B. Pascal [47].

As we have seen, the above mentioned three instances all lead to the same mathematical notion, namely, the notion of binomial coefficients. We can conclude that

$$F_n^k = \binom{n+k-1}{k-1} \tag{25}$$

for $n \geq 0$ and $k \geq 1$, and

$$C_n^k = \binom{n}{k} \tag{26}$$

for $0 \leq k \leq n$.

It should be noted that in those early times no mathematical notation was used for these numbers. It seems that L. Euler [22 p. 78], [23 p. 33] was the first who used the notation $[\frac{a}{k}]$ and later $(\frac{a}{k})$ for (19). The notation $\binom{a}{k}$, which is a slight modification of Euler's second notation, was introduced in 1851 by J. L. Raabe [53 p. 350].

Finally, we note that if n , k and $n-k$ are all large, then we can use Stirling's formula in finding a good approximation for the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ .} \tag{27}$$

According to Stirling's formula we have

$$n! \sim \sqrt{2\pi n}\ (\frac{n}{e})^n \tag{28}$$

as $n \to \infty$, that is, $n!$ is asymptotically equal to the right-hand side of (28). If we divide $n!$ by the right-hand side of (28), then the ratio tends to 1 as $n \to \infty$. This result was found in 1730 by J. Stirling [61 p. 135]. It should be noted, however, that preceding Stirling, in 1730 A. De Moivre [17 p. 170] discovered that

$$n! \sim C\sqrt{n}\ (\frac{n}{e})^n \tag{29}$$

as $n \to \infty$ where C is a constant which he found numerically by using the asymptotic series

$$\log C = 1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680} - \cdots . \tag{30}$$

By the inspiration of De Moivre his friend Stirling studied the problem and demonstrated that $C = \sqrt{2\pi}$. This fact can easily be deduced from the product representation of $4/\pi$ which was found in 1655 by J. Wallis [66]. The unnoticed fact that the series (30) is divergent was pointed out by Th. Bayes [5].

As a refinement of (28) we can write that

$$n! = \sqrt{2\pi n}\ (\frac{n}{e})^n e^{\Theta_n} \tag{31}$$

where $1/(12n+1) < \Theta_n < 1/12n$. For the proof of (31) we refer to H. Robbins [55].

In proving the limit theorem (17) A. De Moivre used formula (28) to find a good approximation for (1). He found numerically $\Phi(1)$, $\Phi(2)$ and $\Phi(3)$.

36. Classical Problems. It seems the oldest knwon problem in the theory of probability is the division problem (the problem of points). This problem is of considerable interest, it had a great influence on the development of probability theory and it is the predecessor of two other important problems, namely, the ruin problem and the problem of the duration of plays. In what follows we shall give a survey of the aforementioned three problems.

The Division Problem. We can formulate this problem in the following general form:

Two players A and B play a sequence of games. In each game, independently of the others, either A wins a point with probability p, or B wins a point with probability q where $p+q = 1$. The players agree to continue the games until one has won a predetermined number of games. However, the match has to stop when A still needs a points and B still needs b points to win the series. In what proportion should the stakes be divided?

Denote by $P_A(a,b)$ the probability that A wins the series and by $P_B(a,b)$ the probability that B wins the series. Obviously $P_A(a,b) + P_B(a,b) = 1$.

It is evident that in the case of fair sharing the stakes should be divided in the proportion of

$$P_A(a,b) : P_B(a,b) = P_A(a,b) : [1-P_A(a,b)] . \tag{1}$$

Thus the problem is to find the probability $P_A(a,b)$ for $a = 0,1,2,\ldots$ and $b = 0,1,2,\ldots$. In the particular case of $p = q = \frac{1}{2}$ the following table contains $P_A(a,b)$ for $0 \leq a \leq 6$, and $0 \leq b \leq 6$.

$P_A(a,b)$

a \ b	0	1	2	3	4	5	6
0	--	1	1	1	1	1	1
1	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	$\frac{63}{64}$
2	0	$\frac{1}{4}$	$\frac{4}{8}$	$\frac{11}{16}$	$\frac{26}{32}$	$\frac{57}{64}$	$\frac{120}{128}$
3	0	$\frac{1}{8}$	$\frac{5}{16}$	$\frac{16}{32}$	$\frac{42}{64}$	$\frac{99}{128}$	$\frac{219}{256}$
4	0	$\frac{1}{16}$	$\frac{6}{32}$	$\frac{22}{64}$	$\frac{64}{128}$	$\frac{163}{256}$	$\frac{382}{512}$
5	0	$\frac{1}{32}$	$\frac{7}{64}$	$\frac{29}{128}$	$\frac{93}{256}$	$\frac{256}{512}$	$\frac{638}{1024}$
6	0	$\frac{1}{64}$	$\frac{8}{128}$	$\frac{37}{256}$	$\frac{130}{512}$	$\frac{386}{1024}$	$\frac{1024}{2048}$

According to O. Ore [92] it seems likely that the problem is of Arabic origin. He found some particular versions of the aforementioned problem in Italian mathematical manuscripts dating from as early as 1380. The problem appears for the first time in printed form in 1494 in the book of Lucas dal Burgo Pacioli [94 p. 197]. In Pacioli's version $p = q = 1/2$, the players have agreed to play 6 games and $a = 1$ and $b = 3$.

Pacioli gave the incorrect answer 5:3 which is simply the ratio of the number of games already won by the two players. The correct answer is 7:1.

In 1556 Nicolo Tartaglia [64] (Part I, p. 265.) discussed the problem and he too gave a wrong answer, namely, 2:1 .

In 1558 Francesco Peverone [48 p. 40] posed the same problem, with the irrelevant modification that the players have agreed to play 10 games, and got the wrong answer, 2:12 .

In 1603 L. Forestani [25] posed the same problem, with the modification that the players have agreed to play 8 games, and $a = 3$ and $b = 5$.

It is interesting to mention that L. Forestani [25] formulated the same problem for the case of three players too. Three players agreed to play 14 games, but they have to interrupt the match when they have won respectively ten, eight and five games. The proper shares of the stakes should again be determined.

In 1654 Antoine Gombauld chevalier de Méré (1607-1684) a distinguished philosopher and a prominent figure at the court of Louis XIV called the attention of Blaise Pascal (1623-1662) to the division problem. The division problem has been mentioned also in some old French books, and de Méré may have read it somewhere.

It seems that Pascal provided an incorrect solution for this problem and communicated it to Pierre de Fermat (1601-1665). In reply, Fermat found a remarkably elegant solution of the problem. He determined $P_A(a,b)$ in the case when $p = q = \frac{1}{2}$. Fermat reasoned in the following way: If A needs a points and B needs b points, then in at most $a+b-1$ games it can be decided who wins the series. Let us assume that the players actually play $a+b-1$ games regardless of the possibility that one of them already won the series. Then the number of possible series is 2^{a+b-1}, and they are equally probable because $p = q = \frac{1}{2}$. Player A wins the series if and only if he wins at least a games among the $a+b-1$ games. The number of all those sequences in which A wins exactly k games is equal to the number of combinations without repetition of $a+b-1$ elements taken k at a time. If we add these combinations for $k = a, a+1, \ldots, a+b-1$, then we obtain the number of favorable cases and $P_A(a,b)$ is equal to the number of favorable cases divided by the number of possible cases. What Fermat said in words can be expressed by the following mathematical formula:

$$P_A(a,b) = \frac{1}{2^{a+b-1}} \sum_{k=a}^{a+b-1} \binom{a+b-1}{k} . \tag{2}$$

In Fermat's formula (2) the binomial coefficient $\binom{a+b-1}{k}$ is interpreted as the number of combinations of $a+b-1$ elements taken k at a time. It is not clear whether Fermat was familiar with Hérigone's formula for $\binom{n}{k}$ [formula (35.24) in the previous section] or whether he enumerated the combinations in another way. It should be noted that Fermat discovered already in 1636 that the figurate numbers $F_n^k = \binom{n+k-1}{k-1}$ satisfy the relation $kF_n^{k+1} = (n+1)F_{n+1}^k$. (See L. E. Dickson [19] II. p. 7.) Possibly Fermat used this recurrence formula to find $\binom{n}{k}$ too. This part of the correspondence between Pascal and Fermat has unfortunately not been preserved. The above information is taken from a letter written by Pascal to Fermat on July 29, 1654. (See P. Fermat [81], B. Pascal [93], and P. R. Montmort [91].) In this letter Pascal recalls Fermat's solution. He writes that he admires Fermat's method of solution, and admits that he himself was wrong. In this letter Pascal discloses also that he has found another solution which is short and neat. Pascal explains in examples how to calculate $P_A(a,b)$ if $a = 1, b = 2$; $a = 1, b = 3$; $a = 2, b = 3$. Actually, Pascal calculated expectations instead of probabilities, but this does not make any essential difference. We can express Pascal's discovery by the following recurrence formula

$$P_A(a,b) = \frac{1}{2} P_A(a-1,b) + \frac{1}{2} P_A(a,b-1) \tag{3}$$

for $a \geqq 1$ and $b \geqq 1$ where $P_A(a,0) = 0$ for $a = 1,2,\ldots$ and $P_A(0,b) = 1$ for $b = 1,2,\ldots$. Pascal's formula (3) makes it possible to calculate quickly $P_A(a,b)$ for small values of a and b.

Pascal's formula (3) can easily be seen to be true for any $a \geqq 1$ and $b \geqq 1$. If A needs a points and B needs b points, then A can win the series in the following two mutually exclusive ways: A wins the next game which has probability $\frac{1}{2}$ and he wins the series which has probability $P_A(a-1,b)$ or B wins the next game which has probability $\frac{1}{2}$ and A wins the series which has probability $P_A(a,b-1)$. Then (3) follows by the theorem of total probability.

Pascal introduced the notion of "the value of a point." If A needs a points and B needs b points and A wins the next game, then the value of the point for A is

$$p_A(a,b) = P_A(a-1,b) - P_A(a,b) \tag{4}$$

in the case of a unit stake; otherwise, the right-hand side of (4) should be multiplied by the total number of stakes. Pascal observed that $p_A(a.b)$ can also be obtained by the same recurrence formula as $P_A(a,b)$, that is,

$$p_A(a,b) = \frac{1}{2}\, p_A(a-1,b) + \frac{1}{2}\, p_A(a,b-1) \tag{5}$$

for $a \geqq 2$ and $b \geqq 2$ where $p_A(a,1) = 1/2^a$ for $a \geqq 1$ and $p_A(1,b) = 1/2^b$ for $b \geqq 1$. Pascal also observed that $p_A(a,b)$ can easily be obtained with the aid of the arithmetic triangle. Indeed we have

$$p_A(a,b) = \binom{a+b-2}{a-1} \frac{1}{2^{a+b-1}} \tag{6}$$

for $a \geqq 1$ and $b \geqq 1$. Obviously, this discovery led Pascal to declare that Fermat and himself had found the same solution. "The truth is the same at Toulouse and at Paris."

It seems that in a missing letter Fermat indicated that his method can also be applied in the case of three or more players. Apparently, Pascal misunderstood Fermat and believed that Fermat's solution for two players can be applied verbatim for three players, which is evidently not what Fermat meant. Pascal expressed his opposite view in his letter to Fermat dated August 24, 1654. (See P. Fermat [81], B. Pascal [93], and P. R. Montmort [91] pp. 232-244.) In his letter to Pascal, dated September 25, 1654, Fermat brilliantly explained that the method of requiring that the players continue to play a particular number of games even if one of them might have already won the series, serves only to simplify the rules and to make all the possible sequences equally probable, or to state it more intelligibly, "to reduce all the fractions to the same denominator." Fermat explained how his method should be applied correctly in the case of three or more players.

Fermat noted that the same result can be obtained without the artifice of the continuation of the games after winning; however, in this case the possible sequences will not be equally probable. He illustrated this method for the case of three players, but it can equally be applied also for the case of two players. By this method of Fermat we can express the probability (2) in the following equivalent form

$$(7) \qquad P_A(a,b) = \sum_{n=a-1}^{a+b-2} \binom{n}{a-1} \frac{1}{2^{n+1}} .$$

For A can win the series in $n+1$ games where $n = a, a+1, \ldots, a+b-2$. Player A wins the series in $n+1$ games if he wins $a-1$ games among the first n games which has probability $\binom{n}{a-1} \frac{1}{2^n}$ and he wins the $n+1$-st game which has probability $\frac{1}{2}$. Since the events in question are independent, the probabilities multiply. If we add the product for every $n = a, a+1, \ldots, a+b-2$, then we get $P_A(a,b)$.

It is interesting to note that Fermat's second solution which is given by formula (7), and Pascal's solution which can be obtained by formulas (4) and (6) show complete agreement. Obviously this agreement prompted Pascal to reply to Fermat in his letter of October 27, 1654, "I admire your method for the division problem all the more because I understand it very well. It is entirely yours, and has nothing in common with mine, and it reaches the same end easily."

Although Fermat calculated probabilities skillfully even if the possible cases were not equally probable, he did not consider the problem of finding $P_A(a,b)$ in the case when $p \neq q$. This generalization has been given only after Jakob Bernoulli's results concerning repeated trials become widely known. It should be mentioned that Jakob Bernoulli [7](pp. 107-112) gives demonstrations for formulas (2) and (7) both, but he does not make any suggestions for a possible extension to the case of $p \neq q$. Possibly he worked out these proofs before discovering his celebrated formula (35.1).

Pascal and Fermat did not write down explicit formulas for $P_A(a,b)$. They explained only in words how $P_A(a,b)$ can be obtained, and illustrated their results by examples. An explicit formula for $P_A(a,b)$ was given only in 1708 by P. R. Montmort [90 p. 177] in the case of $p = q = \frac{1}{2}$. In 1713 in the second edition of his book P. R. Montmort [91] (pp. 244-246) gave two explicit expressions for $P_A(a,b)$ in the general case too. These formulas are the counterparts of (2) and (7). In the general case (2) becomes

$$(8) \qquad P_A(a,b) = \sum_{k=a}^{a+b-1} \binom{a+b-1}{k} p^k q^{a+b-1-k}$$

and (7) becomes

$$(9) \qquad P_A(a,b) = \sum_{n=a-1}^{a+b-2} \binom{n}{a-1} p^a q^{n-a+1} .$$

The proofs of (8) and (9) follow on the same lines as the proofs of (2) and (7) except that now Bernoulli's formula (35.1) should be used. Formula (8) was communicated to P. R. Montmort by Johann Bernoulli in a letter dated March 17, 1710. (See P. R. Montmort [91] pp. 294-295.) Formula (9) seems to have been found by Montmort himself.

Pascal's recurrence formula (3) in the general case becomes

$$(10) \qquad P_A(a,b) = pP_A(a-1,b) + qP_A(a,b-1)$$

for $a \geq 1$ and $b \geq 1$ where $P_A(a,0) = 0$ for $a \geq 1$ and $P_A(0,b) = 1$

for $b \geqq 1$. For small values of a and b the probability $P_A(a,b)$ can be calculated quickly by (10). However, the general solution of the difference equation (9) can be obtained only by using more advanced methods which were developed in 1773 by P. S. Laplace [86], [39] and in 1775 by J. L. Lagrange [88] .

The Ruin Problem. The first known ruin problem was proposed by B. Pascal in 1655 to Pierre de Carcavy for the purpose of transmitting it to Christiaan Huygens (1629-1695). The problem is as follows:

"Two players A and B play a sequence of games with three dice and fixed points fourteen and eleven respectively. Each player has twelve counters, and receives one counter from the other every time his own number of points turns up. What are the odds for one player to ruin the other?"

We can state this problem more generally as follows:

Two players, A and B, play a series of games. In each game independently of the others, either A wins a counter from B with probability p or B wins a counter from A with probability q where $p > 0$, $q > 0$ and $p+q = 1$. The series ends if either A wins a total number of a counters from B or B wins a total number of b counters from A. If initially A has b counters and B has a counters, then the games are continued until one of the two players wins all the counters of his adversary, in other words, until one of the two players is ruined.

Denote by $P(a,b)$ the probability that A wins the series, and by $Q(a,b)$ the probability that B wins the series. The problem is to find the ratio $Q(a,b)/P(a,b)$.

In the aforementioned problem of Pascal the probability of throwing 14 points with three dice is 15/216 ad the probability of throwing 11 points with three dice is 27/216, and therefore $p/q = 5/9$. Furthermore $a = b = 12$.

The probabilities $P(a,b)$ and $Q(a,b)$ are given by the following formulas

$$(11)\qquad P(a,b) = \begin{cases} \dfrac{p^a(p^b - q^b)}{p^{a+b} - q^{a+b}} & \text{if } p \neq q\,, \\[2ex] \dfrac{b}{a+b} & \text{if } p = q\,, \end{cases}$$

and

$$(12)\qquad Q(a,b) = \begin{cases} \dfrac{q^b(q^a - p^a)}{q^{a+b} - p^{a+b}} & \text{if } p \neq q\,, \\[2ex] \dfrac{a}{a+b} & \text{if } p = q\,. \end{cases}$$

We have $P(a,b) + Q(a,b) = 1$, and

$$(13)\qquad \frac{Q(a,b)}{P(a,b)} = \begin{cases} \dfrac{1 - (\frac{p}{q})^a}{(\frac{p}{q})^a - (\frac{p}{q})^{a+b}} & \text{if } p \neq q\,, \\[2ex] \dfrac{a}{b} & \text{if } p = q\,. \end{cases}$$

In 1657 C. Huygens [32] found $Q(a,b)/P(a,b)$ in the aforementioned particular case when $a = b = 12$ and $p/q = 5/9$. C. Huygens [32] included this problem as the last one in his collection of exercises for the reader.

About 1680 or so, Jakob Bernoulli [7 pp. 67-71], [8 , I-II pp. 71-75 and p. 138] found $Q(a,b)/P(a,b)$ in the general case. Formula (13) was proved only in 1711 by A. De Moivre [76 pp. 227-228], [77 pp. 23-24], [78 pp. 44-47], [79 pp. 51-54] . A.De Moivre's proof for (13) is a very ingenious direct proof which we shall present here in the following simple way. Let us imagine that in each game A receives or pays a certain amount of money depending on his accumulated gain. If at the beginning of a game A's accumulated gain is j counters $(j = a-1,\ldots,-b+1)$ and if he wins, he receives $(q/p)^j$ units of money; if he loses, he pays $(q/p)^{j-1}$ units of money. In each game A's expected receipt is 0 because $p(q/p)^j - q(q/p)^{j-1} = 0$ for all j. Thus the expected total receipt of A at the end of the series is also 0, that is,

$$(14) \qquad P(a,b)[1+(\tfrac{q}{p})+\ldots+(\tfrac{q}{p})^{a-1}] - Q(a,b)[(\tfrac{p}{q})+\ldots+(\tfrac{p}{q})^{b}] = 0 ,$$

whence

$$(15) \qquad \frac{Q(a,b)}{P(a,b)} = \begin{cases} \dfrac{q^b(p^a - q^a)}{p^a(p^b - q^b)} & \text{if } p \neq q , \\ \dfrac{a}{b} & \text{if } p = q . \end{cases}$$

This is in agreement with (13). If we can show that $P(a,b) + Q(a,b) = 1$, then (13) implies both (11) and (12).

The solution of the ruin problem can also be found in the book of P. R. Montmort [90 p. 178], [91 p. 277, pp. 295-296, p. 311].

In 1780 P. S. Laplace [87 pp. 387-390] proved (11) and (12) by showing that the probabilities $\pi_j = P(j,a+b-j)$ $(j = 1,2,\ldots, a+b-1)$ satisfy the recurrence formula

$$(16) \qquad \pi_j = p\pi_{j-1} + q\pi_{j+1}$$

where $\pi_0 = 1$ and $\pi_{a+b} = 0$. Here π_j is the probability that A wins the series provided that he has j counters in his possession. The event of winning the series under this condition can occur in two mutually exclusive ways: he wins the next game in which case his capital increases by one counter or he loses the next game in which case his capital decreases by one counter. Thus we get (16).

If $p = q$, then the general solution of (16) is

(17) $$\pi_j = \alpha + \beta j$$

for $j = 0,1,\ldots,a+b$. Since $\pi_0 = 1$ and $\pi_{a+b} = 0$ we obtain that

(18) $$\pi_j = 1 - \frac{j}{a+b}$$

for $j = 0,1,\ldots, a+b$ and $P(a,b) = \pi_a = b/(a+b)$.

If $p \neq q$, then the general solution of (16) is

(19) $$\pi_j = \alpha + \beta\left(\frac{p}{q}\right)^j$$

for $j = 0,1,\ldots, a+b$. Since $\pi_0 = 1$ and $\pi_{a+b} = 0$, we obtain that

(20) $$\pi_j = \frac{\left(\frac{p}{q}\right)^j - \left(\frac{p}{q}\right)^{a+b}}{1 - \left(\frac{p}{q}\right)^{a+b}}$$

for $j = 0,1,\ldots, a+b$. Since $P(a,b) = \pi_a$, we get (11) for $p \neq q$. This completes the proof of (11). In a similar way we can prove (12).

It is interesting to note that as a byproduct we obtain that $P(a,b) + Q(a,b) = 1$. This implies that the probability that the series never ends is 0.

<u>The Problem of the Duration of Plays.</u> The ruin problem which we discussed before leads in a natural way to more general problems. One such problem is to find the probability that the series ends in at most n games. This problem can be reduced to the problem of finding the probability that A wins the series in at most n games, and the probability that B wins the series in at most n games.

Our objective is to mention the solutions of these problems. Let us formulate the problems precisely:

Two players, A and B, play a series of games. In each game, independently of the others, either A wins a counter from B with probability p or B wins a counter from A with probability q where $p > 0$, $q > 0$ and $p+q = 1$. The series ends if either A wins a total number of a counters from B or B wins a total number of b counters from A.

Denote by $P_n(a,b)$ the probability that A wins the series in at most n games.

Denote by $Q_n(a,b)$ the probability that B wins the series in at most n games.

Denote by ρ the duration of the games, that is, the number of games played until the series ends. Then

$$\mathbf{P}\{\rho \leq n\} = P_n(a,b) + Q_n(a,b) . \tag{21}$$

Obviously

$$\lim_{n \to \infty} P_n(a,b) = P(a,b) \quad \text{and} \quad \lim_{n \to \infty} Q_n(a,b) = Q(a,b) \tag{22}$$

where the right-hand sides are given by (11) and (12) respectively. Since $P(a,b) + Q(a,b) = 1$, it follows that

$$\underset{\sim}{P}\{\rho < \infty\} = 1 . \tag{23}$$

We obtain an interesting variant of the series of games mentioned above if we suppose that $b = \infty$, that is, if we suppose that the series ends if A wins a total number of a counters from B regardless of how many counters B won from A. In this case we may assume that initially A has an unlimited number of counters, and B has a counters, and the series ends if B is ruined. Then B cannot win the series and therefore $Q_n(a, \infty) = 0$ and $Q(a, \infty) = 0$.

In this case

$$\underset{\sim}{P}\{\rho \leqq n\} = P_n(a, \infty) \tag{24}$$

and

$$\underset{\sim}{P}\{\rho < \infty\} = P(a, \infty) . \tag{25}$$

If $P(a, \infty) < 1$, then there is a positive probability that the series does not end in a finite number of games. Actually, we have

$$P(a, \infty) = \begin{cases} (\frac{p}{q})^a & \text{if } p < q , \\ 1 & \text{if } p \geqq q . \end{cases} \tag{26}$$

The probabilities $P_n(a,b)$ and $Q_n(a,b)$ are given by the following explicit formulas:

(27)
$$P_n(a,b) = \sum_{i \leq \frac{n+a}{2}} \left[\sum_{k=0}^{\infty} \binom{n}{i-a-k(a+b)} - \sum_{k=1}^{\infty} \binom{n}{i-k(a+b)} \right] p^i q^{n-i} + \sum_{i > \frac{n+a}{2}} \left[\sum_{k=0}^{\infty} \binom{n}{i+k(a+b)} - \sum_{k=1}^{\infty} \binom{n}{i-a+k(a+b)} \right] p^i q^{n-i}$$

and

(28)
$$Q_n(a,b) = \sum_{i \geq (n-b)/2} \left[\sum_{k=1}^{\infty} \binom{n}{i-a-k(a+b)} - \sum_{k=1}^{\infty} \binom{n}{i+k(a+b)} \right] p^i q^{n-i} + \sum_{i < (n-b)/2} \left[\sum_{k=0}^{\infty} \binom{n}{i-k(a+b)} - \sum_{k=0}^{\infty} \binom{n}{i-a-k(a+b)} \right] p^i q^{n-i} .$$

Probability $Q_n(a,b)$ can easily be obtained from $P_n(a,b)$ by interchanging the roles of A and B . Actually,

(29)
$$Q_n(a,b) = P_n(a,b)(q/p)^b .$$

We note that if we interchange the order of summation in (27), then we obtain the following equivalent expresssion:

(30)
$$P_n(a,b) = \sum_{k=0}^{\infty} p^{k(a+b)+a} q^{k(a+b)} \left[\sum_{j \leq \frac{n-a-2k(a+b)}{2}} \binom{n}{j} p^j q^{n-a-j-2k(a+b)} + \sum_{j < \frac{n-a-2k(a+b)}{2}} \binom{n}{j} q^j p^{n-a-j-2k(a+b)} \right] - \sum_{k=1}^{\infty} p^{k(a+b)} q^{k(a+b)-a} \cdot \left[\sum_{j \leq \frac{n+a-2k(a+b)}{2}} \binom{n}{j} p^j q^{n+a-j-2k(a+b)} + \sum_{j < \frac{n+a-2k(a+b)}{2}} \binom{n}{j} q^j p^{n+a-j-2k(a+b)} \right] .$$

Probability $Q_n(a,b)$ can also be expressed in a similar way.

The probability that the series does not end in n games is given by

$$(31) \qquad \underset{\sim}{P}\{\rho > n\} = \sum_{\frac{n-b}{2} < i < \frac{n+a}{2}} \left[\sum_{k=-\infty}^{\infty} \binom{n}{i+k(a+b)} - \sum_{k=-\infty}^{\infty} \binom{n}{i-a+k(a+b)} \right] p^i q^{n-i} .$$

Formulas (27), (28), (30), (31) contain only a finite number of terms. If k is a sufficiently large positive or negative integer, then the corresponding binomial coefficients vanish.

We note that we use the following definition of the binomial coefficient $\binom{x}{k}$. For any x

$$(32) \qquad \binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$$

if $k = 1,2,\dots$; $\binom{x}{0} \equiv 1$ and $\binom{x}{k} \equiv 0$ if $k = -1,-2,\dots$.

In 1708 P. R. Montmort [90 , p. 184] showed that if $a = b = 3$ and $p = q = \frac{1}{2}$, then

$$(33) \qquad \underset{\sim}{P}\{\rho \leq 2m+1\} = 1 - \left(\frac{3}{4}\right)^m$$

for $m = 0,1,2,\dots$.

In 1711 A. De Moivre [76 , p. 261] published a practical procedure for finding $\underset{\sim}{P}\{\rho > n\}$ in the general case. See also A. De Moivre [77 pp. 113-114], [78 p. 173] , [79 p. 203] . A. De Moivre observed that if we multiply $(p+q)$ n-times with itself in such a way that after each multiplication we remove those terms (if any) which have the forms $p^{a+j} q^j$ $(j = 0,1,2,\dots)$ and $p^j q^{b+j}$ $(j = 0,1,2,\dots)$, then finally we get $\underset{\sim}{P}\{\rho > n\}$.

In a letter dated November 15, 1710 and addressed to Johann Bernoulli (1667-1748), P. R. Montmort (1678-1719) mentioned that he obtained a general solution for the problem of the duration of games, and he also suggested the problem for the consideration of his nephew Niclaus Bernoulli (1687-1759). (See [91] pp. 303-307.) In his letter to P. R. Montmort dated February 26, 1711, N. Bernoulli gave an explicit expression for $P_n(a,b)$. (See [91] pp. 308-314.) N. Bernoulli obtained exactly formula (30) for $P_n(a,b)$ in the case when $n = a+2m$ $(m = 0,1,2,\ldots)$. (See [91] p. 310.) Since obviously, $P_{a+2m+1}(a,b) = P_{a+2m}(a,b)$, N. Bernoulli's formula gives a complete solution of the problem. In his letter to N. Bernoulli dated April 10, 1711, P. R. Montmort replied that he admired N. Bernoulli's formula, but he could not understand it. (See [91] pp. 315-323.) N. Bernoulli in his letter to P. R. Montmort dated November 10, 1711 gave examples for the application of his formula. (See [91] pp. 323-337.) Afterwards, in his letter to N. Bernoulli dated March 1, 1712, P. R. Montmort wrote that he found that N. Bernoulli's result and his own result were the same except that P. R. Montmort had considered only the particular case $p = q = \frac{1}{2}$. (See [91] pp. 337-347.)

In 1713 P. R. Montmort [91 pp. 268-277] published N. Bernoulli's general formula for $P_n(a,b)$. (See also [91] p. 275, p. 310, p. 324.)

In 1718 A. De Moivre [77 pp. 122-124] also published N. Bernoulli's general solution and he attributed it to P. R. Montmort and N. Bernoulli. A. De Moivre remarked also that the same solution can be obtained by using his own method published in 1711. (See [76 p. 262], [77 pp. 119-122] , [78 pp. 179-181], [79 pp. 208-210].) In 1738 A. De Moivre [78 , pp. 181-184], [79 pp. 210-213] published this solution again; however, at this time, as his own result. See his Remark [78 , pp. 181-182], [79 . pp. 210-211] in which he gives a somewhat questionable explanation for changing his attitude. A. De Moivre [77 , pp. 122-123], [78 , pp. 182-184], [79 , pp. 211-213] expressed in words how $P_n(a,b)$ and $Q_n(a,b)$ can be found. If we transform his words into mathematical formulas, then we obtain formula (30) for $P_n(a,b)$ and an analogous formula for $Q_n(a,b)$. No doubt A. De Moivre must be given the credit for noticing that these formulas are valid for any n . The dilemma that both $P_n(a,b)$ and $Q_n(a,b)$ can be obtained by two apparently different formulas might explain A. De Moivre's argument. For $P_{n+1}(a,b) = P_n(a,b)$ if $n = a+2m$ $(m = 0,1,2,\ldots)$ and $Q_{n+1}(a,b) = Q_n(a,b)$ if $n = b+2m$ $(m = 0,1,2,\ldots)$.

In 1718 A. De Moivre [77 , pp. 115-119], [78 , pp. 174-179], [79 , pp. 204-208] published another solution for finding the distribution of ρ . Let us write

$$(34) \qquad \underset{\sim}{P}\{\rho = n\} = S_n(a,b)p^{(n+a)/2}\, q^{(n-a)/2} + T_n(a,b)p^{(n-b)/2}\, q^{(n+b)/2}$$

where the first term on the right-hand side of (34) is the probability that A wins the series in exactly n games and the second term on the right-hand side of (34) is the probability that B wins the series in exactly n games. A. De Moivre [77 , pp. 118-119], [78 , p. 178], [79 , p. 207] found that

$$(35) \qquad S_n(a,b) = \sum_{j=0}^{\infty} \frac{2j(a+b)+a}{a+2m} \binom{a+2m}{m-j(a+b)} - \sum_{j=0}^{\infty} \frac{(2j+1)(a+b)+b}{a+2m} \binom{a+2m}{m-b-j(a+b)}$$

if $n = a+2m$ $(n = 0,1,\ldots)$ and $S_n(a,b) = 0$ if $n = a+2m+1$ $(m = 0,1,\ldots)$. Furthermore, $T_n(a,b) = S_n(b,a)$.

We note that in 1738 A. De Moivre [78 , pp. 190-191], [79 ,pp. 219-220] expressed $S_n(a,a)$ also with the aid of trigonometric functions. A. De Moivre's formula is a particular case of the following more general one:

$$(36) \qquad S_n(a,b) = \frac{2^n}{a+b} \sum_{k=0}^{a+b-1} \left(\cos \frac{k\pi}{a+b}\right)^{n-1} \sin \frac{ka\pi}{a+b} \sin \frac{k\pi}{a+b} .$$

Accordingly, we can write that

$$(37) \qquad P_n(a,b) = \sum_{m=0}^{[\frac{n-a}{2}]} S_{a+2m}(a,b) p^{a+m} q^m$$

where $S_n(a,b)$ is given either by (35) or by (36).

Furthermore,

$$(38) \qquad Q_n(a,b) = \sum_{m=0}^{[\frac{n-b}{2}]} T_{b+2m}(a,b) p^m q^{b+m}$$

where $T_n(a,b) = S_n(b,a)$.

We note that by (34) and (36) we obtain that

$$(39) \qquad \mathbf{P}\{\rho > n\} = \frac{(4pq)^{\frac{n+1}{2}}}{a+b} \sum_{k=1}^{a+b-1} \frac{\left(\cos \frac{k\pi}{a+b}\right)^n \sin \frac{k\pi}{a+b}}{1-2\sqrt{pq}\cos \frac{k\pi}{a+b}} \left[\left(\frac{p}{q}\right)^{\frac{a}{2}} \sin \frac{ka\pi}{a+b} + \left(\frac{q}{p}\right)^{\frac{b}{2}} \sin \frac{kb\pi}{a+b}\right] .$$

According to the investigations of A. De Moivre, P. R. Montmort, and N. Bernoulli we have three expressions for the probabilities $P_n(a,b)$ and $Q_n(a,b)$ and hence we have also three expressions for the distribution of ρ . These authors did not provide proofs for their results, and did not indicate how they obtained their results.

Rigorous proofs for (30) and for (37), where $S_n(a,b)$ is given by (36), were given only in 1776 by J. L. Lagrange [88 , pp. 238-249]. J. L. Lagrange has obtained his results by solving a linear difference equation. In 1812 P. S. Laplace [39, pp. 228-242] proved (37), where $S_n(a,b)$ is given by (36), by using the method of generating functions. Actually, P. S. Laplace considered the problem of finding $P_n(a,b)$ as early as 1773 and he obtained partial results in his papers [85 , pp. 11-16], and [86 , pp. 176-188]. For other proofs we refer to A. M. Ampère [69], R. L. Ellis [128], L. Bachelier [72], [73], D. Arany [70], J. V. Uspensky [99, pp. 154-158], W. Feller [80, pp. 344-354], K. Jordan [83, pp. 397-420], and E. C. Fieller [82] .

In what follows we shall give simple elementary proofs for the above mentioned three formulas for $P_n(a,b)$ and $Q_n(a,b)$. The proofs presented here are based on the reflection principle and on the method of inclusion and exclusion. It is probable that the proofs we shall give in this section are closely related to the original methods of A. De Moivre, P. R. Montmort, and N. Bernoulli.

If we suppose that $b = \infty$, that is, A has an unlimited number of counters, then the probability that A wins the series in a finite number of games is $\underset{\sim}{P}\{\rho < \infty\} = 1$ whenever $p \geqq q$ and $\underset{\sim}{P}\{\rho < \infty\} = (p/q)^a < 1$ whenever $p < q$. If $p < q$, then $1-(p/q)^a$ is the probability that the series does not end in a finite number of games.

If a is finite and $b = \infty$, then we have

$$(40) \qquad \underset{\sim}{P}\{\rho \leqq n\} = \sum_{i \leqq \frac{n+a}{2}} \binom{n}{i-a} p^i q^{n-i} + \sum_{i > \frac{n+a}{2}} \binom{n}{i} p^i q^{n-i} ,$$

or in another form

$$(41) \qquad \underset{\sim}{P}\{\rho \leqq n\} = \sum_{m=0}^{[\frac{n-a}{2}]} \frac{a}{a+2m} \binom{a+2m}{m} p^{a+m} q^m .$$

By (41) we can write that

$$(42) \qquad \underset{\sim}{P}\{\rho = a+2m\} = \frac{a}{a+2m} \binom{a+2m}{m} p^{a+m} q^m$$

for $m = 0,1,2,\ldots$.

The formula (40) was found by A. De Moivre in 1708 and published in 1711. (See A. De Moivre [76 p. 262], [77 pp. 119-122], [78 pp. 179-181], [79 pp. 208-210].)

A. De Moivre did not mention how he obtained his result, but it is probable that he essentially used the method of reflection. The second form, (41) can be obtained from (40) by simple transformations. Formula (41) was published by A. De Moivre in 1718. (See A. De Moivre [77 p. 121], [78 p. 181], [79 p. 210].) These results of A. De Moivre are very significant. As A. De Moivre stated himself, these solutions led him to the solution of the general problem of the duration of plays. (See A. De Moivre [78 p. 181], [79 p. 210].) In fact P. R. Montmort and N. Bernoulli had preceded A. De Moivre in obtaining an explicit formula for $P_n(a,b)$ and $Q_n(a,b)$.

Formula (41) was proved only in 1773 by P. S. Laplace [86, pp. 188-193], [39, p. 235] and both (40) and (41) were proved in 1776 by J. L. Lagrange [88, pp. 230-238].

An elementary proof for (22) was found in 1887 by D. André [160]. See also J. V. Uspensky [99, pp. 147-153] and the author [63, pp. 2-9].

It is interesting to recall A. M. Ampère [69, p. 9] who comments formula (42) as remarkable for its simplicity and elegance.

First we shall prove formulas (40) and (41) for $P_n(a,\infty)$. Suppose that the players actually play n games regardless of wether A has already won the series or not. Denote by η_n the gain of A at the end of the n-th game, that is, the total number of counters won by A during the n games. Obviously we have

$$\mathbf{P}\{\eta_n = 2i-n\} = \binom{n}{i} p^i q^{n-i} \tag{43}$$

for $i = 0,1,\ldots,n$. For $\eta_n = 2i-n$ if and only if A wins i games and B wins $n-i$ games. The number of such series is $\binom{n}{i}$ and each series has probability $p^i q^{n-i}$. This implies (43).

Now we shall prove that for $i = 0,1,\ldots, n$

$$(44) \qquad \underset{\sim}{P}\{\rho \leq n \text{ and } \eta_n = 2i-n\} = \begin{cases} \binom{n}{i} p^i q^{n-i} & \text{if } 2i \geq n+a , \\ \binom{n}{i-a} p^i q^{n-i} & \text{if } 2i \leq n+a . \end{cases}$$

If $2i \geq n+a$, then $\eta_n = 2i-n \geq a$ and consequently $\rho \leq n$ necessarily occurs. Thus $\underset{\sim}{P}\{\rho \leq n$ and $\eta_n = 2i-n\} = \underset{\sim}{P}\{\eta_n = 2i-n\}$ given by (43). This proves (44) for $2i \geq n+a$.

If $i < a$, then (44) is evidently 0. It remains to consider the case when $2a \leq 2i \leq n+a$. Denote by C_1 the set of the series of games in which A wins i games, B wins $n-i$ games, and A wins at least once a counters from B. Denote by R_{n+a-i} the set of the series of games in which A wins $n+a-i$ games and B wins $i-a$ games. There is a one-to-one correspondence between the series in the two sets C_1 and R_{n+a-i}. For if in each series we change the results of all those games into their opposites which follow the game in which A wins a total number of a counters from B for the first time, then each series in C_1 is mapped into a series in R_{n+a-i}, and conversely each series in R_{n+a-i} is mapped into a series in C_1, and different series correspond to different series. Thus the number of series in C_1 is equal to the number of series in R_{n+a-i} which is evidently $\binom{n}{i-a}$. Since each series in C_1 has probability $p^i q^{n-i}$, (44) follows for $2a \leq 2i \leq n+a$.

If we add (44) for $i = 0,1,\ldots, n$, then we get (40).

We note that by (44)

$$(45) \qquad P\{\rho \leqq n \ \text{ and } \ \eta_n = 2i-n\} = \begin{cases} P\{\eta_n = 2i-n\} & \text{if} \quad 2i \geqq n+a \,, \\ (\frac{p}{q})^a P\{\eta_n = 2i-2a-n\} & \text{if} \quad 2i \leqq n+a \,, \end{cases}$$

whence it follows that

$$(46) \qquad P\{\rho \leqq n\} = P\{\eta_n \geqq a\} + (\frac{p}{q})^a P\{\eta_n < -a\} \,.$$

By (44) we have also

$$(47) \qquad P\{\rho > n \ \text{ and } \ \eta_n = 2i-n\} = \begin{cases} [\binom{n}{i} - \binom{n}{i-a}]p^i q^{n-i} & \text{if} \quad 2i \leqq n+a \,, \\ 0 & \text{if} \quad 2i \leqq n+a \,. \end{cases}$$

Since evidently

$$(48) \qquad P\{\rho = n\} = pP\{\rho > n-1 \ \text{ and } \ \eta_{n-1} = a-1\} \,,$$

it follows from (47) that

$$(49) \qquad P\{\rho = a+2m\} = [\binom{a+2m-1}{a+m-1} - \binom{a+2m-1}{m-1}]p^{a+m} q^m =$$

$$= \frac{a}{a+2m} \binom{a+2m}{m} p^{a+m} q^m$$

for $m = 0,1,2,\ldots$ which is in agreement with (42). If we add (49) for $m \leqq (n-a)/2$, then we obtain (41).

Finally, we shall prove formulas (27) and (28) for $P_n(a,b)$ and $Q_n(a,b)$ respectively, and we shall show that $S_n(a,b)$ can be expressed by (35) or by (36).

Denote by A_n the event that A wins the series in at most n games, and by B_n the event that B wins the series in at most n games. Then $P_n(a,b) = \underset{\sim}{P}\{A_n\}$ and $Q_n(a,b) = \underset{\sim}{P}\{B_n\}$.

In finding the probabilities $P_n(a,b)$ and $Q_n(a,b)$ we may assume, without loss of generality, that the players actually play n games regardless of whether one of them already has won the series.

We shall show that if we apply repeatedly the same reflection principle which we used in proving (40), then we obtain formula (27) for $P_n(a,b)$.

If the players actually play n games, then it may happen more than once that A's gain reaches a and B's gain reaches b . In this case A_n can be interpreted as the event that A's gain reaches a before B's gain reaches b (if at all) in the n games.

Denote by η_n the gain of A at the end of the n-th game. We have

(50) $$\underset{\sim}{P}\{\eta_n = 2i-n\} = \binom{n}{i}p^i q^{n-i}$$

for $i = 0,1,\dots, n$ because $\eta_n = 2i-n$ if A wins i games and B wins $n-i$ games.

Denote by $U_n(a,b,i)$ the number of series of length n in which A wins a total number of i games and A's gain reaches a before B's gain reaches b (if at all). Then

(51) $$\underset{\sim}{P}\{A_n \text{ and } \eta_n = 2i-n\} = U_n(a,b,i)p^i q^{n-i} .$$

Now we are going to find $U_n(a,b,i)$. Denote by C_{2k} $(k = 1,2,\dots)$ the set of all those series of length n in which A wins i games and A's gain at least k times passes from $-b$ to a. Furthermore, denote by C_{2k+1} $(k = 0,1,2,\dots)$ the set of all those series of length n in which A wins i games, A's gain at least once reaches a and subsequently at least k times passes from $-b$ to a. Let $N(C_j)$ $(j = 1,2,\dots)$ denote the number of series in the set C_j. Then by the method of inclusion and exclusion we obtain that

(52) $$U_n(a,b,i) = \sum_{j=1}^{\infty} (-1)^{j-1} N(C_j) .$$

If $2i \leqq n+a$, then we have

(53) $$N(C_{2k}) = \binom{n}{i-k(a+b)}$$

and

(54) $$N(C_{2k+1}) = \binom{n}{i-a-k(a+b)}.$$

If $2i \geqq n+a$, then we have

(55) $$N(C_{2k}) = \binom{n}{i-a+k(a+b)}$$

and

(56) $$N(C_{2k+1}) = \binom{n}{i+k(a+b)} .$$

These formulas can be proved by using the method of reflection. We shall prove only (53). Formulas (54), (55), (56) can be proved in a similar way.

Let $2i \leq n+a$. We shall show that there is a one-to-one correspondence between the series in C_{2k} and the series in the set $R_{i-k(a+b)}$ where $R_{i-k(a+b)}$ contains all those series in which A wins exactly $i-k(a+b)$ games in the n games. The number of series in $R_{i-k(a+b)}$ is evidently $N(R_{i-k(a+b)}) = \binom{n}{i-k(a+b)}$.

Consider a series in C_{2k} and let us mark $2k$ games as follows: First, we mark the game in which A's gain first attains $-b$. Second, we mark the game in which A's gain first attains a afterwards. Third, we mark the game in which A's gain first attains $-b$ again afterwards. We continue this process through $2k$ games. Now starting from the first marked game let us change the results of all the subsequent games into their opposites. Then starting from the second marked game let us again change the results of all the subsequent games into their opposites. Continuing this process, finally, starting from the $2k$-th marked game let us change the results of all the subsequent games into their opposites. Thus we obtain a series which belongs to $R_{i-k(a+b)}$. By this mapping, to every series in C_{2k} there corresponds one series in $R_{i-k(a+b)}$, and to different series in C_{2k} there correspond different series in $R_{i-k(a+b)}$.

Conversely, consider a series in $R_{i-k(a+b)}$ and mark the $2k$ games in which A's gain first reaches $-b$, $-2b-a$, $-3b-2a$, ..., $-2kb-(2k-1)a$. Now starting from the first marked game, let us change the results of all the subsequent games into their opposites. Then starting from the second marked game, let us again change the results of all the subsequent games into their opposites. Continuing this process, finally, starting from the $2k$-th marked game let us change the results of all the subsequent games into their opposites. Thus we obtain a series which belongs to C_{2k}.

By this mapping, to every series in $R_{i-k(a+b)}$ there corresponds one series in C_{2k} and to different series in $R_{i-k(a+b)}$ there correspond different series in C_{2k}.

Accordingly, there is a one-to-one correspondence between the series of the two sets C_{2k} and $R_{i-k(a+b)}$. Thus $N(C_{2k}) = N(R_{i-k(a+b)}) = \binom{n}{i-k(a+b)}$ which was to be proved.

By (52), (53), (54), (55), (56) we obtain that

$$U_n(a,b,i) = \begin{cases} \sum_{k=0}^{\infty} \binom{n}{i-a-k(a+b)} - \sum_{k=1}^{\infty} \binom{n}{i-k(a+b)} & \text{if } 2i \leqq n+a , \\ \sum_{k=0}^{\infty} \binom{n}{i+k(a+b)} - \sum_{k=1}^{\infty} \binom{n}{i-a+k(a+b)} & \text{if } 2i \geqq n+a . \end{cases} \tag{57}$$

Finally, by (51) we get

$$P_n(a,b) = \sum_{i=0}^{n} U_n(a,b,i)p^i q^{n-i} . \tag{58}$$

This proves (27).

If we denote by $V_n(a,b,i)$ the number of series of length n in which A wins a total number of i games and B's gain reaches b before A's gain reaches a (if at all), then

$$\underset{\sim}{P}\{B_n \text{ and } \eta_n = 2i-n\} = V_n(a,b,i)p^i q^{n-i} \tag{59}$$

and

$$Q_n(a,b) = \sum_{i=0}^{n} V_n(a,b,i)p^i q^{n-i} . \tag{60}$$

If we interchange the roles of A and B, then we obtain that

$V_n(a,b,i) = U_n(b, a, n-i)$ and hence by (57) we obtain that

$$(61)\quad V_n(a,b,i) = \begin{cases} \sum_{k=1}^{\infty} \binom{n}{i-a+k(a+b)} - \sum_{k=1}^{\infty} \binom{n}{i+k(a+b)} & \text{if } 2i \geqq n-b , \\ \sum_{k=0}^{\infty} \binom{n}{i-k(a+b)} - \sum_{k=0}^{\infty} \binom{n}{i-a-k(a+b)} & \text{if } 2i \leqq n-b . \end{cases}$$

Formulas (60) and (61) prove (28).

Since

$$(62)\quad \underset{\sim}{P}\{\rho > n \text{ and } \eta_n = 2i-n\} = \underset{\sim}{P}\{\eta_n = 2i-n\} - \underset{\sim}{P}\{A_n \text{ and } \eta_n = 2i-n\} - \underset{\sim}{P}\{B_n \text{ and } \eta_n = 2i-n\}$$

for $i = 0,1,\ldots, n$, by (50), (51), (57), (59), (61) we obtain that

$$(63)\quad \underset{\sim}{P}\{\rho > n \text{ and } \eta_n = 2i-n\} = \left[\sum_{k=-\infty}^{\infty} \binom{n}{i+k(a+b)} - \sum_{k=-\infty}^{\infty} \binom{n}{i-a+k(a+b)}\right] p^i q^{n-i}$$

if $n-b \leqq 2i \leqq n+a$ and 0 otherwise. If we add (63) for $i = 0,1,2,\ldots,$ then we get (31).

The probabilities $P_n(a,b)$ and $Q_n(a,b)$ can also be obtained by (37) and (38) respectively where $S_n(a,b)$ is given either by (35) or by (36) and $T_n(a,b) = S_n(b,a)$. Our next aim is to prove these results.

In (34) we have obviously

$$(64)\quad S_n(a,b)p^{(n+a)/2}q^{(n-a)/2} = \underset{\sim}{P}\{\rho = n \text{ and } \eta_n = a\} = p\underset{\sim}{P}\{\rho > n-1 \text{ and } \eta_{n-1} = a-1\}$$

and

$$(65)\quad T_n(a,b)p^{(n-b)/2}q^{(n+b)/2} = \underset{\sim}{P}\{\rho = n \text{ and } \eta_n = -b\} = q\underset{\sim}{P}\{\rho > n-1 \text{ and } \eta_{n-1} = -b+1\} .$$

The extreme right members of (64) and (65) can be expressed by (63). Thus we get

$$(66)\qquad S_n(a,b) = \sum_{k=-\infty}^{\infty} \binom{a+2m-1}{m+k(a+b)} - \sum_{k=-\infty}^{\infty} \binom{a+2m-1}{m-b+k(a+b)}$$

if $n = a+2m$ $(m = 0,1,2,\ldots)$ and $S_n(a,b) = 0$ if $n = a+2m+1$ $(m = 0,1,2,\ldots)$. Furthermore, $T_n(a,b) = S_n(b,a)$. Formula (66) can easily be expressed in the form (35).

Formula (36) for $S_n(a,b)$ can be obtained from (66) by using the following elementary identity

$$(67)\qquad \sum_{j=0}^{\infty} \binom{n}{r+j(a+b)} = \frac{2^n}{a+b} \sum_{k=0}^{a+b-1} \left(\cos \frac{k\pi}{a+b}\right)^n \cos \frac{k(n-2r)\pi}{a+b}$$

which holds if $r < a+b$. If we take into consideration that

$$(68)\qquad \sum_{k=0}^{a+b-1} (1+\omega^k)^n \omega^{-kr} = \sum_{k=0}^{a+b-1} (\omega^{k/2} + \omega^{-k/2})^n \omega^{k(n-2r)/2}$$

and if we put

$$(69)\qquad \omega = e^{2\pi i/(a+b)} = \cos \frac{2\pi}{a+b} + i \sin \frac{2\pi}{a+b}$$

in (68), then we obtain (67). This proof for (67) was given in 1834 by C. Ramus [54]. (See also E. Netto [45] pp. 19-20.)

37. Random Walks. The classical problems of games of chance discussed in the preceding section can also be described imaginatively by using the following model:

Suppose that a particle performs a random walk on the x-axis. Starting at $x = 0$ the particle takes a sequence of steps. In each step, independently of the others, it can move either a unit distance to the right with probability p or a unit distance to the left with probability q where $p > 0$, $q > 0$ and $p+q = 1$. Denote by η_n the position of the particle at the end of the n-th step. Let $\eta_0 = 0$.

This random walk process has the same stochastic properties as the series of games considered in the preceding section. Let us suppose that if A wins a game, then the particle moves a unit distance to the right and if B wins a game, then the particle moves a unit distance to the left. Then η_n can be interpreted as the total gain of A at the end of the n-th game. We have

$$(1) \qquad \mathbf{P}\{\eta_n = 2i-n\} = \binom{n}{i} p^i q^{n-i}$$

for $i = 0,1,\ldots, n$.

All the results of the preceding section have simple interpretations in the terminology of random walks.

We can interpret $P(a,b)$ as the probability that the particle sooner or later reaches the point $x = a$ before reaching $x = -b$ (if at all). By (36.11) we have

$$(2) \qquad P(a,b) = \begin{cases} \dfrac{p^a(p^b - q^b)}{p^{a+b} - q^{a+b}} & \text{if } p \neq p , \\ \dfrac{b}{a+b} & \text{if } p = q . \end{cases}$$

Furthermore, $P_n(a,b)$ can be interpreted as the probability that in n steps the particle reaches the point $x = a$, before reaching $x = -b$ (if at all). If we use the notation (1), then by (36.27) we can write that

$$(3) \qquad \begin{aligned} P_n(a,b) = & \sum_{k=0}^{\infty} \left(\frac{p}{q}\right)^{k(a+b)+a} \underset{\sim}{P}\{\eta_n \leq -2k(a+b)-a\} \\ & - \sum_{k=1}^{\infty} \left(\frac{p}{q}\right)^{k(a+b)} \underset{\sim}{P}\{\eta_n \leq -2k(a+b)+a\} \\ & + \sum_{k=0}^{\infty} \left(\frac{p}{q}\right)^{-k(a+b)} \underset{\sim}{P}\{\eta_n > 2k(a+b)+a\} \\ & - \sum_{k=1}^{\infty} \left(\frac{p}{q}\right)^{-k(a+b)+a} \underset{\sim}{P}\{\eta_n > 2k(a+b)-a\} . \end{aligned}$$

If $b = \infty$ in (2) and in (3), then we obtain that

$$(4) \qquad P(a,\infty) = \begin{cases} \left(\frac{p}{q}\right)^a & \text{for } p < q , \\ 1 & \text{for } p \geq q , \end{cases}$$

which is in agreement with (36.26) and

$$(5) \qquad P_n(a,\infty) = \underset{\sim}{P}\{\eta_n > a\} + \left(\frac{p}{q}\right)^a \underset{\sim}{P}\{\eta_n \leq -a\}$$

which is in agreement with (36.40) and (36.46) respectively.

The probability that the n-th step takes the particle to the point $x = 2i-n$ and during the first n steps the particle never reaches the points $x = a$ and $x = -b$ is given by

$$(6)\quad \begin{aligned} &\underset{\sim}{P}\{\eta_n = 2i-n \text{ and } -b < \eta_r < a \text{ for } r = 0,1,\ldots, n\} = \\ &\Big[\sum_{k=-\infty}^{\infty} \binom{n}{i+k(a+b)} - \sum_{k=-\infty}^{\infty} \binom{n}{i-a+k(a+b)}\Big] p^i q^{n-i} \end{aligned}$$

for $-b < 2i-n < a$ and 0 otherwise. This follows from (36.63).

By (6) we can write that

$$(7)\quad \begin{aligned} &\underset{\sim}{P}\{\eta_n = j \text{ and } -b < \eta_r < a \text{ for } r = 0,1,\ldots, n\} = \\ &= \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{-k(a+b)} \underset{\sim}{P}\{\eta_n = 2k(a+b)+j\} - \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{k(a+b)+a} \underset{\sim}{P}\{\eta_n = -2k(a+b)-2a-j\} \end{aligned}$$

for $-b < j < a$ and 0 otherwise. If we add (7) for $-b < j < a$, then we obtain that

$$(8)\quad \begin{aligned} &\underset{\sim}{P}\{-b < \eta_r < a \text{ for } r = 0,1,\ldots, n\} = \\ &= \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{-k(a+b)} \underset{\sim}{P}\{2k(a+b)-b < \eta_n < 2k(a+b)+a\} - \\ &- \sum_{k=-\infty}^{\infty} \left(\frac{p}{q}\right)^{k(a+b)+a} \underset{\sim}{P}\{-2(k+1)(a+b)+b < \eta_n < -2k(a+b)-a\} . \end{aligned}$$

If $b = \infty$ in (6), then we obtain that

$$(9)\quad \begin{aligned} &\underset{\sim}{P}\{\eta_n = 2i-n \text{ and } \eta_r < a \text{ for } r = 0,1,\ldots, n\} = \\ &= \Big[\binom{n}{i} - \binom{n}{i-a}\Big] p^i q^{n-i} \end{aligned}$$

for $i < (n+a)/2$ and 0 otherwise. This is in agreement with (36.47). By (9) we can write that

$$\mathbf{P}\{\eta_n = j \text{ and } \eta_r < a \text{ for } r = 0,1,\dots, n\} =$$

(10)

$$= \mathbf{P}\{\eta_n = j\} - \left(\frac{p}{q}\right)^a \mathbf{P}\{\eta_n = j-2a\}$$

for $j < a$ and 0 otherwise. If we add (10) for $j < a$, then we obtain that

(11) $$\mathbf{P}\{\eta_r < a \text{ for } r = 0,1,\dots, n\} = \mathbf{P}\{\eta_n < a\} - \left(\frac{p}{q}\right)^a \mathbf{P}\{\eta_n < -a\} .$$

Random walk interpretations of the results of games of chance have some interest of their own, and probably the classical researchers have used some geometric descriptions to visualize the possible outcomes of a sequence of games. In a two-dimensional coordinate system the sequence (r, η_r) for $r = 0,1,\dots, n$ describes the path of the random walk during the first n steps or the results of the first n games. If we join the successive vertices (r, η_r) by straight lines, then we obtain an easily visualizable space-time diagram which can be seen on this page.

Space-Time Diagram of a Random Walk.

In fact random walk problems did not originate in the theory of games of chance. At the end of the nineteenth century new discoveries in physics attracted attention to problems which we now call random walk problems.

In the beginning of the nineteenth century John Dalton (1766-1844) revived the atomic theory, according to which matter (solid, liquid or gaseous) consists of a large number of corpuscles. In the middle of the nineteenth century Rudolph Clausius (1822-1888) had succeeded in explaining thermical phenomena with the aid of the molecular motion of matter. In 1860 James Clerk Maxwell (1831-1879) determined the probability distribution of the velocities of particles (molecules) in perfect gases and found that

$$f(v) = e^{-\frac{mv^2}{2kT}} \frac{\sqrt{2m}\, v}{\sqrt{\pi (kT)^3}} \tag{12}$$

for $0 < v < \infty$ is the density function of the velocity of a particle (molecule) where m is the mass of the particle (molecule), T is the absolute temperature and $k = R/N$ where R is the constant of a perfect gas, and N is Avogadro's constant. The constant k is called Boltzmann's constant and $k = 1.34 \times 10^{-16}$ erg/grad. (See J. C. Maxwell [110], [111] and L. Boltzmann [100].)

While the molecular motion of matter cannot be observed directly, small particles suspended in fluids or floating in gases perform peculiarly rapid and irregular movement which can be observed by a microscope. Apparently this phenomenon was described for the first time in 1828 by a botanist, Robert Brown [101], who observed the motion of particles of pollen in water. He was surprised by the result and repeated the same experiment with various kinds of organic and inorganic particles and in each case observed the same phenomenon. In the following decades a number of unsatisfactory attempts have been made to explain this phenomenon. (See D'Arcy W. Thompson [118] pp. 44-48 and the Notes of R. Fürth in the book of A. Einstein [106] pp. 86-119.) D'Arcy Thompson mentions that in 1863 Christian Wiener [119] expressed his view that the Brownian movement has its origin in the impacts of the molecules of the liquid on the particles. (See also Siegmund Exner [107].) The first discoveries concerning the characteristic nature of the Brownian motion were in 1888 by G. Gouy [109]. The precise mathematical laws governing Brownian motion were discovered in 1905 by A. Einstein [104], [105], and in 1906 by M. Smoluchowski [114], [115]. A. Einstein showed that the probability density function of the displacement of a particle in a given direction during a time interval of length t is

$$f(x,t) = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}} \tag{13}$$

where D is the coefficient of diffusion. If spherical particles of constant radius a are subjected to the Brownian movement, then

$$D = \frac{kT}{12\pi a\eta} \tag{14}$$

where $k = 1.34 \times 10^{-16}$ erg/grad is Boltzmann's constant, T is the absolute temperature, and η is the viscosity of the fluid containing the suspension.

The above mentioned physical phenomena led in a natural way to the investigation of mathematical models for random walks. The problem of random walks was first mentioned in 1905 by K. Pearson [146]. He posed the following problem:

"A man starts from a point O and walks ℓ yards in a straight line; he then turns through any angle whatever and walks another ℓ yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is a distance between r and r+dr from his starting point, O."

In response to this problem G. J. Bennett found that for $n = 3$ the problem can be solved by elliptic integrals, and Lord Rayleigh (J. W. Strutt) found an approximate solution for large n values, namely, he showed that the probability is approximately

$$\frac{2r}{n\ell^2} e^{-\frac{r^2}{n\ell^2}} dr \, . \tag{15}$$

(See also Lord Rayleigh [150].) Actually, Lord Rayleigh [149] found this result in 1880 when he considered the problem of finding the distribution of the resultant amplitude of n isoperiodic vibrations of unit amplitude and random phases. If we denote by $F_n(r)$ the probability that after n stretches the distance $\leq r$, then by Lord Rayleigh's result we have

$$\lim_{n \to \infty} F_n(r\sqrt{n}) = 1 - e^{-\frac{r^2}{\ell^2}} \tag{16}$$

for $r \geq 0$. In 1906 J. C. Kluyver [139] showed that precisely

$$F_n(r) = r\int_0^\infty [J_0(\ell x)]^n J_1(rx)\,dx \tag{17}$$

where

$$J_\nu(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\nu\theta+ix\sin\theta}\,d\theta = \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{\nu+2m}}{m!(m+\nu)!} \tag{18}$$

for $\nu = 0,1,2,\ldots$ is the Bessel function of the first kind and order ν.

In 1919 Lord Rayleigh [159] extended this result to three-dimensional random flights. He considered the case where a particle takes n random stretches. In each stretch it moves a distance ℓ in a random direction having a uniform distribution independently of the other stretches. Denote by $P_n(r)$ the probability that after n stretches the distance from the starting point is $\leq r$. He showed that

$$\frac{dP_n(r)}{dr} = \frac{2r}{\pi\ell^n}\int_0^\infty \frac{(\sin\ell x)^n \sin rx}{x^{n-1}}\,dx \tag{19}$$

for $r > 0$ and $n \geq 2$. We note that $dP_n(r)/dr$ can be expressed by the following explicit formula

$$\frac{dP_n(r)}{dr} = \frac{r}{\ell^2}\left[h_{n-1}\left(\frac{r}{\ell}+1\right) + h_{n-1}\left(\frac{r}{\ell}-1\right)\right] \tag{20}$$

for $r > 0$ and $n \geq 2$ where

$$h_m(x) = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^m \cos tx\,dt = \frac{1}{2^m(m-1)!}\sum_{j=0}^{\left[\frac{m-x}{2}\right]} (-1)^j \binom{m}{j}(m-x-2j)^{m-1} \tag{21}$$

for $m = 1,2,\ldots$ is the density function of the sum of m mutually independent random variables having a uniform distribution over the interval $(-1, 1)$.

In 1905 Maryan Smoluchowski [114] (see also [115], [116], [117]) investigated random walk models in studying the Brownian motion phenomenon.

The studies of the Brownian motioh of small particles suggested various mathematical models for random walks. A particle may perform a Brownian motion subjected to no force, or constant force, or central force and so on. The case of a free particle leads to the model of a symmetric random walk. The case of particles subjected to a constant force leads to the model of an asymmetric random walk. The case of particles under the influence of a central force can be described by an urn model of P. Ehrenfest and T. Ehrenfest [103] . (See also M. Kac [137].) One- , two- , and three-dimensional random walk models appear naturally. Simulating the effect of a container we are led to the models of random walks with absorbing barriers and with reflecting barriers. See M. Smoluchowski [116], [117], S. Chandrasekhar [123] , and M. Kac [136] . Discrete time models and continuous time models have been investigated simultaneously from the beginning. In later years various limit theorems have been discovered for random walk processes. In what follows we shall mention only a few selected results.

One Dimensional Random Walks. Suppose that a particle performs a random walk on the x-axis. Starting at $x = 0$ the particle takes a sequence of steps. In each step, independently of the others, it can move either a unit distance to the right with probability p or a unit distance to the left with probability q where $p > 0$, $q > 0$ and $p+q = 1$.

Denote by $P(n,j)$ the probability that the n-th step takes the particle to point $x = j$ where $j = 0, \pm 1, \pm 2, \ldots$. Usually we say also that $P(n,j)$ is the probability that at time n the particle is at $x = j$. Obviously, we can write that

$$(22) \qquad P(n,j) = pP(n-1, j-1)+qP(n-1, j+1)$$

for $n = 1,2,\ldots$ and $j = 0, \pm 1, \pm 2,\ldots$ where $P(0, 0) = 1$ and $P(0,j) = 0$ if $j \neq 0$. The recurrence formula (22) determines $P(n,j)$ for $n = 1,2,\ldots$ and we obtain easily that

$$(23) \qquad P(n,j) = \binom{n}{\frac{n+j}{2}} p^{\frac{n+j}{2}} q^{\frac{n-j}{2}}$$

for $j = n, n-2,\ldots, -n+2, -n$ and 0 otherwise.

Now let us assume that the particle moves in exactly the same way as above except that there are two <u>absorbing barriers</u> at the points $x = a$ and $x = -b$, where a and b are positive integers, and if the particle reaches the point $x = a$ or the point $x = -b$, then it remains forever at this point. Denote by $P^*(n,j)$ the probability that at time n the position of the particle is $x = j$. By (6) we have

$$(24) \quad P^*(n,j) = \left[\sum_k \binom{n}{\frac{n+j}{2} + k(a+b)} - \sum_k \binom{n}{\frac{n+j}{2} -a+k(a+b)} \right] p^{\frac{n+j}{2}} q^{\frac{n-j}{2}}$$

for $-b < j < a$ and $j = n, n-2,\ldots, -n+2, -n$. By applying the identity (36.67) we can write also that

$$(25) \quad P^*(n,j) = \frac{2^n p^{\frac{n+j}{2}} q^{\frac{n-j}{2}}}{a+b} \sum_{k=0}^{a+b-1} \left(\cos \frac{k\pi}{a+b}\right)^n \left(\cos \frac{kj\pi}{a+b} - \cos \frac{k(2a-j)\pi}{a+b}\right)$$

for $-b < j < a$. Obviously, we have

$$P^*(n,a) = p \sum_{m=0}^{n-1} P^*(m,a-1) \tag{26}$$

and

$$P^*(n,-b) = q \sum_{m=0}^{n-1} P^*(m,-b+1) \tag{27}$$

for $n = 1,2,\ldots$.

The probabilities $P^*(n,j)$ $(n = 1,2,\ldots,\ j = 0, \pm 1, \pm 2,\ldots)$ satisfy the recurrence formulas

$$P^*(n,j) = pP^*(n-1,\ j-1) + qP^*(n-1,\ j+1) \tag{28}$$

if $-b+1 < j < a-1$, $P^*(n,\ a-1) = pP^*(n-1,\ a-2)$, and $P^*(n,\ -b+1) = qP^*(n-1,\ -b+2)$. Furthermore, they satisfy (26) and (27) too. The above recurrence formulas completely determine $P^*(n,j)$ for $n = 1,2,\ldots$ and $j = 0, \pm 1, \pm 2,\ldots$ if we take into consideration that $P^*(0,0) = 1$ and $P^*(0,j) = 0$ for $j \neq 0$.

We deduced formulas (24) and (25) from the results of P. R. Montmort, N. Bernoulli and A. De Moivre. These authors did not provide proofs for their results. Proofs were given only in 1776 by J. L. Lagrange [88 , pp. 238-249], in 1812 by P. S. Laplace [39 , pp. 225-238], [41 , pp. 228-242], and in 1844 by R. L. Ellis [128] . All these authors noticed that $P^*(n,j)$ for $-b < j < a$ can be obtained as the solution of the difference equation

$$P^*(n,j) = pP^*(n-1,\ j-1) + qP^*(n-1,\ j+1) \tag{29}$$

for $n = 1,2,\ldots$ and $-b < j < a$ with the initial conditions $P^*(0,0) = 1$,

$P^*(0,j) = 0$ for $j \neq 0$, and the boundary conditions $P^*(n,a) = P^*(n, -b) = 0$ for $n = 1,2,\ldots$. The above mentioned authors used various ingenious methods for solving (29).

Limit Distributions. In studying the fluctuations of prices in a stock exchange in 1900 L. Bachelier [71], [72], [73] introduced a stochastic process which we call now a Brownian motion process. He also showed that the behavior of this process can be determined by using an approximating sequence of random walk processes. This procedure gained full justification only in the 1950's. (See Section 52.)

In what follows we shall deduce some limiting distributions for the random walk process $\{\eta_r ; r = 0,1,2,\dots\}$ studied in this section. For each $n = 1,2,\dots$ define a family of random variables $\{\xi_n(u) , 0 \leq u \leq 1\}$ by the following formula

$$\xi_n(u) = \frac{\sigma\eta_{[nu]}}{\sqrt{n}} \tag{30}$$

where σ is a given positive constant. We can interpret $\xi_n(u)$ as the position of a particle at time u if the particle starts at $x = 0$ and at times $u = \frac{1}{n}, \frac{2}{n},\dots, \frac{n}{n}$ it moves a distance $\sigma/\sqrt{n}$ to the right or to the left with probabilities p and q respectively. Let us suppose also that p and q depend on n and let

$$p = p_n = \frac{1}{2} + \frac{\alpha}{2\sigma\sqrt{n}} \quad \text{and} \quad q = q_n = \frac{1}{2} - \frac{\alpha}{2\sigma\sqrt{n}} \tag{31}$$

for $n > \alpha^2/\sigma^2$ where α is a given real number.

First we shall prove the following limit theorem.

Theorem 1. If $x \geq 0$, then

(32) $$\lim_{n\to\infty} P\{\xi_n(u) \leq x \text{ for } 0 \leq u \leq 1\} = \Phi(\frac{x-\alpha}{\sigma}) - e^{\frac{2\alpha x}{\sigma^2}} \Phi(\frac{-x-\alpha}{\sigma})$$

where $\Phi(x)$ is the normal distribution function defined by (35.18).

Proof. We can write that

(33) $$P\{\max_{0\leq u\leq 1} \xi_n(u) \leq x\} = P\{\max_{0\leq r\leq n} \eta_r < a_n\}$$

where a_n is the smallest integer greater than $x\sqrt{n}/\sigma$. By (11) it follows that

(34) $$P\{\max_{0\leq u\leq 1} \xi_n(u) \leq x\} = P\{\eta_n < a_n\} - (\frac{p_n}{q_n})^{a_n} P\{\eta_n < -a_n\} .$$

Since by (35.17)

(35) $$\lim_{n\to\infty} P\{\frac{\eta_n - n(p_n - q_n)}{\sqrt{4np_nq_n}} \leq x\} = \lim_{n\to\infty} P\{\frac{\eta_n - \frac{\alpha\sqrt{n}}{\sigma}}{\sqrt{n}} \leq x\} = \Phi(x)$$

for any x , and since $\lim_{n\to\infty} a_n/\sqrt{n} = x/\sigma$, and

(36) $$\lim_{n\to\infty} (\frac{p_n}{q_n})^{\sqrt{n}} = e^{\frac{2\alpha}{\sigma}} ,$$

we obtain (32) by (34).

In a similar way we can prove more general limit theorems. First, however, let us prove a useful auxiliary theorem.

Lemma 1. *Let* $p_j(n) \geq 0$ *for* $j = 0, \pm 1, \pm 2, \ldots$ *and* $n = 1,2,\ldots,$ *and suppose that* $\lim_{n \to \infty} p_j(n) = p_j$ *exists for* $j = 0, \pm 1, \pm 2, \ldots$. *Furthermore, let us suppose that*

$$\sum_{j=-\infty}^{\infty} p_j(n) = \sum_{j=-\infty}^{\infty} p_j = 1 \tag{37}$$

for $n = 1,2,\ldots$. *If* $|c_j| < M$ *for* $j = 0, \pm 1, \pm 2, \ldots,$ *then*

$$\lim_{n \to \infty} \sum_{j=-\infty}^{\infty} c_j p_j(n) = \sum_{j=-\infty}^{\infty} c_j p_j . \tag{38}$$

Proof. This lemma is a discrete version of a result of *E. Helly* [28] and, actually, it can be deduced from his result.

We shall prove that for any $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that

$$\left| \sum_{j=-\infty}^{\infty} c_j p_j - \sum_{j=-\infty}^{\infty} c_j p_j(n) \right| < \varepsilon \tag{39}$$

whenever $n > N$. This follows from the following inequalities

$$\begin{aligned} \left| \sum_{j=-\infty}^{\infty} c_j p_j - \sum_{j=-\infty}^{\infty} c_j p_j(n) \right| &\leq M \sum_{j=-\infty}^{\infty} |p_j - p_j(n)| = \\ = 2M \sum_{j=-\infty}^{\infty} [p_j - p_j(n)]^+ &\leq 2M \sum_{|j| \leq m} |p_j - p_j(n)| + 2M \sum_{|j| > m} p_j \end{aligned} \tag{40}$$

where m is any positive integer. Here the equality between the second and third expressions follows from (37). First, let us choose m so large that the last member be $< \varepsilon/2$. Since for any m

(41) $$\lim_{n\to\infty} 2M \sum_{|j|\leq m} |p_j - p_j(n)| = 0 ,$$

we can find an N such that

(42) $$2M \sum_{|j|\leq m} |p_j - p_j(n)| < \frac{\varepsilon}{2}$$

if $n > N$. Hence (39) follows, which proves (38).

<u>Theorem 2.</u> <u>If</u> $x > 0$ <u>and</u> $y > 0$, <u>then</u>

(43) $$\lim_{n\to\infty} \mathbf{P}\{-y \leq \xi_n(u) \leq x \quad \text{for} \quad 0 \leq u \leq 1\} = F_{\alpha/\sigma}\left(\frac{x}{\sigma}, \frac{y}{\sigma}\right)$$

<u>where</u>

(44) $$\begin{aligned} F_\alpha(x,y) = &\sum_{k=-\infty}^{\infty} e^{-2k\alpha(x+y)}[\Phi(2k(x+y)+x-\alpha)-\Phi(2k(x+y)-y-\alpha)] \\ &- e^{2\alpha x} \sum_{k=-\infty}^{\infty} e^{2k\alpha(x+y)}[\Phi(-2k(x+y)-x-\alpha)-\Phi(-2(k+1)(x+y)+y-\alpha)] . \end{aligned}$$

<u>Proof.</u> We can write that

(45) $$\mathbf{P}\{-y \leq \xi_n(u) \leq x \quad \text{for} \quad 0 \leq u \leq 1\} = \mathbf{P}\{-b_n < \eta_r < a_n \quad \text{for} \quad r = 0,1,\dots,n\}$$

where a_n is the smallest integer greater than $x\sqrt{n}/\sigma$ and b_n is the smallest integer greater than $y\sqrt{n}/\sigma$. If in (8) we put $a = a_n$, $b = b_n$, $p = p_n$, $q = q_n$ and let $n \to \infty$, then we obtain (43). In (8) we can interchange the limit and summation. If $\alpha = 0$ and if in Lemma 1 we choose $c_j = 1, 0, -1$ depending on j , then we get (43) for $\alpha = 0$.

We note that in the particular case of $\alpha = 0$ (45) reduces to

$$(46) \qquad F_0(x,y) = \sum_{k=-\infty}^{\infty} (-1)^k [\Phi(k(x+y)+x)-\Phi(k(x+y)-y)] \ .$$

If, in particular, $p = q = \frac{1}{2}$, then by using (36.67) we can write that

$$(47) \qquad \underset{\sim}{P}\{-b < \eta_r < a \text{ for } r = 0,1,\ldots, n\} = \frac{2}{a+b} \sum_{k=0}^{a+b} [\frac{1+(-1)^k}{2}](\cos \frac{k\pi}{a+b})^{n+1} \frac{\sin \frac{ka\pi}{a+b}}{\sin \frac{k\pi}{a+b}} \ .$$

In this case if we use (47) instead of (8) in (45), then we obtain that

$$(48) \qquad F_0(x,y) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{1}{2j+1} e^{-\frac{(2j+1)^2\pi^2}{2(x+y)^2}} \frac{\sin(2j+1)\pi x}{2(x+y)}$$

for $x > 0$ and $y > 0$.

Finally, we note that the process $\{\xi_n(u), 0 \leq u \leq 1\}$ has independent increments and by (35.17) we have

$$(49) \qquad \lim_{n \to \infty} \underset{\sim}{P}\{ \frac{\xi_n(t)- \xi_n(u)-\alpha(t-u)}{\sigma \sqrt{t-u}} \leq x\} = \Phi(x)$$

for $0 \leq u < t \leq 1$. Since obviously $\lim_{n \to \infty} \underset{\sim}{E}\{\xi_n(u)\} = \alpha u$, $\lim_{n \to \infty} \underset{\sim}{Var}\{\xi_n(u)\} = \sigma^2 u$ and $\lim_{n \to \infty} \underset{\sim}{Cov}\{\xi_n(u), \xi_n(t)\} = \sigma^2 \min(u,t)$ for $0 \leq u \leq 1$ and $0 \leq t \leq 1$, we can conclude that for $0 < t_1 < t_2 < \ldots < t_k \leq 1$, the random variables $\xi_n(t_1)$, $\xi_n(t_2),\ldots, \xi_n(t_k)$ have a k-dimensional limiting normal distribution

$$(50) \qquad N\left(\alpha \begin{Vmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{Vmatrix}, \quad \sigma^2 \begin{Vmatrix} t_1, & t_1, \ldots, & t_1 \\ t_1, & t_2, \ldots, & t_2 \\ \cdot & \cdot \quad \cdots & \cdot \\ t_1, & t_2, \ldots, & t_k \end{Vmatrix}\right).$$

If a stochastic process $\{\xi(t), 0 \leq t < \infty\}$ has the property that for any $k = 1,2,\ldots$ and $0 < t_1 < t_2 < \ldots < t_k < \infty$ the random variables $\xi(t_1), \xi(t_2), \ldots, \xi(t_k)$ have a k-dimensional normal distribution, then we say that $\{\xi(t), 0 \leq t < \infty\}$ is a Gaussian process. If, in particular, $E\{\xi(t)\} = \alpha t$ for $t \geq 0$ and $Cov\{\xi(u), \xi(t)\} = \sigma^2 \min(u, t)$ for $0 \leq u$ and $0 \leq t$, then we say that $\{\xi(t), 0 \leq t < \infty\}$ is a Brownian motion process.

If $\{\xi(t), 0 \leq t < \infty\}$ is a separable Brownian motion process, then we can prove that

$$(51) \quad P\{-y \leq \xi(u) \leq x \text{ for } 0 \leq u \leq 1\} = \lim_{n \to \infty} P\{-y \leq \xi_n(u) \leq x \text{ for } 0 \leq u \leq 1\} = = F_{\alpha/\sigma}\left(\frac{x}{\sigma}, \frac{y}{\sigma}\right)$$

for $x > 0$ and $y > 0$ where the right-hand side is given by (44). Hence it follows immediately that

$$(52) \qquad P\{-y \leq \xi(u) \leq x \text{ for } 0 \leq u \leq t\} = F_{\frac{\alpha\sqrt{t}}{\sigma}}\left(\frac{x}{\sigma\sqrt{t}}, \frac{y}{\sigma\sqrt{t}}\right)$$

for any $t > 0$, $x > 0$ and $y > 0$.

Random Walks in Euclidean Spaces. In 1919 Lord Rayleigh [150] studied random flights in one, two, and three dimensions. The one-dimensional case discussed in this section can be extended in a natural way to random walks in multidimensional periodic lattices. The first extensive study of such random walks was given in 1921 by G. Pólya [148].

Here we shall consider only symmetric random walks. Let us suppose that a particle performs a random walk in an r-dimensional Euclidean space. Starting from the origin in each step the particle moves a unit distance in one of the $2r$ directions parallel to the coordinate axes. We suppose that the successive displacements are independent and each of the $2r$ directions has the same probability.

The probability that the n-th step takes the particle to the point $(x_1, x_2, \ldots, x_r)$ is

$$(53)\quad P_n(x_1,x_2,\ldots,x_r) = \left(\frac{1}{2r}\right)^n \sum_{\substack{j_i-k_i=x_i \\ (i=1,2,\ldots,r)}} \frac{n!}{j_1!j_2!\ldots j_r!k_1!k_2!\ldots k_r!}$$

where the summation is extended over all nonnegative integers $j_1, j_2, \ldots, j_r, k_1, k_2, \ldots, k_r$ satisfying the conditions $j_i - k_i = x_i$ for $i = 1,2,\ldots, r$. For the number of possible paths is $(2r)^n$. If we denote by j_i the number of steps taken in the positive direction parallel to the i-th coordinate axis, and by k_i the number of steps taken in the negative direction parallel to the i-th coordinate axis, then a path is favorable if it satisfies the requirements $j_i - k_i = x_i$ for $i = 1,2,\ldots, r$. The number of such paths is given by the sum in (53).

If $r = 1$, then (53) reduces to

$$P_n(x) = \binom{n}{\frac{n+x}{2}} \frac{1}{2^n} \tag{54}$$

for $x = n, n-2, \ldots, -n+2, -n$. If $r = 2$, then (53) reduces to

$$P_n(x,y) = \binom{n}{\frac{n+x+y}{2}} \binom{n}{\frac{n+x-y}{2}} \frac{1}{4^n} \tag{55}$$

for $x+y \equiv 0 \pmod{2}$ and $|x+y| \leq n$, $|x-y| \leq n$.

Denote by the vector $\underset{\sim}{\eta}_n(r) = (\eta_n(1), \ldots, \eta_n(r))$ the position of the particle at the n-th step. The characteristic function of $\underset{\sim}{\eta}_n(r) = (\eta_n(1), \ldots, \eta_n(r))$ is given by

$$\underset{\sim}{E}\{e^{it_1\eta_n(1)+\ldots+it_r\eta_n(r)}\} = \left(\frac{e^{it_1}+e^{-it_1}+\ldots+e^{it_r}+e^{-it_r}}{2r}\right)^n \tag{56}$$

$$= \left(\frac{\cos t_1 + \ldots + \cos t_r}{r}\right)^n$$

for real $t_1, t_2, \ldots, t_r$. Hence by inversion we obtain that

$$P_n(x_1, \ldots, x_r) = \frac{1}{(2\pi)^r} \int_0^{2\pi} \ldots \int_0^{2\pi} \left(\frac{\cos t_1 + \ldots + \cos t_r}{r}\right)^n e^{-it_1x_1 - \ldots - it_rx_r} dt_1 \ldots dt_r \tag{57}$$

for any $(x_1, x_2, \ldots, x_r)$.

In the particular case when $x_1 = x_2 = \ldots = x_r = 0$, let us write

$$Q_n(r) = P_n(0, 0, \ldots, 0), \tag{58}$$

that is, $Q_n(r)$ is the probability that in an r-dimensional symmetric random walk the particle returns to the origin at the n-th step. By (53) we have

$$Q_{2m}(r) = \left(\frac{1}{2r}\right)^m \sum_{j_1+j_2+\dots+j_r=m} \frac{(2m)!}{(j_1!j_2!\dots j_r!)^2} \tag{59}$$

and $Q_{2m+1}(r) = 0$. Let us write also $Q_0(r) = 1$.

In particular,

$$Q_{2m}(1) = \binom{2m}{m} \frac{1}{2^{2m}} , \tag{60}$$

and an elementary inequality (see Problem 403) shows that

$$\frac{1}{\sqrt{(m+\frac{1}{2})\pi}} < \binom{2m}{m} \frac{1}{2^{2m}} < \frac{1}{\sqrt{m\pi}} \tag{61}$$

for $m = 1,2,\dots$. Accordingly

$$\lim_{m\to\infty} Q_{2m}(1)\sqrt{m\pi} = 1 . \tag{62}$$

If $r = 2$, then we obtain that

$$Q_{2m}(2) = \left[\binom{2m}{m} \frac{1}{2^{2m}}\right]^2 \tag{63}$$

and thus

$$\lim_{m\to\infty} Q_{2m}(2)m\pi = 1 . \tag{64}$$

By using Stirling's formula,

$$n! = \sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n e^{\theta_n}\,, \tag{65}$$

where $1/(12n+1) < \theta_n < 1/12n$, (H. Robbins [55]), we can prove that

$$Q_{2m}(r) < 2^{\frac{r+1}{2}}\left(\frac{r}{4\pi m}\right)^{\frac{r}{2}} \tag{66}$$

for $m = 1,2,\ldots$. For we have

$$Q_{2m}(r) = \frac{\binom{2m}{m}}{(2r)^{2m}} \sum_{j_1+\ldots+j_r=m} \left(\frac{m!}{j_1!j_2!\ldots j_r!}\right)^2 \leq \frac{\binom{2m}{m}}{(2r)^{2m}} \max_{j_1+\ldots+j_r=m} \frac{m!}{j_1!j_2!\ldots j_r!}\,\cdot \tag{67}$$
$$\sum_{j_1+\ldots j_r=m} \frac{m!}{j_1!j_2!\ldots j_r!} = \frac{\binom{2m}{m}}{(4r)^m} \max_{j_1+\ldots+j_r=m} \frac{m!}{j_1!j_2!\ldots j_r!} \leq \binom{2m}{m} \frac{m!}{(4r)^m\left(\left[\frac{m}{r}\right]!\right)^r}\,.$$

Here we used that $m!/j_1!j_2!\ldots j_r!$ attains its maximum if $|j_s - j_t| \leq 1$ for all s and t . By applying (61) and (65) we obtain (66) for $m = 1,2,\ldots$.

Following G. Pólya [148] we can prove that

$$Q_{2m}(r) \sim 2\left(\frac{r}{4\pi m}\right)^{\frac{r}{2}} \tag{68}$$

as $m \to \infty$. By (57) we have

$$Q_{2m}(r) = \frac{1}{(2\pi)^r} \int_0^{2\pi}\!\!\ldots\!\int_0^{2\pi} \left(\frac{\cos t_1+\ldots+\cos t_r}{r}\right)^{2m} dt_1\ldots dt_r\,. \tag{69}$$

Since the integrand in (69) is a periodic function in each variable with period 2π we can replace the domain of integration in (69) by $D = \{(t_1,\ldots, t_r):\, -\frac{\pi}{2} \leq t_k \leq \frac{3\pi}{2}$ for $k = 1,2,\ldots, r\}$ without changing the value of the integral. Thus we can write that

$$(70) \qquad m^{\frac{r}{2}} Q_{2m}(r) = \frac{m^{\frac{r}{2}}}{(2\pi)^r} \int\cdots\int_D \left(\frac{\cos t_1+\ldots+\cos t_r}{r}\right)^{2m} dt_1\ldots dt_r \,.$$

We observe that in the domain D the function

$$(71) \qquad \left|\frac{\cos t_1+\ldots+\cos t_r}{r}\right|$$

equals 1 if $t_1 = t_2 = \ldots = t_r = 0$ or if $t_1 = t_2 = \ldots = t_r = \pi$ and < 1 otherwise. Let $D_1(\varepsilon) = \{(t_1,\ldots,t_r)\colon -\varepsilon < t_k < \varepsilon \,,\ k = 1,\ldots, r\}$ and $D_2(\varepsilon) = \{(t_1,\ldots, t_r)\colon -\varepsilon < t_k - \pi < \varepsilon \,,\ k = 1,\ldots, r\}$ for some small $\varepsilon > 0$. Then if $i = 1$ or $i = 2$, we can write that

$$m^{\frac{r}{2}} \int\cdots\int_{D_i(\varepsilon)} \left(\frac{\cos t_1+\ldots+\cos t_r}{r}\right)^{2m} dt_1\ldots dt_r = \int_{-\varepsilon\sqrt{m}}^{\varepsilon\sqrt{m}}\cdots\int_{-\varepsilon\sqrt{m}}^{\varepsilon\sqrt{m}} \left(\frac{\cos\frac{u_1}{\sqrt{m}}+\ldots+\cos\frac{u_r}{\sqrt{m}}}{r}\right)^{2m} du_1\ldots du_r$$

$$(72) \qquad \sim \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{u_1^2+\ldots+u_r^2}{r}} du_1\ldots du_r = (r\pi)^{r/2}$$

as $m \to \infty$.

Denote by $D^*(\varepsilon)$ the set of all those points of D which do not belong to $D_1(\varepsilon)$ or $D_2(\varepsilon)$. In the closed set $D^*(\varepsilon)$ the function (71) has a maximum $\rho < 1$ and therefore

$$(73) \qquad m^{\frac{r}{2}} \int\cdots\int_{D^*(\varepsilon)} \left(\frac{\cos t_1+\ldots+\cos t_r}{r}\right)^{2m} dt_1\ldots dt_r \leqq (2\pi)^r m^{r/2} \rho^{2m} \to 0 \quad \text{as} \quad m \to \infty \,.$$

If we add (72) for $i = 1,2$, and (73) and divide the sum by $(2\pi)^r$, then we obtain (68) which was to be proved.

Let us denote by A_n $(n = 1,2,\ldots)$ the event that the particle returns to the initial position at the n-th step. In the random walk process discussed above, and in many other random walk processes a return to the initial position is a recurrent event, that is, if the particle returns to the initial position, then the future stochastic behavior of the process is independent of the past and is the same as the stochastic behavior of the whole process. Briefly we can say that after each return to the initial position the process starts anew independently of the past. In this case the events $A_1, A_2, \ldots, A_n, \ldots$ satisfy the following property: If k and m are positive integers, and $1 \leq n_1 < n_2 < \ldots < n_k$ then

$$(74) \qquad \underset{\sim}{P}\{A_m A_{m+n_1} \ldots A_{m+n_k}\} = \underset{\sim}{P}\{A_m\}\underset{\sim}{P}\{A_{n_1} \ldots A_{n_k}\} .$$

As far as the theory of recurrent events is concerned we refer to W. Feller [24].

Denote by ν the number of events occurring in the sequence $A_1, A_2, \ldots, A_n, \ldots$. Then ν is a discrete random variable taking on nonnegative integers (possibly ∞) . We are interested in studying the distribution of ν .

Let $P = \underset{\sim}{P}\{\nu \geq 1\}$, that is P is the probability that at least one event occurs in the sequence $A_1, A_2, \ldots, A_n, \ldots$. We can write that

$$(75) \qquad P = \underset{\sim}{P}\{A_1 + A_2 + \ldots + A_n + \ldots\} = \underset{\sim}{P}\{A_1\} + \underset{\sim}{P}\{\bar{A}_1 A_2\} + \ldots + \underset{\sim}{P}\{\bar{A}_1 \ldots \bar{A}_{n-1} A_n\} + \ldots .$$

Let $M = \underset{\sim}{E}\{\nu\}$, that is M is the expectation of the number of events occurring in the sequence $A_1, A_2, \ldots, A_n, \ldots$. We can write that

(76) $$M = \sum_{n=1}^{\infty} \underset{\sim}{P}\{A_n\} .$$

Denote by A^* the event that infinitely many events occur in the sequence $A_1, A_2, \ldots, A_n, \ldots$, that is, $A^* = \{\nu = \infty\}$. We can write that

(77) $$A^* = \prod_{n=1}^{\infty} \sum_{i=n}^{\infty} A_i ,$$

and by the continuity theorem for probabilities we can state that

(78) $$\underset{\sim}{P}\{A^*\} = \lim_{n \to \infty} \underset{\sim}{P}\{\sum_{i=n}^{\infty} A_i\} .$$

<u>Theorem 3</u>. <u>Let</u> $A_1, A_2, \ldots, A_n, \ldots$ <u>be a sequence of events satisfying the condition</u>

(79) $$\underset{\sim}{P}\{A_m A_{m+n_1} \cdots A_{m+n_k}\} = \underset{\sim}{P}\{A_m\} \underset{\sim}{P}\{A_{n_1} \cdots A_{n_k}\}$$

<u>for</u> $1 \leq n_1 < n_2 < \ldots < n_k$ <u>and</u> $k \geq 1$ <u>and</u> $m \geq 1$. <u>Then we have</u>

(80) $$P = \underset{\sim}{P}\{A_1 + A_2 + \ldots + A_n + \ldots\} = \frac{M}{1+M}$$

<u>where</u> M <u>is given by (76) and the right-hand side of (80) should be taken</u> 1 <u>if</u> $M = \infty$. <u>Furthermore, we have</u>

(81) $$\underset{\sim}{P}\{A^*\} = \begin{cases} 1 & \text{if } M = \infty , \\ 0 & \text{if } M < \infty . \end{cases}$$

Proof. By using (79) we can prove that

$$(82)\qquad \begin{aligned} \mathbf{P}\{A_i + A_{i+1} + \dots\} &= \mathbf{P}\{A_i\} + \mathbf{P}\{A_{i+1} + A_{i+2} + \dots\} - \mathbf{P}\{A_i A_{i+1} + A_i A_{i+2} + \dots\} = \\ &= \mathbf{P}\{A_i\} + \mathbf{P}\{A_{i+1} + A_{i+2} + \dots\} - \mathbf{P}\{A_i\}\mathbf{P}\{A_1 + A_2 + \dots\} \end{aligned}$$

for $i = 1,2,\dots$. In proving (82) we need the relation

$$(83)\qquad \mathbf{P}\{A_i \overline{A}_{i+1} \dots \overline{A}_{i+k-1} A_{i+k}\} = \mathbf{P}\{A_i\}\mathbf{P}\{\overline{A}_1 \dots \overline{A}_{k-1} A_k\}$$

for $k = 2,3,\dots$. We shall prove (83) here for $k = 2$. For $k = 3,4,\dots$ we can prove (83) similarly. If we use (79), then we can write that

$$(84)\qquad \begin{aligned} &\mathbf{P}\{A_i \overline{A}_{i+1} A_{i+2}\} = \mathbf{P}\{A_i A_{i+2}\} - \mathbf{P}\{A_i A_{i+1} A_{i+2}\} = \\ &= \mathbf{P}\{A_i\}\mathbf{P}\{A_2\} - \mathbf{P}\{A_i\}\mathbf{P}\{A_1 A_2\} = \mathbf{P}\{A_i\}\mathbf{P}\{\overline{A}_1 A_2\} , \end{aligned}$$

which proves (83) for $k = 2$. By (83) we get

$$(85)\qquad \begin{aligned} &\mathbf{P}\{A_i A_{i+1} + A_i A_{i+2} + \dots\} = \mathbf{P}\{A_i A_{i+1} + A_i \overline{A}_{i+1} A_{i+2} + \dots\} = \\ &= \mathbf{P}\{A_i A_{i+1}\} + \mathbf{P}\{A_i \overline{A}_{i+1} A_{i+2}\} + \dots = \mathbf{P}\{A_i\}\mathbf{P}\{A_1\} + \mathbf{P}\{A_i\}\mathbf{P}\{\overline{A}_1 A_2\} + \dots = \\ &= \mathbf{P}\{A_i\}\mathbf{P}\{A_1 + A_2 + \dots\} \end{aligned}$$

which we used in (82). Accordingly (82) is indeed true.

If we add (82) for $i = 1,2,\dots, n$, then we obtain that

$$(86)\qquad P = (1-P) \sum_{i=1}^{n} \mathbf{P}\{A_i\} + \mathbf{P}\{ \sum_{i=n+1}^{\infty} A_i\} .$$

First, let $M < \infty$. Then

$$(87) \qquad 0 \leq \underset{\sim}{P}\{\sum_{i=n+1}^{\infty} A_i\} \leq \sum_{i=n+1}^{\infty} \underset{\sim}{P}\{A_i\} \to 0 \quad \text{as} \quad n \to \infty \ .$$

Thus it follows from (86) that

$$(88) \qquad P = (1-P) \sum_{i=1}^{\infty} \underset{\sim}{P}\{A_i\} = (1-P)M$$

and from (78) that $\underset{\sim}{P}\{A^*\} = 0$. This proves (80) and (81) in this case.

If $M = \infty$, then $P < 1$ is impossible, because in this case the right hand side of (86) would tend to ∞ as $n \to \infty$ which is obviously false. Thus if $M = \infty$, then necessarily $P = 1$. Furthermore, by (86) it follows that

$$(89) \qquad \underset{\sim}{P}\{\sum_{i=n+1}^{\infty} A_i\} = 1$$

for every $n = 1,2,\ldots$. Consequently by (78) we get $\underset{\sim}{P}\{A^*\} = 1$. This completes the proof of the theorem.

Bt Theorem 3 we have $\underset{\sim}{P}\{\nu < \infty\} = 1$ if $M < \infty$ and $\underset{\sim}{P}\{\nu = \infty\} = 1$ if $M = \infty$. Now let us determine the distribution of ν if $M < \infty$.

<u>Theorem 4. If</u> $A_1, A_2, \ldots, A_n, \ldots$ <u>satisfy (79), and</u> $M < \infty$, <u>then</u>

$$(90) \qquad \underset{\sim}{P}\{\nu = k\} = \frac{M^k}{(1+M)^{k+1}}$$

<u>for</u> $k = 0,1,2,\ldots$.

<u>Proof.</u> We shall prove that

(91) $$\underset{\sim}{P}\{\nu \geqq k\} = \left(\frac{M}{1+M}\right)^k$$

for $k = 0,1,2,\ldots$. Hence (90) follows because $\underset{\sim}{P}\{\nu = k\} = \underset{\sim}{P}\{\nu \geqq k\} - \underset{\sim}{P}\{\nu \geqq k+1\}$ for $k = 0,1,2,\ldots$. If $k = 0$, then (91) is trivially true. If $k = 1$, then (91) is precisely (80) . For any $k = 1,2,\ldots$ we have

(92) $$\underset{\sim}{P}\{\nu \geqq k\} = [\underset{\sim}{P}\{\nu \geqq 1\}]^k .$$

We shall prove (92) only for $k = 2$. The general case can be proved similarly. By (79) it follows that

(93) $$\underset{\sim}{P}\{\bar{A}_1\ldots\bar{A}_{m-1}A_m\bar{A}_{m+1}\ldots\bar{A}_{m+n-1}A_{m+n}\} = \underset{\sim}{P}\{\bar{A}_1\ldots\bar{A}_{m-1}A_m\}\underset{\sim}{P}\{\bar{A}_1\ldots\bar{A}_{n-1}A_n\}$$

for $m = 1,2,\ldots$ and $n = 1,2,\ldots$. If we add (93) for $m = 1,2,\ldots$ and $n = 1,2,\ldots$, then we get

(94) $$\underset{\sim}{P}\{\nu \geqq 2\} = \underset{\sim}{P}\{\nu \geqq 1\}\underset{\sim}{P}\{\nu \geqq 1\}$$

which is (92) for $k = 2$. Since by (80) $\underset{\sim}{P}\{\nu \geqq 1\} = M/(1+M)$, therefore (92) implies (91).

Finally, let us consider the problem of finding the distribution of ν_n , the number of events occurring among $A_1, A_2,\ldots, A_n$, in the case when $A_1, A_2,\ldots, A_n,\ldots$ satisfies (79). Let us define the random variables τ_k $(k = 1,2,\ldots)$ in the following way: $\tau_k = n$ if and only if the k-th event which occurs in the sequence is A_n . Let $\tau_0 = 0$. If the sequence $A_1, A_2,\ldots, A_n, \ldots$ satisfies (79), then it follows that $\tau_k - \tau_{k-1}$ $(k = 1,2,\ldots)$ is a sequence of mutually independent and identically distributed random variables taking on positive integers (possibly ∞) . Let

(95) $$\underset{\sim}{P}\{\tau_k - \tau_{k-1} = j\} = f_j$$

for $j = 1,2,3,\ldots$.

If we know the probabilities $\{f_j\}$, then the distribution of ν_n can easily be obtained. For we have

(96) $$\underset{\sim}{P}\{\nu_n < k\} = \underset{\sim}{P}\{\tau_k > n\}$$

whenever $n \geqq 1$ and $k \geqq 0$, and τ_k is the sum of k mutually independent and identically distributed random variables having the distribution (95). Thus the problem of finding the distribution of ν_n can be reduced to the problem of finding the distribution $\{f_j\}$. This is given by the following theorem. Let us introduce the notation

(97) $$u_n = \underset{\sim}{P}\{A_n\}$$

for $n = 1,2,\ldots$ and $u_0 = 1$. Let

(98) $$U(z) = \sum_{n=0}^{\infty} u_n z^n$$

for $|z| < 1$. Obviously

(99) $$f_n = \underset{\sim}{P}\{\bar{A}_1 \ldots \bar{A}_{n-1} A_n\}$$

for $n = 1,2,\ldots$. Let

(100) $$F(z) = \sum_{n=1}^{\infty} f_n z^n$$

for $|z| \leq 1$.

Theorem 5. If $|z| < 1$, then we have

(101) $$F(z) = 1 - \frac{1}{U(z)} .$$

Proof. Since obviously

(102) $$\mathbf{P}\{A_n\} = \sum_{j=1}^{n} \mathbf{P}\{\bar{A}_1 \dots \bar{A}_{j-1} A_j A_n\}$$

for $n = 1,2,\dots,$ it follows from (79) that

(103) $$u_n = \sum_{j=1}^{n} f_j u_{n-j}$$

for $n = 1,2,\dots$. If we multiply (103) by z^n and add for $n = 1,2,\dots,$ then we get

(104) $$U(z)-1 = F(z)U(z)$$

for $|z| < 1$ and this proves (101). The definition of $F(z)$ for $|z| \leq 1$ can be extended by continuity.

Now let us return to the random walk processes studied previously and let us give a few examples for the use of the above theorems.

First we shall prove an interesting theorem due to G. Pólya [148].

Theorem 6. In one- and two-dimensional symmetric random walks the particle sooner or later returns to its initial position with probability 1. In three- and higher dimensional random walks, however, this probability is less than 1.

Proof. A return to the origin is a recurrent event in each case. If we denote by A_n the event that the particle returns to the origin at the n-th step, then we can apply Theorems 3, 4 and 5 to the sequence $\{A_n\}$. For an r-dimensional symmetric random walk $(r = 1,2,\ldots)$ denote by $Q_n(r)$ the probability that the particle returns to the origin at the n-th step. A return cannot occur at the $2m+1$-st step. By (67) we have

$$Q_{2m}(r) \sim 2\left(\frac{r}{4\pi m}\right)^{r/2} \tag{105}$$

as $m \to \infty$. Thus

$$\sum_{m=1}^{\infty} Q_{2m}(r) \begin{cases} = \infty & \text{if } r = 1,2, \\ < \infty & \text{if } r \geq 3, \end{cases} \tag{106}$$

and Theorem 6 follows from (80). If $r = 1$ or $r = 2$, then the particle infinitely often returns to its initial position with probability 1. If $r \geq 3$, then this probability is 0.

If we want to find the probability that the particle returns precisely k times $(k = 0,1,2,\ldots)$ to the origin in an r-dimensional symmetric random walk where $r \geq 3$, then we should determine the sum

$$Q(r) = \sum_{n=0}^{\infty} Q_n(r) \tag{107}$$

where $Q_n(r)$ is defined by (58) and can be expressed by (57). If $r = 3$, then by (57) it follows that

(108) $$Q(3) = \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{dt_1dt_2dt_3}{1-\frac{1}{3}(\cos t_1+\cos t_2+\cos t_3)} .$$

In 1939 G. N. Watson [153] found that

(109) $$Q(3) = \frac{4}{3\pi^2}(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})K^2((2-\sqrt{3})(\sqrt{3}-\sqrt{2}))$$

where

(110) $$K(k) = \int_0^{\pi/2}\frac{du}{\sqrt{1-k^2\sin^2u}} = \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

is the complete elliptic integral of the second kind. Numerically,

(111) $$Q(3) = 1.51638\ 60591\ \ldots .$$

We can analyse in a similar way various random walks on Euclidean symmetric lattices. First let us consider a two-dimensional symmetric random walk on a triangular lattice. Suppose that starting at the origin (0, 0) a particle takes a series of steps on the plane. In each step the particle moves according to one of the following six vectors

(112) $$(0,1),\ (0,-1),\ (\tfrac{\sqrt{3}}{2},\tfrac{1}{2}),\ (\tfrac{\sqrt{3}}{2},-\tfrac{1}{2}),\ (-\tfrac{\sqrt{3}}{2},\tfrac{1}{2}),\ (-\tfrac{\sqrt{3}}{2},-\tfrac{1}{2})$$

with probability 1/6 . Suppose that the successive displacements are independent. Denote by $(\eta_n(1), \eta_n(2))$ the position of the particle at the n-th step. Now we have

$$(113)\qquad \underset{\sim}{E}\{e^{it_1\eta_n(1)+it_2\eta_n(2)}\} = \frac{1}{3^n}\left(2\cos\frac{t_1\sqrt{3}}{2}\cos\frac{t_2}{2} + \cos t_2\right)^n$$

for real t_1 and t_2 values. Hence it follows that

$$(114)\quad \underset{\sim}{P}\{\eta_n(1)=0,\ \eta_n(2)=0\} = \frac{\sqrt{3}}{16\pi^2 3^n}\int_0^{\frac{4\pi}{\sqrt{3}}}\int_0^{4\pi}\left(2\cos\frac{t_1\sqrt{3}}{2}\cos\frac{t_2}{2} + \cos t_2\right)^n dt_1 dt_2 .$$

In a similar way as we proved (67), it follows that

$$(115)\qquad \lim_{n\to\infty} n\underset{\sim}{P}\{\eta_n(1) = 0,\ \eta_n(2) = 0\} = \frac{3\sqrt{2}}{8\pi} .$$

Since the integrand in (114) is periodic with periods $t_1 = 4\pi/\sqrt{3}$ and $t_2 = 4\pi$, we can replace the domain of integration in (114) by

$$(116)\qquad D = \{(t_1, t_2): -\frac{\pi}{\sqrt{3}} \leqq t_1 \leqq \frac{3\pi}{\sqrt{3}},\ -\pi \leqq t_2 \leqq 3\pi\}$$

without changing the value of the integral. In this domain the function

$$(117)\qquad \left|\frac{2}{3}\cos\frac{t_1\sqrt{3}}{2}\cos\frac{t_2}{2} + \cos t_2\right|$$

equals 1 if $t_1 = t_2 = 0$ or if $t_1 = \frac{2\pi}{\sqrt{3}}$ and $t_2 = 2\pi$ and it is < 1 otherwise. Let ε be a sufficiently small positive number and define $D_1(\varepsilon) = \{(t_1, t_2): |t_1| < \varepsilon,\ |t_2| < \varepsilon\}$ and $D_2(\varepsilon) = \{(t_1, t_2): |t_1 - \frac{2\pi}{\sqrt{3}}| < \varepsilon,\ |t_2 - 2\pi| < \varepsilon\}$. Denote by $D^*(\varepsilon)$ the set of all those points of D which do not belong to $D_1(\varepsilon)$ or $D_2(\varepsilon)$. In the closed set $D^*(\varepsilon)$ the function (117) has a maximum $\rho < 1$.

If we take into consideration that for $i = 1$ and $i = 2$

(118)
$$n \iint_{D_i(\varepsilon)} \left(\frac{2}{3}\cos\frac{t_1\sqrt{3}}{2}\cos\frac{t_2}{2} + \cos t_2\right)^n dt_1 dt_2 = \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \left(\frac{2}{3}\cos\frac{u_1\sqrt{3}}{2\sqrt{n}}\cos\frac{u_2}{2\sqrt{n}} + \frac{1}{3}\cos\frac{u_2}{n}\right)^n du_1 du_2 \sim \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \left(1 - \frac{u_1^2}{2n} - \frac{u_2^2}{3n}\right)^n du_1 du_2 \sim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{u_1^2}{2} - \frac{u_2^2}{3}} du_1 du_2 = \sqrt{6}\,\pi$$

as $n \to \infty$, and

(119)
$$n \iint_{D^*(\varepsilon)} \left(\frac{2}{3}\cos\frac{t_1\sqrt{3}}{2}\cos\frac{t_2}{2} + \cos t_2\right)^n dt_1 dt_2 \leqq \frac{16\pi^2}{3} n\rho^n \to 0$$

as $n \to \infty$, then we obtain (115). Since a return to the origin is a recurrent event, it follows from Theorem 3 that the particle infinitely often returns to its initial position with probability 1.

Now let us consider some three-dimensional random walks which were analyzed in 1956 by <u>E. W. Montroll</u> [144].

First, let us suppose that starting from the origin in a three-dimensional Euclidean space a particle takes a sequence of steps and in each step it moves in accordance with one of the eight vectors $(\pm 1, \pm 1, \pm 1)$ with probability $1/8$. Let us suppose that the successive displacements are independent. Denote by $\underset{\sim}{\eta}_n(3) = (\eta_n(1), \eta_n(2), \eta_n(3))$ the position of the particle at the n-th step. Now we have

$$(120) \qquad \mathop{E}_{\sim}\{e^{it_1\eta_n(1)+it_2\eta_n(2)+it_3\eta_n(3)}\} = (\cos t_1 \cos t_2 \cos t_3)^n$$

for real t_1, t_2, t_3. In this case

$$(121) \qquad \mathop{P}_{\sim}\{\eta_n(1) = 0, \eta_n(2) = 0, \eta_n(3) = 0\} =$$

$$= \frac{1}{(2\pi)^3} \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} (\cos t_1 \cos t_2 \cos t_3)^n dt_1 dt_2 dt_3 ,$$

whence we obtain that

$$(122) \qquad \mathop{P}_{\sim}\{\underset{\sim}{\eta}_n(3) = \underset{\sim}{0}\} = \begin{cases} [\binom{2m}{m} \frac{1}{2^{2m}}]^3 & \text{if } n = 2m , \\ 0 & \text{if } n = 2m+1 . \end{cases}$$

Here we used the notation $\underset{\sim}{0} = (0, 0, 0)$. By (61) it follows that

$$(123) \qquad \mathop{P}_{\sim}\{\underset{\sim}{\eta}_{2m}(3) = \underset{\sim}{0}\} \sim \frac{1}{(m\pi)^{3/2}}$$

as $m \to \infty$. Since a return to the origin is a recurrent event, it follows from Theorem 3 that with probability 1 the particle returns to the origin only finitely many times. To find the probability distribution of the number of returns to the origin we should determine the sum

$$(124) \qquad \sum_{n=0}^{\infty} \mathop{P}_{\sim}\{\underset{\sim}{\eta}_n(3) = \underset{\sim}{0}\} = \frac{1}{(2\pi)^3} \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{1}{1-\cos t_1 \cos t_2 \cos t_3} dt_1 dt_2 dt_3 .$$

<u>G. N. Watson</u> [153] showed that the right-hand side of (124) can be

expressed as

$$(125)\qquad \frac{[\Gamma(\frac{1}{4})]^4}{4\pi^3} = 1.39320\ \ 39297\ldots\ .$$

Second, let us suppose that starting from the origin in a three-dimensional Euclidean space a particle takes a sequence of steps and in each step it moves in accordance with one of the twelve vectors: $(\pm 1, \pm 1, 0)$, $(\pm 1, 0, \pm 1)$, $(0, \pm 1, \pm 1)$ with probability $1/12$. Let us suppose that the successive displacements are independent. Denote by $\underset{\sim}{\eta}_n(3) = (\eta_n(1), \eta_n(2), \eta_n(3))$ the position of the particle at the n-th step. Now we have

$$(126)\qquad \underset{\sim}{E}\{e^{it_1\eta_n(1)+it_2\eta_n(2)+it_3\eta_n(3)}\} = \frac{1}{3^n}(\cos t_1\cos t_2 + \cos t_1\cos t_3 + \cos t_2\cos t_3)^n$$

for real t_1, t_2, t_3. In this case

$$(127)\qquad \underset{\sim}{P}\{\underset{\sim}{\eta}_n(3) = \underset{\sim}{0}\} = \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{\cos t_1\cos t_2+\cos t_1\cos t_3+\cos t_2\cos t_3}{3}\right)^n dt_1dt_2dt_3\ .$$

In a similar way as we proved (67), it follows that

$$(128)\qquad \underset{\sim}{P}\{\underset{\sim}{\eta}_{2m}(3) = \underset{\sim}{0}\} \sim \frac{3\sqrt{3}}{8\sqrt{2}\,\pi^{3/2}m^{3/2}}\ .$$

Obviously, $\underset{\sim}{P}\{\eta_{2m+1}(3) = \underset{\sim}{0}\} = 0$.

First, in (127) we can replace the domain of integration by

$$(129)\qquad D = \{(t_1, t_2, t_3):\ -\frac{\pi}{2} \leq t_k \leq \frac{3\pi}{2}\ ,\quad k = 1,2,3\}$$

without changing the value of the integral. In this domain the integrand in (127) equals 1 if $t_1 = t_2 = t_3 = 0$ or if $t_1 = t_2 = t_3 = \pi$ and has absolute value <1 otherwise. For a sufficiently small positive ε let us define $D_1(\varepsilon) = \{(t_1, t_2, t_3): |t_k| < \varepsilon$ for $k = 1, 2, 3\}$ and $D_2(\varepsilon) = \{(t_1, t_2, t_3): |t_k - \pi| < \varepsilon$ for $k = 1, 2, 3\}$. Let $D^*(\varepsilon)$ the set of all those points of D which do not belong to $D_1(\varepsilon)$ or $D_2(\varepsilon)$. If $n = 2m$ and $i = 1$ or $i = 2$, then

$$\text{(130)} \quad \begin{aligned} & m^{3/2} \iiint_{D_i(\varepsilon)} \left(\frac{\cos t_1 \cos t_2 + \cos t_1 \cos t_3 + \cos t_2 \cos t_3}{3}\right)^{2m} dt_1 dt_2 dt_3 = \\ & = \int_{-\varepsilon\sqrt{m}}^{\varepsilon\sqrt{m}} \int_{-\varepsilon\sqrt{m}}^{\varepsilon\sqrt{m}} \int_{-\varepsilon\sqrt{m}}^{\varepsilon\sqrt{m}} \left(\frac{1}{3}\cos\frac{u_1}{\sqrt{m}}\cos\frac{u_2}{\sqrt{m}} + \frac{1}{3}\cos\frac{u_1}{\sqrt{m}}\cos\frac{u_2}{\sqrt{m}} + \frac{1}{3}\cos\frac{u_2}{\sqrt{m}}\cos\frac{u_3}{\sqrt{m}}\right)^{2m} du_1 du_2 du_3 \end{aligned}$$

$$\sim \iiint_{-\varepsilon\sqrt{m}}^{\varepsilon\sqrt{m}} \left(1 - \frac{u_1^2+u_2^2+u_3^2}{3\sqrt{m}}\right)^{2m} du_1 du_2 du_3 \sim \iiint_{-\infty}^{\infty} e^{-\frac{2(u_1^2+u_2^2+u_3^2)}{3}} du_1 du_2 du_3 = \left(\frac{3\pi}{2}\right)^{3/2}$$

as $m \to \infty$. The integral over the domain $D^*(\varepsilon)$ tends to 0 as $m \to \infty$. Thus by (130) we obtain (128).

Accordingly, with probability 1, the particle returns to the origin only a finite number of times. To find the distribution of the number of returns to the origin we should determine the sum

$$\text{(131)} \quad \begin{aligned} & \sum_{n=0}^{\infty} \mathbf{P}\{\underset{\sim}{\eta}_n(3) = \underset{\sim}{0}\} = \\ & = \frac{1}{(2\pi)^3} \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{dt_1 dt_2 dt_3}{1 - \frac{1}{3}(\cos t_1 \cos t_2 + \cos t_1 \cos t_3 + \cos t_2 \cos t_3)} . \end{aligned}$$

G. N. Watson [153] showed that the right-hand side of (131) can be expressed as

$$\frac{9[\Gamma(\frac{1}{3})]^8}{2^{14/3}\pi^4} = 1.34466\ 11832\ \ldots \tag{132}$$

Further Results for One-Dimensional Random Walks. A particle performs a random walk on the x-axis. It starts at $x=0$ and in each step independently of the others, it moves a unit distance to the right with probability p or a unit distance to the left with probability q where $p>0$, $q>0$, and $p+q=1$. Denote by η_n the position of the particle at the n-th step $(n=1,2,\ldots)$ and let $\eta_0 = 0$. Denote by τ_k the time when the particle returns to the initial position for the k-th time $(k = 1, 2, \ldots)$ and let $\tau_0 = 0$.

The differences $\tau_k - \tau_{k-1}$ $(k = 1, 2, \ldots)$ form a sequence of mutually independent and identically distributed random variables taking on positive integers (possibly ∞). Let

$$\underset{\sim}{P}\{\tau_k - \tau_{k-1} = j\} = f_j \tag{133}$$

for $j = 1,2,\ldots$. Obviously, $f_{2m+1} = 0$ for $m = 0,1,\ldots$. Now we shall prove that

$$f_{2m} = \frac{2}{m}\binom{2m-2}{m-1}(pq)^m \tag{134}$$

for $m = 1,2,\ldots,$ and

$$\sum_{m=1}^{\infty} f_{2m} = 1-|p-q| . \tag{135}$$

Let $u_n = \mathrm{P}\{\eta_n = 0\}$ for $n = 0,1,2,\dots$. We have

$$(136) \qquad u_{2m} = \binom{2m}{m} p^m q^m = \binom{-\frac{1}{2}}{m} (-4pq)^m$$

for $m = 0,1,2,\dots$ and $u_{2m+1} = 0$ for $m = 0,1,2,\dots$. Furthermore,

$$(137) \qquad U(z) = \sum_{n=0}^{\infty} u_n z^n = \sum_{m=0}^{\infty} \binom{2m}{m} p^m q^m z^{2m} = \frac{1}{\sqrt{1-4pqz^2}}$$

for $|z| < 1$.

Let

$$(138) \qquad F(z) = \sum_{j=1}^{\infty} f_j z^j$$

for $|z| \leq 1$. Since a return to the origin is a recurrent event, we obtain by Theorem 5 that

$$(139) \qquad F(z) = 1 - \frac{1}{U(z)} = 1 - \sqrt{1-4pqz^2} = \sum_{m=1}^{\infty} (-1)^{m-1} \binom{\frac{1}{2}}{m} (4pqz^2)^m$$

for $|z| < 1$ and by continuity (139) holds for $|z| \leq 1$ too. Hence

$$(140) \qquad f_{2m} = (-1)^{m-1} \binom{\frac{1}{2}}{m} (4pq)^m = \frac{2}{m} \binom{2m-2}{m-1} (pq)^m$$

if $m = 1,2,\dots,$ and $f_{2m+1} = 0$ if $m = 0,1,2,\dots$. Furthermore, we have

$$(141) \qquad \sum_{j=1}^{\infty} f_j = F(1) = 1 - \sqrt{1-4pq} = 1 - |p-q| \ .$$

Accordingly, if $p = q$, then $\{f_j\}$ is a proper distribution. If $p \neq q$, then $\{f_j\}$ is defective. The recurrence time may be ∞ with probability $|p-q|$. If $p = q$, then the expected recurrence time

$$(142) \qquad \sum_{j=1}^{\infty} j f_j = \infty .$$

For in this case $f_{2m} \sim 2/\pi m^{3/2}$, or more precisely,

$$(143) \qquad \frac{1}{\sqrt{(m-\frac{1}{2})\pi}} < 2mf_{2m} < \frac{1}{\sqrt{(m-1)\pi}}$$

for $m = 2,3,\ldots$.

We note that

$$(144) \qquad \underset{\sim}{P}\{\tau_k = 2m\} = \frac{k2^k}{2m-k} \binom{2m-k}{m} (pq)^m$$

for $m = k, k+1,\ldots$. This can be obtained from the generating function

$$(145) \qquad \sum_{m=1}^{\infty} \underset{\sim}{P}\{\tau_k = 2m\} z^{2m} = [F(z)]^k = (1-\sqrt{1-4pqz^2})^k$$

by Lagrange's expansion. If we take into consideration that for $|z| < 1$ the equation

$$(146) \qquad w^2 - 2w + 4pqz^2 = 0$$

has exactly one root $w = F(z)$ in the unit circle $|w| \leq 1$, and we form the Lagrange expansion of $[F(z)]^k$ for $k = 1,2,\ldots,$ then we obtain (144).

In the particular case when $p = q$, we obtain easily from (144) that

$$\lim_{k \to \infty} P\{\frac{\tau_k}{k^2} \leq x\} = \begin{cases} 2[1-\Phi\left(x^{-\frac{1}{2}}\right)] & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \tag{147}$$

where $\Phi(x)$ is the normal distribution function defined by (35.18).

Now denote by ν_n $(n = 1,2,\ldots)$ the number of returns to the origin in the first n steps and let $\nu_0 = 0$. Evidently we have $\{\nu_n < k\} \equiv \{\tau_k > n\}$ for $n \geq 0$ and $k \geq 0$. Thus

$$P\{\nu_n < k\} = P\{\tau_k > n\} \tag{148}$$

for $n \geq 0$ and $k \geq 0$. The probability on the right-hand side of (148) can be obtained by (144) and thus the distribution of ν_n is determined by (148).

If, in particular, $p = q$, then we obtain that

$$P\{\nu_{2m} = r\} = \binom{2m-r}{m} \frac{1}{2^{2m-r}} \tag{149}$$

for $m = 1,2,\ldots,$ and obviously $P\{\nu_{2m+1} = r\} = P\{\nu_{2m} = r\}$. In this case the expectation of ν_n is given by

$$E\{\nu_n\} = \binom{n}{[\frac{n}{2}]} \frac{n+1}{2^n} - 1 \tag{150}$$

for $n = 0,1,2,\ldots$. For

$$E\{\nu_{2m+1}\} = E\{\nu_{2m}\} = \sum_{j=1}^{m} u_{2j} = \sum_{j=1}^{m} \binom{2j}{j} \frac{1}{2^{2j}} = \sum_{j=1}^{m} (-1)^j \binom{-\frac{1}{2}}{j} =$$

(151)

$$= (-1)^m \binom{-\frac{3}{2}}{m} - 1 = \binom{2m}{m} \frac{(2m+1)}{2^{2m}} - 1 = \binom{2m+1}{m} \frac{(2m+2)}{2^{2m+1}} - 1$$

for $m = 1,2,\ldots$ which proves (150).

Accordingly, we have the following interesting identity

(152)

$$E\{\nu_{2m}\} = \sum_{r=0}^{m} r\binom{2m-r}{m} \frac{1}{2^{2m-r}} = \binom{2m}{m} \frac{2m+1}{2^{2m}} - 1$$

for $m = 1,2,\ldots$.

If $p = q$, then by (150) it follows that

(153) $$E\{\nu_n\} \sim \sqrt{\frac{2n}{\pi}}$$

as $n \to \infty$, and by (147) and (148) we can prove that

(154) $$\lim_{n\to\infty} P\left\{\frac{\nu_n}{\sqrt{n}} \leq x\right\} = \begin{cases} 2\Phi(x)-1 \text{ for } x > 0 , \\ 0 \quad \text{for } x \leq 0 . \end{cases}$$

For by (148) we have

(155) $$P\left\{\frac{\nu_n}{\sqrt{n}} < \frac{k}{\sqrt{n}}\right\} = P\left\{\frac{\tau_k}{k^2} > \frac{n}{k^2}\right\}$$

for any $n \geq 0$ and $k \geq 0$. Let $x > 0$ and choose n and k such that $n \sim k^2 x$ as $k \to \infty$. Then $k \sim \sqrt{n/x}$ as $n \to \infty$. If we choose n and k in such a way, then by (155) we obtain that

(156) $$\lim_{n\to\infty} \underset{\sim}{P}\left\{\frac{\nu_n}{\sqrt{n}} \le x\right\} = \lim_{k\to\infty} \underset{\sim}{P}\left\{\frac{\tau_k}{k^2} \le \frac{1}{x^2}\right\} = 2\,\Phi(x) - 1$$

for $x > 0$. This proves (154) for $x > 0$. The case of $x \leqq 0$ is trivial.

Finally, denote by Δ_n the number of positive elements in the sequence $\eta_1, \eta_2, \ldots, \eta_n$ and let $\Delta_0 = 0$. If $p = q = \frac{1}{2}$, then we have

(157) $$\underset{\sim}{P}\{\Delta_n = j\} = \binom{j-1}{[\frac{j-1}{2}]}\binom{n-j}{[\frac{n-j}{2}]}\frac{1}{2^n}$$

for $0 \le j \le n$ where $\binom{0}{0} = \binom{-1}{-1} = 1$. More generally, if $0 < p < 1$, then we have $\underset{\sim}{P}\{\Delta_n = 0\} = a_n(p)$ for $n = 0,1,2,\ldots,$ and

(158) $$\underset{\sim}{P}\{\Delta_n = j\} = p\, a_{j-1}(q)\, a_{n-j}(p)$$

for $j = 1,2,\ldots,n$ where

(159) $$a_n(p) = 1 - p\sum_{m=0}^{[\frac{n-1}{2}]} \binom{2m}{m}\frac{(pq)^m}{m+1}$$

for $n = 1,2,\ldots$ and $0 < p < 1$ and $a_0(p) = 1$.

In what follows we shall prove (158). Write $\underset{\sim}{P}\{\Delta_n = 0\} = a_n(p)$. Then $a_0(p) = 1$ for $0 < p < 1$ and by (10) we have

(160) $$a_n(p) = \underset{\sim}{P}\{\eta_r < 1 \text{ for } 0 < r \le n\} = \underset{\sim}{P}\{\eta_n < 1\} - \frac{p}{q}\underset{\sim}{P}\{\eta_n < -1\}$$

for $n = 1,2,\ldots$. Thus

(161) $$a_n(p) = \left(1 - \frac{p}{q}\right)\sum_{j=0}^{[\frac{n}{2}]}\binom{n}{j} p^j q^{n-j} + \frac{p}{q}\binom{n}{[\frac{n}{2}]} p^{[\frac{n}{2}]} q^{[\frac{n+1}{2}]}$$

for $n = 1,2,\ldots$ and (161) can also be expressed in the form of (159).

Furthermore, we have

(162) $$\underset{\sim}{P}\{\Delta_n = n\} = p\, a_{n-1}(q)$$

for $n = 1,2,\ldots$ which follows from the relations

(163) $$\begin{aligned}\underset{\sim}{P}\{\Delta_n = n\} &= \underset{\sim}{P}\{\eta_1 = 1 \text{ and } \eta_i - \eta_1 \geq 0 \quad \text{for} \quad 1 \leq i \leq n\} = \\ &= \underset{\sim}{P}\{\eta_1 = 1\}\,\underset{\sim}{P}\{-\eta_r \leq 0 \quad \text{for} \quad 0 \leq r \leq n-1\}\ .\end{aligned}$$

By Theorem 22.1 we have

(164) $$\underset{\sim}{P}\{\Delta_n = j\} = \underset{\sim}{P}\{\Delta_j = j\}\,\underset{\sim}{P}\{\Delta_{n-j} = 0\}$$

for $0 \leq j \leq n$ and this implies (158).

If $p = q = \frac{1}{2}$, then (161) reduces to

(165) $$a_n(\tfrac{1}{2}) = \binom{n}{[\frac{n}{2}]}\frac{1}{2^n} = \binom{2[\frac{n+1}{2}]}{[\frac{n+1}{2}]}\frac{1}{2^{2[\frac{n+1}{2}]}}$$

and as a particular case of (158) we obtain (157).

If $p = q = \frac{1}{2}$, then we have

(166) $$\lim_{n\to\infty} \underset{\sim}{P}\{\frac{\Delta_n}{n} \leq x\} = \frac{2}{\pi}\text{ arc sin } \sqrt{x}$$

for $0 \leq x \leq 1$. (See P. Erdős and M. Kac [170], and E. S. Andersen [158].) We can prove (166) in the following way. By (61) and (165) we have the inequalities

(167) $$\sqrt{\frac{2}{(n+2)\pi}} < \underset{\sim}{P}\{\Delta_n = 0\} < \sqrt{\frac{2}{n\pi}}$$

for $n = 1,2,\ldots$. Thus by (157) we get

(168) $$\frac{1}{\pi\sqrt{(j+1)(n+2-j)}} < \underset{\sim}{P}\{\Delta_n = j\} < \frac{1}{\pi\sqrt{(j-1)(n-j)}}$$

for $1 < j < n$.

If $0 < \alpha < \beta < 1$, then

$$(169)\quad \lim_{n\to\infty} \sum_{n\alpha \le j \le n\beta} \frac{1}{\sqrt{(j+1)(n+2-j)}} = \lim_{n\to\infty} \sum_{n\alpha \le j \le n\beta} \frac{1}{\sqrt{(j-1)(n-j)}} = \int_{\alpha}^{\beta} \frac{dx}{\sqrt{x(1-x)}}$$

and this implies that

$$(170)\quad \lim_{n\to\infty} \mathbf{P}\{\alpha \le \frac{\Delta_n}{n} \le \beta\} = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{dx}{\sqrt{x(1-x)}} = \frac{2}{\pi}(\text{arc sin} \sqrt{\beta} - \text{arc sin} \sqrt{\alpha})$$

for $0 < \alpha < \beta < 1$. Since (170) holds for all $0 < \alpha < \beta < 1$, it follows that (166) holds for all $0 \le x \le 1$.

In the references of this chapter many other results can be found for the random walk processes discussed in this chapter and for various generalized random walk processes.

38. Ballot Theorems. In this section we shall study various problems connected with the fluctuations of election returns. Throughout this section we assume that two candidates A and B run in an election and candidate A scores a votes, and candidate B scores b votes. We assume that all the possible $\binom{a+b}{a}$ voting records are equally probable.

The first ballot theorem was formulated in 1887 by J. Bertrand [163]. He found the following result:

Theorem 1. If $a \geq b$, then the probability that throughout the counting the number of votes registered for A is always greater than the number of votes registered for B is given by

$$P(a,b) = \frac{a-b}{a+b} . \tag{1}$$

Proof. Denote by $N(a,b)$ the number of voting records satisfying the requirements in Theorem 1. Then

$$P(a,b) = \frac{N(a,b)}{\binom{a+b}{a}} . \tag{2}$$

J. Bertrand noticed that for $a > b$ the function $N(a,b)$ satisfies the following recurrence formula

$$N(a,b) = N(a-1,b) + N(a,b-1) \tag{3}$$

where obviously $N(a,0) = 1$ for $a \geq 1$ and $N(a,b) = 0$ for $b = a$. We can determine $N(a,b)$ for $a \geq b$ by (3). The following table contains $N(a,b)$ for $a \leq 6$ and $b \leq 4$.

N(a,b)

b \ a	0	1	2	3	4	5	6
0	-	1	1	1	1	1	1
1	-	0	1	2	3	4	5
2	-	-	0	2	5	9	14
3	-	-	-	0	5	14	28
4	-	-	-	-	0	14	42

We can easily prove that

$$(4) \qquad N(a,b) = \binom{a+b-1}{a-1} - \binom{a+b-1}{a}$$

for $a \geq b$. Hence (1) follows immediately.

Actually, J. Bertrand did not prove formula (4); however he indicated that probably there is a direct proof for (1). He was right. In the same year D. André [160] provided a direct proof for (1). He reasoned as follows:

Every voting record can be described by a sequence of a letters A and b letters B if A stands for a vote for A and B stands for a vote for B . The number of such voting records is $\binom{a+b}{a}$.

D. André showed that the number of unfavorable voting records is

$$2\binom{a+b-1}{a} . \tag{5}$$

Thus it follows that

$$N(a,b) = \binom{a+b}{a} - 2\binom{a+b-1}{a} \tag{6}$$

which implies (1).

To prove (5) let us observe that the set of unfavorable voting records is the union of two disjoint classes: The first class contains all those voting records which start with a B . The second class contains all those voting records which start with an A and at least once the number of letters B is equal to the number of letters A , if we count the letters from the left.

There is a one-to-one correspondence between the voting records in these two classes. This can be seen as follows: If a voting record belongs to the second class, then counting the letters from the left, there is a shortest subsequence which contains an equal number of letters A and B . The last letter in this shortest sequence is necessarily B . In this shortest sequence let us remove all the letters except the last B and put them at the end of the record in the same order. Then we obtain a voting record which belongs to the first class.

Conversely, if a voting record belongs to the first class, then counting letters from the right, there is a shortest subsequence which contains one more letters A than B . The first letter in this shortest sequence is necessarily A . Let us remove all the letters in this shortest sequence and put them at the beginning of the record in the same order. Then we obtain a voting record which belongs to the second class.

It is easy to see that this mapping is one-to-one, and therefore both classes contain the same number of voting records. The first class evidently contains $\binom{a+b-1}{a}$ voting records. Thus the total number of unfavorable voting records is $2\binom{a+b-1}{a}$ which proves (5).

It should be added that Theorem 1 can also be deduced from a result of duration of plays which was found in 1708 by A. De Moivre [76 p. 262], and in a different form in 1718 also by A. De Moivre [77p. 121]. De Moivre did not give proofs of his results. Proofs for De Moivre's results were given only in 1773 by P. S. Laplace [86 pp. 188-193] and in 1776 by J. L. Lagrange [88 pp. 230-238].

This result is the following: Suppose that two players A and B play a sequence of games. In each game, independently of the others, either A wins a coin from B with probability p or B wins a coin from A with probability q where $p > 0$, $q > 0$ and $p+q = 1$. Suppose that A has an initial capital of $a-b$ coins, and B has an unlimited number of coins. De Moivre found that the probability that A will be ruined at the $(a+b)$-th game is

$$(7) \qquad \frac{a-b}{a+b}\binom{a+b}{a}q^a p^b .$$

(See formula (36.42).) The probability that in the $a+b$ games A loses a games and B loses b games is

$$(8) \qquad \binom{a+b}{a} q^a b^b \ .$$

The conditional probability that A will be ruined at the $(a+b)$-th game, given that in the $(a+b)$ games A loses a games and B loses b games, is accordingly $(a-b)/(a+b)$.

If we consider the $(a+b)$ games in reverse order, and a loss for A corresponds to a vote for A , and a loss for B corresponds to a vote for B , then we can see immediately that $(a-b)/(a+b)$ is the probability that A is leading throughout the counting of the $a+b$ votes.

We can use the same reflection principle in proving (1) as we used in Section 36 in proving (36.42). Equivalently, we can prove (1) by using a random walk interpretation.

Suppose that a particle performs a random walk on the x-axis. It starts at $x = 0$ and moves a steps to right and b steps to the left in random order. Each step consists of a unit distance displacement. If every path has the same probability, then the probability that the particle never returns to the point $x = 0$ is given by $(a-b)/(a+b)$ for $a \geq b$.

For the total number of possible paths is $\binom{a+b}{a}$. The number of paths in which the particle never returns to the point $x = 0$ is

$$(9) \qquad \binom{a+b-1}{a-1} - \binom{a+b-1}{a} = \frac{a-b}{a+b}\binom{a+b}{a}$$

which can be obtained by using the method of reflection. If a step to the right corresponds to a vote for A and a step to the left corresponds to a vote for B , then it follows that $(a-b)/(a+b)$ is the probability that candidate A leads throughout the counting.

For other proofs of (1) we refer to J. Aebly [155], D. Mirimanoff [181], A. Aeppli [157], P. Erdős and I. Kaplansky [171], and H. D. Grossman [173].

The following result is an easy consequence of Theorem 1.

Theorem 2. *If $a \geq b$, then the probability that throughout the counting the number of votes registered for A is greater than or equal to the number of votes registered for B is given by*

$$(10) \qquad Q(a,b) = \frac{a+1-b}{a+1} .$$

Proof. We have the obvious relation

$$(11) \qquad P(a+1,b) = \frac{a+1}{a+1+b} Q(a,b) .$$

For if we add one more vote for A to the $a+b$ votes, then the probability that A leads throughout the counting is $P(a+1,b) = (a+1-b)/(a+1+b)$. Since A leads throughout the counting if and only if the first vote is for A , and even if we disregard this vote, he never loses throughout the counting. Thus we obtain (11), whence (10) follows. Conversely, (10) implies (1) too.

If we interpret the process of counting as a random walk, then $Q(a,b)$ is the probability that the particle never reaches the point $x = -1$. By using the reflection principle we obtain easily that the number of favorable paths is $\binom{a+b}{a} - \binom{a+b}{a+1}$. If we divide this by the number of possible paths, $\binom{a+b}{a}$, then we obtain (10).

The proof of Theorem 2 can be deduced from a combinatorial result which was found in 1879 by W. A. Whitworth [196]. See also W. A. Whitworth [197], Chapter V, and P. A. MacMahon [179], [180].

In 1887 É. Barbier [161] generalized Theorem 1 in the following way:

Theorem 3. *If $a \geq \mu b$ where μ is a nonnegative integer, then the probability that throughout the counting the number of votes registered for A is always greater than μ times the number of votes registered for B is given by*

$$P(a,b;\mu) = \frac{a-\mu b}{a+b} , \tag{12}$$

and the probability that the number of votes registered for A is always at least μ times the number of votes registered for B is given by

$$Q(a,b;\mu) = \frac{a+1-\mu b}{a+1} . \tag{13}$$

Proof. Since we have the obvious relation

$$(14) \qquad P(a+1,\ b;\mu) = \frac{a+1}{a+1+b}\,Q(a,b;\mu)\ ,$$

it is sufficient to prove one of the two formulas (12) and (13).

Let us prove (12). First, more generally, we suppose that μ is a nonnegative real number, and then we consider the particular case when μ is a nonnegative integer.

Denote by $N(a,b;\mu)$ the number of voting records which satisfy the requirements that throughout the counting the number of votes registered for A is always greater than the number of votes registered for B . If $a > b\mu$, then we have

$$(15) \qquad N(a,b;\mu) = N(a-1,\ b;\mu) + N(a,b-1;\mu)$$

where obviously $N(a,0;\mu) = 1$ for $a \geqq 1$ and $N(a,b;\mu) = 0$ for $a = [b\mu]$. The equation (15) is obvious, it reflects only the fact that the last vote counted may be either a vote for A or a vote for B . We can obtain $N(a,b;\mu)$ recursively from (15) if take into consideration the boundary conditions $N(a,0;\mu) = 1$ for $a \geqq 1$, and $N([b\mu],\ b;\mu) = 0$ for $b \geqq 1$. The following table contains $N(a,b;2)$ for $a \leqq 8$ and $b \leqq 4$.

N(a,b;2)

b \ a	0	1	2	3	4	5	6	7	8
0	-	1	1	1	1	1	1	1	1
1	-	-	0	1	2	3	4	5	6
2	-	-	-	-	0	3	7	12	18
3	-	-	-	-	-	-	0	12	30
4	-	-	-	-	-	-	-	-	0

The general solution of the difference equation (15) which satisfies the boundary conditions $N(a,0;\mu) = 1$ for $a \geq 1$ is given by

$$N(a,b;\mu) = \sum_{j=0}^{b} C_j(\mu)\binom{a+b-1-j}{b-j} \tag{16}$$

where $C_0(\mu) = 1$. (See Ch. Jordan [33] p. 607.) The constants $C_j(\mu)$ $(j = 1,2,\ldots)$ are determined by the boundary conditions

$$N([r\mu],r;\mu) = \sum_{j=0}^{r} C_j(\mu)\binom{[r\mu]+r-1-j}{r-j} = 0 \tag{17}$$

for $r = 1,2,\ldots$. Thus we obtain that

$$P(a,b;\mu) = \sum_{j=0}^{b} C_j(\mu)\frac{\binom{a+b-1-j}{b-j}}{\binom{a+b}{a}} = \frac{a}{a+b}\sum_{j=0}^{b} C_j(\mu)\frac{\binom{b}{j}}{\binom{a+b-1}{j}} \tag{18}$$

for $a \geq b\mu$ where $C_0(\mu) = 1$ and $C_j(\mu)$ $(j = 1,2,\ldots)$ are determined by (17).

If, in particular, μ is a nonnegative integer, then by (17) we obtain that $C_j(\mu) = -\mu$ for $j = 1,2,\ldots$, and in this case (18) reduces to (12).

In the general case $C_1(\mu) = -[\mu]$, $C_2(\mu) = -[2\mu](1-2[\mu]+[2\mu])/2$, and so on.

The first proof for (12) was given only in 1924 by A. Aeppli [157]. Other proofs for (12) were given in 1947 by A. Dvoretzky and Th. Motzkin [166], in 1950 by H. D. Grossman [175], in 1961 by S. G. Mohamty and T. V. Narayana [183], and in 1960 by the author [191].

Theorem 3 is a particular case of the corollary of Lemma 20.1. This can be seen as follows:

Let us suppose that a box contains a cards marked by 0 and b cards marked by $\mu+1$. We draw all the $a+b$ cards from the box without replacement. Suppose that all the possible outcomes are equally probable. Then $P(a,b;\mu)$ can be interpreted as the probability that for every $r = 1,2,\ldots, a+b$ the sum of the first r numbers drawn is less than r, and $Q(a,b;\mu)$, as the probability that for every $r = 1,2,\ldots, a+b$ the sum of the first r numbers drawn is less than or equal to r. For if among the first r drawings there are α_r zeros and β_r $(\mu+1)$'s, then $\alpha_r 0 + \beta_r(\mu+1) < \alpha_r + \beta_r$ holds if and only if $\alpha_r > \beta_r\mu$ and $\alpha_r 0 + \beta_r(\mu+1) \leqq \alpha_r + \beta_r$ holds if and only if $\alpha_r \geqq \beta_r\mu$. Thus by Lemma 20.1 we obtain that

$$P(a,b;\mu) = 1- \frac{b(\mu+1)}{a+b} \tag{19}$$

if $a \geqq b\mu$ and μ is a nonnegative integer. Formula (13) follows immediately from (12).

The problems discussed above can be considered as particular cases of the following more general problems:

As previously, let us suppose that in a ballot candidate A scores a votes and candidate B scores b votes and that all the possible $\binom{a+b}{a}$ voting records are equally probable. Denote by α_r and β_r the number of votes registered for A and B respectively among the first r votes recorded. Denote by $P_j(a,b;\mu)$ the probability that the inequality $\alpha_r > \mu\beta_r$ holds for precisely j subscripts $r = 1,2,\ldots, a+b$ and by $Q_j(a,b;\mu)$ the probability that the inequality $\alpha_r \geqq \mu\beta_r$ holds for precisely j subscripts $r = 1,2,\ldots, a+b$. Here μ is a nonnegative real number.

It has some importance to find the probabilities

(20) $$P_j(a,b;\mu) = \underset{\sim}{P}\{\alpha_r > \mu\beta_r \text{ for } j \text{ subscripts } r = 1,2,\ldots, a+b\}$$

and

(21) $$Q_j(a,b;\mu) = \underset{\sim}{P}\{\alpha_r \geqq \mu\beta_r \text{ for } j \text{ subscripts } r = 1,2,\ldots, a+b\}$$

for $j = 0,1,\ldots, a+b$.

If we denote by $N_j(a,b;\mu)$ the number of voting records satisfying the condition $\alpha_r > \mu\beta_r$ for precisely j subscripts $r = 1,2,\ldots, a+b$, then

(22) $$P_j(a,b;\mu) = \frac{N_j(a,b;\mu)}{\binom{a+b}{a}}$$

and if $M_j(a,b;\mu)$ denotes the number of voting records satisfying the condition $\alpha_r \geq \mu\beta_r$ for precisely j subscripts $r = 1,2,\ldots, a+b$, then

$$(23) \qquad Q_j(a,b;\mu) = \frac{M_j(a,b;\mu)}{\binom{a+b}{a}} .$$

We shall determine the probability distributions $\{P_j(a,b;\mu)\}$ and $\{Q_j(a,b;\mu)\}$ in two particular cases when either μ is a positive integer or $\mu = a/b$.

The following theorem was found in 1964 by the author [194].

Theorem 4. *If μ is a nonnegative integer, then we have*

$$(24) \qquad P_j(a,b;\mu) = \sum_{0 \leq s \leq \frac{j}{\mu+1}} \frac{\binom{j}{s}\binom{a+b-j}{b-s}}{\binom{a+b}{a}} P_0(a+s-j, b-s;\mu) P_j(j-s,s;\mu)$$

for $j = 0,1,\ldots, a+b$ *where*

$$(25) \qquad P_0(a,b;\mu) = 1 - \frac{1}{\binom{a+b}{a}} \sum_{0 \leq s \leq \frac{a+b-1}{\mu+1}} \frac{1}{[s(\mu+1)+1]} \binom{s\mu+s+1}{s}\binom{a+b-s\mu-s-1}{b-s}$$

for $a \leq \mu b$ *and* $P_0(a,b;\mu) = 0$ *for* $a > \mu b$, *and*

$$(26) \qquad P_{a+b}(a,b;\mu) = \frac{a-\mu b}{a+b}$$

for $a > \mu b$ *and* $P_{a+b}(a,b;\mu) = 0$ *for* $a \leq \mu b$.

Proof. Define the random variables ν_r $(r = 1,2,\ldots, a+b)$ in the following way: $\nu_r = 0$ if the r-th vote is cast for A and $\nu_r = (\mu+1)$ if the r-th vote is cast for B. Set $N_r = \nu_1+\ldots+\nu_r$ for $r = 1,2,\ldots, a+b$ and $N_0 = 0$. Now $\nu_1, \nu_2,\ldots, \nu_{a+b}$ are interchangeable random variables taking on nonnegative integers and satisfying the condition $\nu_1+\nu_2+\ldots+\nu_{a+b} = b(\mu+1)$. We have

$$(27) \qquad \underset{\sim}{P}\{N_i = s(\mu+1)\} = \frac{\binom{a}{i-s}\binom{b}{s}}{\binom{a+b}{i}} = \frac{\binom{i}{s}\binom{a+b-i}{b-s}}{\binom{a+b}{a}}$$

for $s = 0,1,\ldots, \min(i,b)$ and $\underset{\sim}{P}\{N_i = j\} = 0$ otherwise.

If we use the above rotation then we can write that

$$(28) \qquad P_j(a,b;\mu) = \underset{\sim}{P}\{N_r < r \text{ for } j \text{ subscripts } r = 1,2,\ldots, a+b\}$$

for $j = 0,1,\ldots, a+b$. Since $N_r = \beta_r(\mu+1)$ and $r = \alpha_r+\beta_r$ for $r = 1,2,\ldots, a+b$, it follows that the inequality $\alpha_r > \mu\beta_r$ holds if and only if $N_r < r$. This proves (28).

By Theorem 22.1 we can conclude that the probability that $N_r < r$ for j subscripts $r = 1,2,\ldots, a+b$ is the same as the probability that the first maximal element in the sequence $r-N_r$ $(r = 0,1,\ldots, a+b)$ is $j-N_j$. Accordingly, it follows that

$$(29) \qquad \begin{aligned} P_j(a,b;\mu) &= \underset{\sim}{P}\{r-N_r < j-N_j \text{ for } 0 \leq r < j \text{ and } r-N_r \leq j-N_j \text{ for } j \leq r \leq a+b\} \\ &= \sum_{s=0}^{b} \underset{\sim}{P}\{N_j = s(\mu+1)\}\underset{\sim}{P}\{N_j - N_r < j-r \text{ for } 0 \leq r < j \mid N_j = s(\mu+1)\} \cdot \\ &\quad \cdot \underset{\sim}{P}\{r-j \leq N_r - N_j \text{ for } j \leq r \leq a+b \mid N_j = s(\mu+1)\} . \end{aligned}$$

By using the representation (28) we can write (29) in the following equivalent form:

$$(30) \qquad P_j(a,b;\mu) = \sum_{s=0}^{b} \underset{\sim}{P}\{N_j = s(\mu+1)\} P_j(j-s,s;\mu) P_0(a+s-j,b-s;\mu)$$

where $j = 0,1,\ldots, a+b$. By (27) this proves (34).

If $j = a+b$ in (28) and if we take into consideration that $N_{a+b} = b(\mu+1)$, then by Lemma 20.2 it follows that

$$(31) \qquad \begin{aligned} P_{a+b}(a,b;\mu) &= \underset{\sim}{P}\{N_r < r \text{ for } r = 1,2,\ldots, a+b \mid N_{a+b} = b(\mu+1)\} = \\ &= \begin{cases} 1 - \dfrac{b(\mu+1)}{a+b} & \text{if } b\mu < a , \\ 0 & \text{if } b\mu \geq a . \end{cases} \end{aligned}$$

(We note that $P_0(0,0;\mu) = 1$.) This proves (26). Obviously $P_{a+b}(a,b;\mu) = P(a,b,\mu)$ defined by (12). Accordingly, in formula (24) we can write $P_j\,(j-s,s;\mu) = (j-s\mu-s)/j$ if $s(\mu+1) < j$ and $P_j(j-s,s;\mu) = 0$ if $s(\mu+1) \geq j$.

It remains to find $P_0(a,b,;\mu)$. By Theorem 20.2 and by (28) it follows that

$$(32) \qquad \begin{aligned} P_0(a,b;\mu) &= \underset{\sim}{P}\{r-N_r \leq 0 \text{ for } r = 1,2,\ldots, a+b\} = \\ &= 1 - \sum_{\ell=1}^{a+b} \frac{1}{\ell} \underset{\sim}{P}\{N_\ell = \ell-1\} = \\ &= 1 - \sum_{0 \leq s \leq \frac{a+b-1}{\mu+1}} \frac{1}{s(\mu+1)+1} \underset{\sim}{P}\{N_{s(\mu+1)+1} = s(\mu+1)\} . \end{aligned}$$

Finally, (27) and (28) prove (25). In formula (24) we can express $P_0(a+s-j,\ b-s;\ \mu)$ by (25).

By (24) we obtain the following relation for $N_j(a,b;\mu)$ defined by (22):

$$N_j(a,b;\mu) = \sum_{0 \leq s < \frac{j}{\mu+1}} N_0(a+s-j,\ b-s) N_j(j-s,s) \,. \tag{33}$$

In some particular cases formula (24) in Theorem 4 can be simplified. The following theorem contains some particular cases of (24).

<u>Theorem 5</u>. <u>Let</u> μ <u>be a positive integer</u>. <u>If</u> $a > \mu b+1$, <u>then</u>

$$P_j(a,b;\mu) = \frac{1}{\binom{a+b}{a}} \sum_{\frac{a+b-j}{\mu+1} \leq s \leq b} \frac{(a-b\mu-1)}{s(b-s)} \binom{s\mu+s}{s-1} \binom{a+b-s\mu-s-2}{b-s-1} \tag{34}$$

<u>for</u> $j = 0,1,\ldots, a+b-1$, <u>and</u> $P_{a+b}(a,b;\mu) = (a-b\mu)/(a+b)$.

<u>If</u> $a = \mu b+1$, <u>then</u>

$$P_j(a,b;\mu) = \frac{1}{a+b} \tag{35}$$

<u>for</u> $j = 1,2,\ldots, a+b$.

<u>If</u> $a = \mu b$, <u>then</u>

$$P_0(a,b;\mu) = 1 - \frac{1}{\binom{a+b}{a}} \sum_{0 \leq s \leq \frac{a+b-2}{\mu+1}} \frac{1}{s} \binom{s\mu+s}{s-1} \binom{a+b-s\mu-s-1}{b-s} \tag{36}$$

and

(37) $$P_j(a,b;\mu) = \frac{1}{\binom{a+b}{a}} \sum_{0 \leq s \leq \frac{a+b-j-1}{\mu+1}} \frac{1}{s(b-s)} \binom{s\mu+s}{s-1}\binom{a+b-s\mu-s-2}{b-s-1}$$

for $j = 1,2,\ldots, a+b-1$.

Proof. If we apply Theorem 26.1 to the random variables $\nu_1, \nu_2,\ldots, \nu_{a+b}$ and if we use (27) then (34), (35), (36) and (37) follow immediately.

Formulas (34) and (35) were proved in 1963 by the author [193], and formula (36) in 1964 also by the author [194].

Now let us determine the probabilities $Q_j(a,b;\mu)$ for $j = 0,1,\ldots, a+b$. We can determine $Q_j(a,b;\mu)$ in a similar way as $P_j(a,b;\mu)$. See the author's note in [194] . The details can be found in [63] .

Theorem 6. If μ is a nonnegative integer, then we have

(38) $$Q_j(a,b;\mu) = \sum_{0 \leq s \leq \frac{j}{\mu+1}} \frac{\binom{j}{s}\binom{a+b-j}{b-s}}{\binom{a+b}{a}} Q_0(a+s-j,b-s;\mu)Q_j(j-s,s;\mu)$$

for $j = 0,1,\ldots, a+b$ where

(39) $$Q_0(a,b;\mu) = \frac{b}{a+b} - \frac{1}{\binom{a+b}{a}} \sum_{\frac{2}{\mu+1} \leq s \leq \frac{a+b}{\mu+1}} \frac{1}{(\mu s+s-1)} \binom{s\mu+s-1}{s}\binom{a+b-s\mu-s}{b-s}$$

for $a < b\mu$ and $Q_0(a,b;\mu) = 0$ for $a \geq b\mu$, ($Q_0(0,0;\mu) = 1$) , and

$$(40) \qquad Q_{a+b}(a,b;\mu) = 1 - \frac{1}{\binom{a+b}{a}} \sum_{\frac{2}{\mu+1} \leq s \leq \frac{a+b}{\mu+1}} \frac{(a+1-b\mu)}{(a+b-s\mu-s+1)} \binom{s\mu+s-1}{s} \binom{a+b-s\mu-s-1}{b-s}$$

__for__ $a \geq b\mu$ __and__ $Q_{a+b}(a,b;\mu) = 0$ __for__ $a < b\mu$.

__Proof.__ Define the random variables $\nu_1, \nu_2, \ldots, \nu_{a+b}$ in exactly the same way as in the proof of Theorem 4. Then we have

$$(41) \qquad \underset{\sim}{P}\{N_i = s(\mu+1)\} = \frac{\binom{i}{s}\binom{a+b-i}{b-s}}{\binom{a+b}{a}}$$

for $1 \leq i \leq a+b$ and

$$(42) \qquad \underset{\sim}{P}\{N_i = s(\mu+1),\ N_k = t(\mu+1)\} = \frac{\binom{i}{s}\binom{k-i}{t-s}\binom{a+b-k}{b-t}}{\binom{a+b}{a}}$$

for $1 \leq i \leq k \leq a+b$.

By using this notation we can write that

$$(43) \qquad Q_j(a,b;\mu) = \underset{\sim}{P}\{N_r \leq r \text{ for } j \text{ subscripts } r = 1,2,\ldots, a+b\}$$

for $j = 0,1,\ldots, a+b$. Since $N_r = \beta_r(\mu+1)$ and $r = \alpha_r + \beta_r$ for $r = 1,2,\ldots, a+b$, it follows that the inequality $\alpha_r \geq \mu\beta_r$ holds if and only if $N_r \leq r$. This proves (43).

By Theorem 22.1 we can conclude that the probability that $N_r \leq r$ for j subscripts $r = 1,2,\ldots, a+b$ is the same as the probability that the last maximal element in the sequence $r-N_r$ $(r = 0,1,\ldots, a+b)$ is $j-N_j$. Accordingly, it follows that

$$Q_j(a,b,;\mu) = \underset{\sim}{P}\{r-N_r \leq j-N_j \quad \text{for} \quad 0 \leq r \leq j \quad \text{and} \quad r-N_r < j-N_j \quad \text{for} \quad j<r\leq a+b\} =$$

(44)

$$= \sum_{s=0}^{b} \underset{\sim}{P}\{N_j = s(\mu+1)\}\underset{\sim}{P}\{N_j - N_r \leq j-r \quad \text{for} \quad 0 \leq r \leq j \mid N_j = s(\mu+1)\} \cdot$$

$$\cdot \underset{\sim}{P}\{r-j < N_r - N_j \quad \text{for} \quad j < r \leq a+b \mid N_j = s(\mu+1)\} .$$

By using the representation (43) we can write (44) in the following equivalent form

$$(45) \qquad Q_j(a,b;\mu) = \sum_{s=0}^{b} \underset{\sim}{P}\{N_j = s(\mu+1)\}Q_j(j-s,s;\mu)Q_0(a+s-j,b-s;\mu)$$

where $j = 0,1,\ldots, a+b$. By (41) this proves (38).

If we prove (39) and (40), then $Q_j(a,b\,;\mu)$ is completely determined by (38).

If $j = a+b$ in (43), then by Theorem 20.1 it follows that

$$Q_{a+b}(a,b;\mu) = \underset{\sim}{P}\{N_r \leq r \quad \text{for} \quad r = 1,2,\ldots, a+b\} =$$

(46)

$$= 1- \sum_{\ell=1}^{a+b-1} \frac{(a+1-b\mu)}{(a+b-\ell)} \underset{\sim}{P}\{N_\ell = \ell+1\}$$

for $a \geq b\mu$ and $Q_{a+b}(a,b;\mu) = 0$ for $a < b\mu$. By (41) this proves (39).

If $j = 0$ in (43), then by Theorem 26.4 it follows that

$$Q_0(a,b\,;\mu) = \underset{\sim}{P}\{N_r > r \quad \text{for} \quad r = 1,2,\ldots, a+b\} =$$

(47)

$$= \underset{\sim}{P}\{N_1 > 1\} - \sum_{i=2}^{a+b} \frac{1}{(i-1)} \underset{\sim}{P}\{N_1 = 0 \quad \text{and} \quad N_i = i\} .$$

If $a \geq b\mu$, then obviously $Q_0(a,b\,;\mu) = 0$. By (42) this proves (40).

By (38) we obtain the following relation for $M_j(a,b;\mu)$ defined by (23):

$$(48) \qquad M_j(a,b;\mu) = \sum_{0 \leq s \leq \frac{j}{\mu+1}} M_0(a+s-j,\ b-s)M_j(j-s,s) \ .$$

In some particular cases formula (38) in Theorem 6 can be simplified. The following theorem contains some particular cases of (38).

Theorem 7. *Let* μ *be a positive integer. If* $a \geq \mu b$, *then* $Q_j(a,b;\mu) = 0$ *for* $j \leq a-b\mu$,

$$(49) \qquad Q_j(a,b;\mu) = \frac{1}{\binom{a+b}{a}} \sum_{\frac{a+b+1-j}{\mu+1} \leq s \leq b} \frac{(a+1-b\mu)}{(s\mu+s-1)(a+b-\mu s-s+1)} \binom{s\mu+s}{s}\binom{a+b-s\mu-s}{b-s}$$

for $a-b\mu < j < a+b$, *and*

$$(50) \qquad Q_{a+b}(a,b;\mu) = \frac{a+1-b\mu}{a+1} \ .$$

If $a = b\mu-1$, *then*

$$(51) \qquad Q_j(a,b;\mu) = \frac{1}{a+b}$$

for $j = 0,1,\ldots, a+b-1$ *and* $Q_{a+b}(a,b;\mu) = 0$.

Proof. If we apply Theorem 26.3 to the random variables $\nu_1,\nu_2,\ldots,\nu_{a+b}$ and if we use (41), then (49) and (51) follow immediately. By Theorem 26.3 we obtain that

$$(52) \qquad Q_{a+b}(a,b;\mu) = 1 - \frac{1}{\binom{a+b}{a}} \sum_{\frac{2}{\mu+1} \leq s \leq b} \frac{(a+1-b\mu)}{(a+b-s\mu-s+1)} \binom{s\mu+s}{s} \binom{a+b-s\mu-s}{b-s}$$

for $a \geq \mu b$. On the other hand by (13) $Q_{a+b}(a,b;\mu) = Q(a,b;\mu) = (a+1-b\mu)/(a+1)$ which is an interesting identity.

For the proofs of (49), (50) and (51) we also refer to O. Engelberg [167] and the author [193], [63] .

Finally, we shall consider the problem of finding the distribution $\{P_j(a,b;\mu)\}$ in the case when $\mu = a/b$. Obviously $P_{a+b}(a,b;\mu) = 0$. If we know $\{P_j(a,b;\mu)\}$, then $\{Q_j(a,b;\mu)\}$ can be obtained immediately by using the following relation

$$(53) \qquad Q_j(a,b;\mu) = P_{a+b-j}(a,b;\mu)$$

which holds for $j = 0,1,\ldots, a+b$ and $\mu = a/b$.

The identity (53) follows simply by symmetry. For if $\mu = a/b$, then we can interpret $P_j(a,b;\mu)$ also as the probability that $\alpha_r < a\beta_r/b$ holds for exactly j subscripts $r = 1,2,\ldots, a+b$. Accordingly, if $Q_j(a,b;\mu)$ (where $\mu = a/b$) denotes the probability that the inequality $\alpha_r \geq a\beta_r/b$ holds for exactly j subscripts $r = 1,2,\ldots, a+b$, then we have $Q_j(a,b;\mu) = P_{a+b-j}(a,b;\mu)$ whenever $\mu = a/b$.

In the particular case when a and b are relatively prime integers and $\mu = a/b$ the problem has a simple solution which was found in 1954 by H. D. Grossman [177] . His result is the following:

<u>Theorem 8. If</u> a <u>and</u> b <u>are relatively prime positive integers, and</u> $\mu = a/b$, <u>then</u>

$$(54) \qquad P_j(a,b;\mu) = \frac{1}{a+b}$$

<u>for</u> $j = 0,1,\ldots, a+b-1$.

<u>Proof</u>. We shall prove that $Q_j(a,b;\mu) = 1/(a+b)$ for $j = 1,2,\ldots, a+b$ which is equivalent to (54). In proving this statement we shall use Lemma 26.2 . Since $(a,b) = 1$, we can choose two positive integers p and q such that $ap-bq = 1$. Define the random variables $\gamma_1, \gamma_2,\ldots, \gamma_{a+b}$ as follows: $\gamma_r = q$ if the r-th vote is cast for A , and $\gamma_r = -p$ if the r-th vote is cast for B . Then $\gamma_1, \gamma_2,\ldots, \gamma_{a+b}$ are interchangeable random variables taking on integers only and satisfying the condition $\gamma_1+\ldots+ \gamma_{a+b} = 1$. Now $\alpha_r \geqq a\beta_r/b$ holds if and only if $\alpha_r > p\beta_r/q$, or equivalently, $\gamma_1+\ldots+ \gamma_r > 0$. Accordingly

$$(55) \qquad Q_j(a,b;\mu) = \underset{\sim}{P}\{\gamma_1+\ldots+ \gamma_r > 0 \text{ for } j \text{ subscripts } r = 1,2,\ldots, a+b\}.$$

By Lemma 26.2 it follows that $Q_j(a,b;\mu) = 1/(a+b)$ for $j = 1,2,\ldots, a+b$. This completes the proof of (54). (See also the author [192], [63].)

If $\mu = a/b$ and a and b are not relatively prime, then it is more complicated to find $\{P_j(a,b;\mu)\}$. The degree of difficulty increases accordi to the magnitude of (a,b) .

Let us mention briefly the historical background for finding $P_j(a,b;\mu)$ in the case when $\mu = a/b$. In 1950 <u>H. D. Grossman</u> [176]

published a conjecture concerning $P_0(a,b;\mu)$. Grossman's formula was proved in 1954 by M. T. L. Bizley [164] . In the same paper Bizley found also $P_{a+b-1}(a,b;\mu)$. In 1966 M. T. L. Bizley [165] made a conjecture concerning the gneral form of $P_j(a,b;\mu)$ for $j = 0,1,\ldots, a+b$. (See also the author [63].) Bizley's formula for $P_j(a,b;\mu)$ was proved in 1969 by the author [195].

In what follows we suppose that

$$\mu = \frac{m}{n} \tag{56}$$

where m and n are relatively prime positive integers and we shall find the probabilities

$$P_j(km,\ kn;\ \mu) \tag{57}$$

for $k = 1,2,3,\ldots$.

Throughout the remaining of this section we assume that m and n are fixed relatively prime positive integers and k varies through the set of positive integers.

Let us write

$$P_j(km,\ kn\ ;\ \mu) = \frac{N_j(km,\ kn\ ;\ \mu)}{\binom{km+kn}{km}} \tag{58}$$

for $j = 0,1,\ldots, a+b$, and $k = 1,2,\ldots,$ and let $N_0(0,0;\mu) = 1$.

Theorem 9. *If* $\mu = m/n$ *where* m *and* n *are relatively prime positive integers, then we have*

$$(59)\quad N_j(km,kn;\mu) = \sum_{0 \leq s < k - \frac{j}{m+n}} N_0(sm,sn;\mu) N_{(k-s)(m+n)-1}((k-s)m,(k-s)n;\mu)$$

for $j = 0,1,\ldots, k(m+n)-1$. *Here* $N_0(0,0;\mu) = 1$.

Proof. Let us define the random variables ξ_r $(r = 1,2,\ldots, k(m+n))$ in the following way: Let $\xi_r = n$ if the r-th vote is cast for A and $\xi_r = -m$ if the r-th vote is cast for B . The random variables ξ_r $(r = 1,2,\ldots, k(m+n))$ are interchangeable and their sum is 0 . Let $\zeta_r = \xi_1+\ldots+\xi_r$ for $r = 1,2,\ldots, k(m+n)$ and $\zeta_0 = 0$. By using this notation we can write that

$$(60)\quad P_j(km,kn;\mu) = \underset{\sim}{P}\{\zeta_r > 0 \text{ for } j \text{ subscripts } r = 1,2,\ldots, k(m+n)\}$$

for $j = 0,1,\ldots, k(m+n)$. Evidently $P_j(km,kn;\mu) = 0$ if $j = k(m+n)$.

By Theorem 22.1 we can conclude that the probability that $\zeta_r > 0$ holds for exactly j subscripts $r = 1,2,\ldots, k(m+n)$ is the same as the probability that the first maximal element in the sequence ζ_r $(r = 0,1,\ldots, k(m+n))$ is ζ_j . Accordingly, it follows that

$$(61)\quad P_j(km,kn;\mu) = \underset{\sim}{P}\{\zeta_r < \zeta_j \text{ for } 0 \leq r < j \text{ and } \zeta_r \leq \zeta_j \text{ for } j \leq r \leq k(m+n)\}$$

for $j = 0,1,\ldots, k(m+n)$. If $j = k(m+n)$, then (61) is 0 .

For $j = 0,1,\ldots, k(m+n)-1$ we can write that

(62) $$P_j(km,kn;\mu) = \underset{\sim}{P}\{\zeta_i \leqq 0 \text{ for } 0 \leqq i < k(m+n)-j \text{ and } \zeta_i < 0 \text{ for } k(m+n)-j \leqq i < k(m+n)\} .$$

If $j = 0$, then (62) is precisely (61). If $0 < j < k(m+n)$ and in (61) we replace the random variables $\xi_1, \ldots, \xi_j, \xi_{j+1}, \ldots, \xi_{k(m+n)}$ by $\xi_{k(m+n)+1-j}, \ldots, \xi_{k(m+n)}, \xi_1, \ldots, \xi_{k(m+n)-j}$ respectively, then $P_j(km,kn;\mu)$ remains unchanged, and the right-hand side becomes (62). This proves (62) for $j = 0,1,\ldots, k(m+n)-1$.

If $j = 0,1,\ldots, k(m+n)-1$ and the event on the right-hand side of (62) occurs, then there is an i $(0 \leqq i < k(m+n)-j)$ such that $\zeta_i = 0$. Denote by r the largest such i. Then necessarily $r = (m+n)s$ where $0 \leqq s < (km+kn-j)/(m+n)$ and furthermore $\zeta_i < 0$ for $s(m+n) < i < k(m+n)$. Accordingly,

(63) $$\begin{aligned} P_j(km,kn;\mu) &= \sum_{0 \leqq s < k - \frac{j}{m+n}} \underset{\sim}{P}\{\zeta_i \leqq 0 \text{ for } 0 \leqq i \leqq s(m+n), \zeta_{s(m+n)} = 0, \zeta_i < 0 \text{ for } s(m+n) < i < k(m+n)\} \\ &= \sum_{0 \leqq s < k - \frac{j}{m+n}} \underset{\sim}{P}\{\zeta_{s(m+n)} = 0\} \underset{\sim}{P}\{\zeta_i \leqq 0 \text{ for } 0 \leqq i \leqq s(m+n) \mid \zeta_{s(m+n)} = 0\} \cdot \\ &\quad \cdot \underset{\sim}{P}\{\zeta_i < 0 \text{ for } s(m+n) < i < k(m+n) \mid \zeta_{s(m+n)} = 0\} \end{aligned}$$

for $j = 0,1,\ldots, k(m+n)-1$. By using the representation (60) we can write (63) in the following equivalent form:

(64) $$\begin{aligned} P_j(km,kn;\mu) = \sum_{0 \leqq s < k - \frac{j}{m+n}} \underset{\sim}{P}\{\zeta_{s(m+n)} = 0\} \cdot P_0(sm,sn;\mu) \cdot \\ \cdot P_{(k-s)(m+n)-1}((k-s)m,(k-s)n;\mu) \end{aligned}$$

for $j = 0,1,\ldots, k(m+n)-1$ where $P_0(0,0;\mu) = 1$. In (64)

(65) $$\underset{\sim}{P}\{\zeta_{s(m+n)} = 0\} = \frac{\binom{sm+sn}{sm}\binom{(k-s)m+(k-s)n}{(k-s)m}}{\binom{km+kn}{km}},$$

and if we multiply (64) by $\binom{km+kn}{km}$ and if we use the notation (58), then we obtain that

(66) $$N_j(km,kn;\mu) = \sum_{0\leq s<k-\frac{j}{m+1}} N_0(sm,sn;\mu)N_{(k-s)(m+n)-1}((k-s)m,(k-s)n;\mu)$$

for $j = 0,1,\ldots, k(m+n)-1$ where $N_0(0,0;\mu) = 1$. This completes the proof of Theorem 9.

For fixed positive integers m and n let us introduce the abbreviation

(67) $$C_k = \binom{km+kn}{km}$$

for $k = 1,2,\ldots$. The generating function

(68) $$C(z) = \sum_{k=1}^{\infty} \binom{km+kn}{km} \frac{z^k}{k(m+n)}$$

is convergent if $|z| \leq \rho = m^m n^n/(m+n)^{m+n}$.

<u>Theorem 10.</u> <u>If</u> $\mu = m/n$ <u>where</u> m <u>and</u> n <u>are relatively prime positive integers, and</u> $k = 1,2,\ldots,$ <u>then we have</u>

(69) $$P_j(km,kn;\mu) = \frac{1}{\binom{km+kn}{km}} \sum_{0\leq s<k-\frac{j}{m+n}} U_s V_{k-s}$$

for $j = 0,1,\ldots, k(m+n)-1$ *and* $P_{k(m+n)}(km,kn;\mu) = 0$. *Here* U_k $(k = 0,1,\ldots)$ *and* V_k $(k = 1,2,\ldots)$ *are given by the generating functions*

$$(70)\qquad U(z) = \sum_{k=0}^{\infty} U_k z^k = e^{C(z)}$$

and

$$(71)\qquad V(z) = \sum_{k=1}^{\infty} V_k z^k = 1-e^{-C(z)}$$

which are convergent for $|z| \leq \rho$.

Proof. For fixed m and n let us introduce the notation

$$(72)\qquad U_k = N_0(km,kn;\mu)$$

for $k = 1,2,\ldots$ and let $U_0 = N_0(0,0;\mu) = 1$. Furthermore, let

$$(73)\qquad V_k = N_{k(m+n)-1}(km,kn;\mu)$$

for $k = 1,2,\ldots$. Then (59) can be expressed as

$$(74)\qquad N_j(km,kn;\mu) = \sum_{0 \leq s < k - \frac{j}{m+n}} U_s V_{k-s}$$

for $j = 0,1,\ldots, k(m+n)-1$.

If we add (74) for $j = 0,1,\ldots, k(m+n)-1$, then on the left-hand side we get the total number of voting records, $\binom{km+kn}{km} = k(m+n)C_k$, and therefore

$$(75)\qquad kC_k = \sum_{s=0}^{k-1} (k-s)U_s V_{k-s}$$

for $k = 1,2,\ldots$. If $j = 0$ in (74), then we get

$$U_k = \sum_{s=0}^{k-1} U_s V_{k-s} \tag{76}$$

for $k = 1,2,\ldots$.

If we form the generating functions of (75) and (76), then we obtain that

$$C'(z) = U(z)V'(z) \tag{77}$$

and

$$U(z)-1 = U(z)V(z) \tag{78}$$

for $|z| < \rho$. The generating functions are convergent for $|z| < \rho$ because evidently $U_k \leqq kC_k$ and $V_k \leqq C_k$ for $k = 1,2,\ldots$. By (77) and (78) $U'(z) = U'(z)V(z) + U(z)V'(z) = U'(z)V(z) + C'(z)$, that is, $C'(z) = = U'(z)[1-V(z)]$. Hence $U(z)C'(z) = U'(z)U(z) \cdot [1-V(z)] = U'(z)$, that is,

$$U'(z) = C'(z)U(z) \tag{79}$$

for $|z| < \rho$ and by definition $U(0) = 1$. The solution of this differential equation is given by

$$U(z) = e^{C(z)} \tag{80}$$

for $|z| \leqq \rho$. Consequently, by (78)

$$V(z) = 1-e^{-C(z)} \tag{81}$$

for $|z| \leqq \rho$.

If we divide (74) by $\binom{km+kn}{km}$, then we obtain $P_j(km,kn;\mu)$ for

$j = 0,1,\ldots, k(m+n)-1$. This completes the proof of the theorem.

Our next aim is to find explicit formulas for U_k $(k = 0,1,2,\ldots)$ and V_k $(k = 1,2,\ldots)$. Then $P_j(km,kn;\mu)$ can be calculated explicitly for $j = 0,1,\ldots, k(m+n)-1$ by (69).

If we form the power series expansions of (70) and (71), then we obtain at once that

$$(82) \qquad U_k = \sum_{i_1+2i_2+\ldots+ki_k=k} \frac{C_1^{i_1} C_2^{i_2} \cdots C_k^{i_k}}{i_1!\, i_2! \cdots i_k!}$$

and

$$(83) \qquad V_k = \sum_{i_1+2i_2+\ldots+ki_k=k} (-1)^{i_1+i_2+\ldots+i_k-1} \frac{C_1^{i_1} C_2^{i_2} \cdots C_k^{i_k}}{i_1!\, i_2! \cdots i_k!}$$

for $k = 1,2,\ldots$.

These formulas have the disadvantage that if k increases, the number of terms in the sums increases tremendously.

In what follows we shall give another method of finding U_k and V_k for $k = 1,2,\ldots$. This method can be applied equally for small and large k values.

<u>Theorem 11.</u> <u>If</u> $|z| \leq \rho$, <u>then we have</u>

$$(84) \qquad U(z) = \prod_{r=1}^{n} \gamma(\varepsilon_r z^{1/n})$$

<u>and</u>

(85) $$V(z) = 1 - \prod_{r=1}^{n} [\gamma(\varepsilon_r z^{1/n})]^{-1}$$

where $\varepsilon_r = e^{2\pi ri/n}$ $(r = 1,2,\dots, n)$ are the n-th roots of unity and

(86) $$[\gamma(z)]^\alpha = 1 + \sum_{\ell=1}^{\infty} \binom{\frac{\ell(m+n)+\alpha}{n}}{\ell} \frac{\alpha z^\ell}{\ell(m+n)+\alpha}$$

for any α.

Proof. Let us consider the equation

(87) $$1 - w^n + zw^{m+n} = 0$$

and denote by $w = \gamma(z)$ that root of this equation for which $\gamma(0) = 1$. If $|z| \leq \rho$, then we can expand $[\gamma(z)]^\alpha$ into the power series (86). (See L. Euler [21] and G. Pólya [50].) From (86) it follows that

(88) $$\log\gamma(z) = \lim_{\alpha \to 0} \frac{[\gamma(z)]^\alpha - 1}{\alpha} = \sum_{\ell=1}^{\infty} \binom{\frac{\ell(m+n)}{n}}{\ell} \frac{z^\ell}{\ell(m+n)}$$

and

(89) $$\sum_{r=1}^{n} \log\gamma(\varepsilon_r z^{1/n}) = \sum_{k=1}^{\infty} \binom{km+kn}{km} \frac{z^k}{k(m+n)} = C(z)$$

for $|z| \leq \rho$. Accordingly,

(90) $$C(z) = \sum_{r=1}^{n} \log\gamma(\varepsilon_r z^{1/n})$$

for $|z| \leq \rho$ in formulas (70) and (71). Hence (84) and (85) follow. This completes the proof.

Note. By (86) it follows that

(91) $$S_\alpha(z) = \sum_{r=1}^{n} [\gamma(\varepsilon_r z^{1/n})]^\alpha = n + \sum_{k=1}^{\infty} \binom{km+kn+\frac{\alpha}{n}}{kn} \frac{\alpha z^k}{km+kn+\frac{\alpha}{n}}$$

for any α and $|z| \leq \rho$. If we apply Waring's formula, then we obtain immediately that

(92) $$\prod_{r=1}^{n} [\gamma(\varepsilon_r z^{1/n})]^\alpha =$$

$$= \sum_{i_1+2i_2+\ldots+ni_n=n} (-1)^{i_1+i_2+\ldots+i_n} \frac{[S_\alpha(z)]^{i_1}[S_{2\alpha}(z)]^{i_2}\ldots[S_{n\alpha}(z)]^{i_n}}{i_1!i_2!\ldots i_n!\, 2^{i_2}\, 3^{i_3}\ldots n^{i_n}}$$

where $i_r = 0,1,2,\ldots$. (See Problem 40.4.)

The generating functions $U(z)$ and $V(z)$ can easily be obtained from (92) by putting $\alpha = 1$ and $\alpha = -1$ in it.

Examples. If $n = 1$, then $U(z) = \gamma(z)$ and $V(z) = 1-[\gamma(z)]^{-1}$ and by (86) we obtain that

(93) $$U_k = \binom{k(m+1)+1}{k} \frac{1}{k(m+1)+1}$$

and

(94) $$V_k = \binom{k(m+1)-1}{k} \frac{1}{k(m+1)-1}$$

for $k = 1,2,\ldots$.

If $n = 2$, then $U(z) = \gamma(z^{\frac{1}{2}})\gamma(-z^{\frac{1}{2}})$ and $V(z) = 1-[\gamma(z^{\frac{1}{2}})\gamma(-z^{\frac{1}{2}})]^{-1}$ and by (86) or by (92) we obtain that

$$(95) \qquad U_k = \sum_{i=0}^{2k} \frac{(-1)^i}{[i(m+2)+1][(2k-i)(m+2)+1]} \binom{\frac{1}{2}i(m+2)+\frac{1}{2}}{i}\binom{\frac{1}{2}(2k-i)(m+2)+\frac{1}{2}}{2k-i}$$

and

$$(96) \qquad V_k = \sum_{i=0}^{2k} \frac{(-1)^{i+1}}{[i(m+2)-1][(2k-i)(m+2)-1]} \binom{\frac{1}{2}i(m+2)-\frac{1}{2}}{i}\binom{\frac{1}{2}(2k-i)(m+2)-\frac{1}{2}}{2k-i}$$

for $k = 1,2,\ldots$.

These formulas make it possible to find $P_j(km,kn;\mu)$ for $j = 0,1,\ldots, k(m+n)-1$ if $\mu = m/n$ and either $n = 1$ or $n = 2$, and m is an odd integer.

Finally, we mention briefly a result concerning the asymptotic behavior of $P_j(km,kn;\mu)$ as $k \to \infty$. Let us denote by $\Delta_k(m,n)$ the number of subscripts $r = 1,2,\ldots, k(m+n)$ for which $\alpha_r > m\beta_r/n$ if candidate A receives $a = km$ votes and candidate B receives $b = kn$ votes. If $0 \leqq x \leqq 1$, then we have

$$(97) \qquad \lim_{k\to\infty} \underset{\sim}{P}\left\{\frac{\Delta_k(m,n)}{k(m+n)} \leqq x\right\} = x .$$

For the proof of this result we refer to [195] .

39. Order Statistics. The objective of this section is to discuss various approaches for the solutions of two main problems in the theory of order statistics.

The first problem is as follows: We have m independent observations $x_1, x_2, \ldots, x_m$ on a random variable ξ. The distribution function $P\{\xi \leq x\}$ is unknown. It is to be decided whether or not the observations $x_1, x_2, \ldots, x_m$ are compatible with the hypothesis that $P\{\xi \leq x\} = F(x)$ where $F(x)$ is a specified distribution function.

Such problems arose in the middle of the nineteenth century in connection with the normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du . \tag{1}$$

The normal distribution has appeared in several instances in the theory of probability and achieved some kind of general recognition. First in 1733 A. De Moivre [18] showed that the normal distribution is a good approximation for the Bernoulli distribution. In 1782 P. S. Laplace [37] demonstrated the usefulness of the normal distribution in the theory of probability. In 1812 P. S. Laplace [39] showed that under suitable normalization the sums of mutually independent and identically distributed symmetric random variables taking on integer values have a limiting normal distribution. In 1809 C. F. Gauss [27] demonstrated that the distribution of errors of observation is normal if we assume that the arithmetic mean of the observations is the most probable result. (See also W. F. Bessel [10], R. L. Ellis [20], A. Cauchy [14], M. W. Crofton [16], and G. Pólya [52].) It should be noted that in 1808 R. Adrain [3] gave an obviously inadequate explanation for the appearance of the normal distribution in the theory of errors of observation. Thus the priority of R. Adrain over C. F. Gauss which C. Abbe [1] claims is unjustified. In 1846 A. Quetelet [312] analysed various statistical data and illustrated, for example, that the distribution of chest measurements of 5,738 Scotch soldiers and the distribution of heights of 100,000 French conscripts fit the normal distribution. In 1884 F. Galton [26] compared the distribution of several physical characteristics of a group of people with a normal distribution and found good agreement. There were, however, examples for asymmetric distributions which of course did not fit a normal distribution. In 1895 K. Pearson [305] classified asymmetric distributions and introduced six basic types which would fit many non-normal distributions. In 1903 J. C. Kapteyn [34] expressed his view that the normal distribution is exceptional and most of the observations have an asymmetric distribution.

These scientists used the method of moments in fitting distributions to empirical observations. First, they chose a suitable type of distribution which depends on a few parameters, and then they determined the unknown parameters by requiring that the same number of moments of the hypothetical distribution and of the observations be equal as the number of unknown parameters. The problem arises naturally to measure the accuracy of this approximation. To answer this problem in 1900 K. Pearson [306] invented the method of χ^2 which provided a solution for discrete distributions. For continuous distributions the method of χ^2 depends on the way in which the observations have been grouped. This defect necessitated the introduction of some general measure for the discrepancy between the empirical observations and the **hypothetical** distribution.

To introduce a measure between the empirical observations and the **hypothetical** distribution it is convenient to define the so-called empirical distribution function. If we have m observations $x_1, x_2, \ldots, x_m$, that is, a sample of size m, then let us define $F_m(x)$ as the number of observations $\leq x$ divided by m. The function $F_m(x)$ defined for $-\infty < x < \infty$ is the empirical distribution function of the sample $(x_1, x_2, \ldots, x_m)$. If $F(x)$ is the theoretical **(hypothetical)** distribution function, then for example

$$\alpha_m = m \int_{-\infty}^{\infty} [F_m(x) - F(x)]^2 dx \tag{2}$$

can be considered as an adequate measure of the discrepancy between $F_m(x)$ and $F(x)$. The smaller α_m, the better the agreement.

The distance (2) was introduced in 1928 by H. Cramér [224] for $F(x) = \Phi(x)$ and in 1931 by R. v. Mises [299 pp. 316-335] for any $F(x)$. We can use α_m defined by (2) in testing the hypothesis that $P\{\xi \leq x\} = F(x)$ where $F(x)$ is a specified distribution function. Knowing the sample $(x_1, x_2, \ldots, x_m)$ we can determine $F_m(x)$ for every x and we can calculate α_m by (2). If α_m is small, then we accept the hypothesis that $\underset{\sim}{P}\{\xi \leq x\} = F(x)$ and if α_n is large, then we reject the hypothesis that $\underset{\sim}{P}\{\xi \leq x\} = F(x)$. How small or how large should α_n be?

To answer the last question we should determine what kind of α_m-values would we obtain if the hypothesis $\underset{\sim}{P}\{\xi \leq x\} = F(x)$ would be correct. Knowing this we can compare the actually calculated α_m- value with the expected α_m- values and make a decision accordingly.

Thus we can proceed in the following way: Let us suppose that $\underset{\sim}{P}\{\xi \leq x\} = F(x)$ where $F(x)$ is the specified distribution function. Let us suppose that we make m independent observations on the random variable ξ. Then we obtain a sample $(\xi_1, \xi_2, \ldots, \xi_m)$ where $\xi_1, \xi_2, \ldots, \xi_m$ are mutually independent random variables having the same distribution function $F(x)$. In this case $F_m(x)$, the empirical distribution function, is a random variable for every x. By definition $F_m(x)$ is equal to the number of variables $\xi_1, \xi_2, \ldots, \xi_m$ less than or equal to x divided by m. We obtain easily that

$$(3) \qquad \underset{\sim}{P}\{F_m(x) = \frac{k}{m}\} = \binom{m}{k}[F(x)]^k[1-F(x)]^{m-k}$$

for $k = 0,1,\ldots,m$ and for every x. Then α_m defined by (2) is also a random variable. Knowing $F(x)$, we can determine the distribution of α_m and we can determine the expectation, the variance and the higher moments of α_m if they exist. If we have this information, then we can make probability statements about the magnitude of α_m . Thus for some ε values (possibly for every ε) where $0 < \varepsilon < 1$ we can find an $a_m = a_m(\varepsilon)$ such that

$$P\{\alpha_m \leqq a_m\} = 1-\varepsilon \ . \tag{4}$$

Then α_m will be $\leqq a_m$ with probability $1-\varepsilon$ and α_m will be $> a_m$ with probability ε .

For any possible ε $(0 < \varepsilon < 1)$ we can design a test in the following way: We observe α_n . If $\alpha_m \leqq a_m$, then we accept the hypothesis that $P\{\xi \leqq x\} = F(x)$, and if $\alpha_m > a_m$, then we reject the hypothesis that $P\{\xi \leqq x\} = F(x)$. If we perform the test on the level ε , then if the hypothesis is correct we accept the hypothesis with probability $1-\varepsilon$ and reject it with probability ε . The largest ε for which we accept the hypothesis can be used as a measure of degree of the goodness of fit.

Although even for a simple $F(x)$ it is complicated to find the distribution of α_m , the moments of α_m can usually be determined easily. Knowing the expectation, the variance, or the higher moments of α_m , we can find lower bounds for the probability $P\{\alpha_m \leqq x\}$ by using Chebyshev's inequality. Thus we can find good upper estimates for $a_m = a_m(\varepsilon)$ which can be used in the above mentioned test.

Let us consider some examples. If $F(x) = \Phi(x)$, the normal distribution function defined by (1), then we have

$$E\{\alpha_m\} = \frac{1}{\sqrt{\pi}} \tag{5}$$

(H. Cramér [224]), and

(6) $$\text{Var}\{\alpha_m\} = \frac{2}{3} + \frac{2}{\pi} - \frac{2\sqrt{3}}{\pi} - \frac{1}{m}\left(\frac{1}{3} + \frac{6}{\pi} - \frac{4\sqrt{3}}{\pi}\right)$$

(R. v. Mises [299], [301]).

If

(7) $$F(x) = \begin{cases} 0 & \text{for } x < 0 , \\ x & \text{for } 0 \leq x \leq 1 , \\ 1 & \text{for } x > 1 , \end{cases}$$

then we have

(8) $$E\{\alpha_m\} = \frac{1}{6}$$

and

(9) $$\text{Var}\{\alpha_m\} = \frac{1}{45} - \frac{1}{60m}$$

(R. v. Mises [301]).

If we want to apply the test described above, then for each $F(x)$ we should determine either exactly or at least approximately the distribution of α_n.

In 1933 A. N. Kolmogorov [283] introduced another distance

(10) $$\delta_m = \sup_{-\infty<x<\infty} |F_m(x) - F(x)|$$

which has many advantages over α_m.

First we mention that in 1933 V. Glivenko [259] proved that if $\xi_1, \xi_2, \ldots, \xi_m, \ldots$ is an infinite sequence of mutually independent and identically distributed random variables with distribution function $F(x)$ and $F_m(x)$ is the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots \xi_m)$, then

$$P\{\lim_{m\to\infty} \delta_m = 0\} = 1 , \tag{11}$$

and this result guarantees that $F(x)$ can be estimated arbitrarily closely by $F_m(x)$ if m is large enough.

Kolmogorov noticed that if $F(x)$ is a continuous distribution function, and the elements of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$ are mutually independent random variables having the same distribution function $F(x)$, then the distribution of the random variable δ_m does not depend on $F(x)$. The random variable δ_m is a so-called distribution-free statistic.

In 1933 A. N. Kolmogorov [283] proved that if $F(x)$ is a continuous distribution function, then

$$\lim_{m\to\infty} P\{\sqrt{m}\ \delta_m \leq z\} = K(z) \tag{12}$$

where

$$K(z) = \sum_{j=-\infty}^{\infty} (-1)^j e^{-2j^2z^2} \tag{13}$$

for $z > 0$ and $K(z) = 0$ for $z \leq 0$.

If $z > 0$, then we can write also that

$$(14) \qquad K(z) = \frac{\sqrt{2\pi}}{2z} \sum_{j=0}^{\infty} e^{-\frac{(2j+1)^2\pi^2}{8z^2}} .$$

We have

$$(15) \qquad M_r = \int_0^{\infty} z^r dK(z) = \frac{r\Gamma(r/2)}{2^{r/2}} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^r}$$

for $r = 1,2,\ldots$. In particular, $M_1 = \sqrt{\pi}\,\log 2/\sqrt{2}$ and $M_2 = \pi^2/12$.

For other proofs of (12) we refer to W. Feller [254], J. L. Doob [246], M. D. Donsker [245], and N. V. Smirnov [328].

In 1939 N. V. Smirnov [327], [330] published a table for the distribution function $K(z)$. Here we reproduce a few values of $K(z)$.

z	K(z)	z	K(z)	z	K(z)
0.5	0.036 055	1.2	0.887 750	1.9	0.998 536
0.6	0.135 718	1.3	0.931 908	2.0	0.999 329
0.7	0.288 765	1.4	0.960 318	2.1	0.999 705
0.8	0.455 857	1.5	0.977 782	2.2	0.999 874
0.9	0.607 270	1.6	0.988 048	2.3	0.999 949
1.0	0.730 000	1.7	0.993 823	2.4	0.999 980
1.1	0.822 282	1.8	0.996 932	2.5	0.999 992

In 1956 P. Schmid [322], [323] determined the limiting distribution of $\sqrt{m}\,\delta_m$ as $m \to \infty$ for an arbitrary $F(x)$. If $F(x)$ has discontinuities, then the limiting distribution of $\sqrt{m}\,\delta_m$ as $m \to \infty$ depends on the positions of the discontinuity points of $F(x)$ and on the magnitudes of the corresponding jumps.

In his paper of 1933 A. N. Kolmogorov [283] deduced a system of recurrence formulas which makes it possible to find the distribution of δ_m for finite m values whenever $F(x)$ is a continuous distribution function. By using these recurrence formulas in 1952 Z. W. Birnbaum [204] tabulted the probabilities $P\{\delta_m < k/m\}$ where $k = 1,2,\dots,m$ for $m \leq 100$ and $k \leq 15$. (See also F. J. Massey [296].)

For a continuous distribution function $F(x)$ and for $m = 1,2,\dots$ the distribution of δ_m was found in 1957 by J. H. B. Kemperman [277], in 1958 by J. Blackman [210], in 1962 by M. Depaix [243], and H. Carnal [212], in 1968 by V. A. Epanechnikov [253], and J. Durbin [248], and in 1971 by G.P. Steck [45 S. G. Mohanty [302], Z. Govindarajulu, R. Alter and L. E. Bragg [267], and K. Sarkadi [320]. By the result of V. A. Epanechnikov [253] for a continuous distribution function $F(x)$ we have

$$(16) \qquad P\{\delta_m \leq z\} = m! \sum_{\nu=0}^{m-1} (-1)^{m-\nu-1} \sum_{\substack{0=k_0<k_1<\dots<k_\nu<k_{\nu+1}=m \\ k_{i+1}-k_i \leq 2mz}} \prod_{i=0}^{\nu} \frac{(\beta_{k_i} - \alpha_{k_{i+1}})^{k_{i+1}-k_i}}{(k_{i+1}-k_i)!}$$

where $\alpha_k = \min\{[(k-mz)/m]^+, 1\}$ and $\beta_k = \min\{[(k+mz)/m]^+, 1\}$. Obviously $P\{\delta_m \leq z\} = 0$ if $z < 1/2m$ and $P\{\delta_m \leq z\} = 1$ if $z \geq 1$.

If we know the distribution function of δ_m, then we can use the same method as before for testing the hypothesis that $P\{\xi \leq x\} = F(x)$. Let us determine a $d_m = d_m(\varepsilon)$ for $0 < \varepsilon < 1$ such that $P\{\delta_m \leq d_m\} = 1-\varepsilon$.

in 1955 by V.S. Korolyuk [286], in 1956 by L.C. Chang [216], Then let us calculate the actual value of δ_m for the specified distribution function $F(x)$ by (10). If $\delta_m \leqq d_m$, then we accept the hypothesis that $\underset{\sim}{P}\{\xi \leqq x\} = F(x)$ on the level ε and if $\delta_m > d_m$, then we reject the hypothesis on the same level. The largest ε for which we accept the hypothesis can be considered as the measure of degree of the goodness of fit.

If $F(x)$ is a continuous distribution function and if m is large, then we can replace the probability $\underset{\sim}{P}\{\delta_m \leqq z\}$ by the approximate value $K(\sqrt{m}\, z)$ and in this way we can find $d_m = d_m(\varepsilon)$ approximately.

Next we shall mention briefly a method of finding the limiting distribution (12). If $F(x)$ is a continuous distribution function, then the distribution of δ_m defined by (10) does not depend on $F(x)$. Therefore in finding the distribution of δ_m we may assume without loss of generality that

$$(17) \qquad F(x) = \begin{cases} 0 & \text{for } x < 0 , \\ x & \text{for } 0 \leqq x \leqq 1 , \\ 1 & \text{for } x > 1 . \end{cases}$$

Then $F_m(x) = 0$ for $x \leqq 0$, $F_m(x) = 1$ for $x \geqq 1$ and $F_m(x)$ for $0 \leqq x \leqq 1$ can be obtained in the following way: Let us choose m points at random in the interval $(0,1)$ in such a way that each point independently of the others has a uniform distribution over the interval $(0,1)$. Denote by $\nu_m(x)$ the number of random points in the interval $(0,x]$ for $0 \leqq x \leqq 1$. Then $F_m(x) = \nu_m(x)/m$ for $0 \leqq x \leqq 1$, and we can write that

$$\delta_m = \max_{0 \leqq x \leqq 1} \left| \frac{\nu_m(x)}{m} - x \right| . \tag{18}$$

Here $\{\nu_m(t) , 0 \leqq t \leqq 1\}$ is a stochastic process with interchangeable increments. If $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$ and $0 = j_0 \leqq j_1 \leqq \ldots \leqq j_k \leqq j_{k+1} = m$, then we have

$$\underset{\sim}{P}\{\nu_m(t_1) = j_1, \nu_m(t_2) = j_2, \ldots, \nu_m(t_k) = j_k\} = m! \prod_{i=0}^{m} \frac{(t_{i+1} - t_i)^{j_{i+1} - j_i}}{(j_{i+1} - j_i)!} . \tag{19}$$

If we write

$$\eta_m(t) = \frac{\nu_m(t) - mt}{\sqrt{m}} \tag{20}$$

for $0 \leqq t \leqq 1$, then by (19) we can easily prove that for $0 < t_1 < t_2 < \ldots < t_k < 1$, the random variables $\eta_m(t_1), \eta_m(t_2), \ldots, \eta_m(t_k)$ have a k-dimensional limiting normal distribution

$$N\left(\begin{Vmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Vmatrix} , \begin{Vmatrix} t_1(1-t_1), & t_1(1-t_2), & \ldots, & t_1(1-t_k) \\ t_1(1-t_2), & t_2(1-t_2), & \ldots, & t_2(1-t_k) \\ \cdot & \cdot & \cdots & \cdot \\ t_1(1-t_k), & t_2(1-t_k), & \ldots, & t_k(1-t_k) \end{Vmatrix} \right) \tag{21}$$

as $m \to \infty$.

If a stochastic process $\{\eta(t) , 0 \leqq t \leqq 1\}$ has the property that for any $k = 1,2,\ldots$ and $0 < t_1 < t_2 < \ldots < t_k < 1$ the random variables $\eta(t_1), \eta(t_2), \ldots, \eta(t_k)$ have a k-dimensional normal distribution, then we say that $\{\eta(t) , 0 \leqq t \leqq 1\}$ is a Gaussian process. Let us suppose that $\{\eta(t)\ ,\ 0 \leq t \leq 1\}$ is a Gaussian process for which $\eta(t_1),\eta(t_2),\ldots,\eta(t_k)$ have a k-dimensional normal distribution defined by (21) whenever $k = 1,2,\ldots$ and $0 < t_1 < t_2 < \ldots < t_k < 1$. Then $E\{\eta(t)\} = 0$ for $0 \leq t \leq 1$ and $\mathrm{Cov}\{\eta(u)\ ,\ \eta(t)\} = \min(u,t)-ut$ for $0 \leq u \leq 1$ and $0 \leq t \leq 1$.

Accordingly we can conclude, that if $m \to \infty$, then the finite dimensional distributions of the process $\{\eta_m(t)\ ,\ 0 \leq t \leq 1\}$ converge to the corresponding finite dimensional distributions of the process $\{\eta(t)\ ,\ 0 \leq t \leq 1\}$.

Since evidently

$$(22)\qquad P\{\sqrt{m}\,\delta_m \leq z\} = P\{\max_{0\leq t\leq 1} |\eta_m(t)| \leq z\}$$

for every z, we expect that

$$(23)\qquad \lim_{m\to\infty} P\{\sqrt{m}\,\delta_m \leq z\} = P\{\sup_{0\leq t\leq 1} |\eta(t)| \leq z\}$$

where $\{\eta(t)\ ,\ 0 \leq t \leq 1\}$ is a separable Gaussian process for which $E\{\eta(t)\} = 0$ and $\mathrm{Cov}\{\eta(u)\ ,\ \eta(t)\} = \min(u,t)-ut$ for $0 \leq u \leq 1$ and $0 \leq t \leq 1$. This is indeed true. The above method was suggested in 1949 by J. L. Doob [246] and was justified in 1952 by M. D. Donsker [245]. Accordingly, we have

$$(24)\qquad P\{\sup_{0\leq t\leq 1} |\eta(t)| \leq z\} = K(z)$$

where $K(z)$ is given by (13) or by (14).

We note that the process $\{\eta(t)\ ,\ 0 \leq t \leq 1\}$ can be represented in the following way

$$\eta(t) = (1-t)\xi\left(\frac{t}{1-t}\right) \tag{25}$$

where $\{\xi(t), 0 \leq t < \infty\}$ is a Brownian motion process for which $\underset{\sim}{E}\{\xi(t)\} = 0$ and $\underset{\sim}{\mathrm{Cov}}\{\xi(u), \xi(t)\} = \min(u,t)$ for $0 \leq u < \infty$ and $0 \leq t < \infty$. Thus we can write down also that

$$\underset{\sim}{P}\{|\xi(u)| \leq (1+u)z \text{ for } 0 \leq u < \infty\} = K(z) \tag{26}$$

where $\{\xi(u), 0 \leq u < \infty\}$ is a separable Brownian motion process for which $\underset{\sim}{E}\{\xi(u)\} = 0$ and $\underset{\sim}{\mathrm{Cov}}\{\xi(u), \xi(t)\} = \min(u,t)$ for $0 \leq u < \infty$ and $0 \leq t < \infty$.

The left hand side of (26) for $z > 0$ can be obtained by solving the diffusion equation (heat equation)

$$\frac{\partial f(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 f(t,x)}{\partial x^2} \tag{27}$$

for $t \geq 0$ and $|x| \leq (1+t)z$ with the boundary conditions $f(t,x) \to 0$ if $x \to \pm(1+t)z$ and $f(t,x) \to 0$ if $t \to 0$ and $x \neq 0$, and further $\int_{-z}^{z} f(t,x)dx \to 1$ if $t \to 0$. Then we obtain

$$K(z) = \lim_{t\to\infty} \int_{-(1+t)z}^{(1+t)z} f(t,x)dx \tag{28}$$

for $z > 0$. This method was actually used by A. N. Kolmogorov [283] in finding $K(z)$. See also A. N. Kolmogorov [35], [36].

More general statistics than δ_n have been considered by G. M. Maniya [293], T. W. Anderson and D. A. Darling [201], and A. Rényi [314]. These authors considered various particular cases of the following statistic

(29) $$\delta_m(h) = \sup_{-\infty<x<\infty} |F_m(x) - F(x)|h(F(x))$$

where $h(u)$ for $0 \leq u \leq 1$ is some preassigned weight function. If $h(u) \equiv 1$, then (29) reduces to Kolmogorov's statistic (10).

If $F(x)$ is a continuous distribution function, then the distribution of $\delta_m(h)$ does not depend on $F(x)$, that is, $\delta_m(h)$ is a distribution-free statistic. In this case the limiting distribution of $\sqrt{m}\,\delta_m(h)$ as $m \to \infty$ has been determined in several particular cases. In 1949 G. M. Maniya [293] and in 1952 T. W. Anderson and D. A. Darling [201] found the limiting distribution of $\sqrt{m}\,\delta_m(h)$ in the case when $h(u) = 1$ for $\alpha \leq u \leq \beta$ and $h(u) = 0$ for $0 \leq u < \alpha$ and $\beta < u \leq 1$ where $0 \leq \alpha < \beta \leq 1$. In the case when $h(u) = 1/\sqrt{u(1-u)}$ for $0 < \alpha \leq u \leq \beta < 1$ and $h(u) = 0$ otherwise, T. W. Anderson and D. A. Darling [201] found the Laplace-Stieltjes transform of the limiting distribution of $\sqrt{m}\,\delta_m(h)$ as $m \to \infty$. In 1953 A. Rényi [314] found the limiting distribution of $\sqrt{m}\,\delta_m(h)$ in the case when $h(u) = u$ for $0 < \alpha \leq u \leq \beta < 1$ and $h(u) = 0$ otherwise.

We can also generalize Kolmogorov's test in several other ways. First let us introduce two statistics

(30) $$\delta_m^+ = \sup_{-\infty<x<\infty} [F_m(x) - F(x)]$$

and

(31) $$\delta_m^- = \sup_{-\infty<x<\infty} [F(x) - F_m(x)] .$$

Obviously we have

(32) $$\delta_m = \max(\delta_m^+, \delta_m^-) .$$

If $F_m(x)$ is the empirical distribution function of a sample $(\xi_1, \xi_2,\ldots, \xi_m)$ where $\xi_1, \xi_2,\ldots, \xi_m$ are mutually independent random variables having the same distribution function $F(x)$, then δ_m^+ and δ_m^-, defined by (30) and (31) respectively, will be random variables. If we know the distributions of these random variables, then we can design various tests for checking the hypothesis that $P\{\xi \leq x\} = F(x)$ where $F(x)$ is a specified distribution function.

If $F(x)$ is a continuous distribution function, then the distributions of δ_m^+, δ_m^- and (δ_m^+, δ_m^-) do not depend on $F(x)$, that is, δ_m^+, δ_m^- and (δ_m^+, δ_m^-) are distribution-free statistics.

Let us suppose that $F(x)$ is a continuous distribution function. Then obviously δ_m^+ and δ_m^- have the same distribution for every $m = 1,2,\ldots$. In 1939 N. V. Smirnov [328] proved that

(33) $$\lim_{m\to\infty} P\{\sqrt{m}\,\delta_m^+ \leq z\} = \begin{cases} 1-e^{-2z^2} & \text{for } z \geq 0 , \\ 0 & \text{for } z < 0 . \end{cases}$$

As a generalization of Kolmogorov's limiting distribution (12) we obtain that for $x > 0$ and $y > 0$

(34) $$\lim_{m\to\infty} P\{\sqrt{m}\,\delta_m^+ \leq x , \sqrt{m}\,\delta_m^- \leq y\} = K(x,y)$$

where

$$(35) \qquad K(x,y) = \sum_{k=-\infty}^{\infty} \left(e^{-2k^2(x+y)^2} - e^{-2(k(x+y)+x)^2}\right)$$

or in another form

$$(36) \qquad K(x,y) = \frac{\sqrt{2\pi}}{x+y} \sum_{j=1}^{\infty} e^{-\frac{j^2\pi^2}{2(x+y)^2}} \sin^2 \frac{j\pi x}{x+y} .$$

(See J. L. Doob [246] and the author [334].) Obviously $K(x,x) = K(x)$ defined by (13) or by (14).

The joint distribution of δ_m^+ and δ_m^- for $m = 1,2,\ldots$ has been determined in 1968 by J. Durbin [248] .

Similarly to (29) we can introduce the statistics

$$(37) \qquad \delta_m^+(h) = \sup_{-\infty<x<\infty} [F_m(x) - F(x)]h(F(x))$$

and

$$(38) \qquad \delta_m^-(h) = \sup_{-\infty<x<\infty} [F(x) - F_m(x)]h(F(x))$$

where $h(u)$ for $0 \leq u \leq 1$ is some preassigned weight function. **Let $\delta_m(h) = \max(\delta_m^+(h), \delta_m^-(h))$.**

It can easily be seen that both $\delta_m^+(h)$ and $\delta_m^-(h)$ are distribution-free statistics, that is, if $F(x)$ is a continuous distribution function, then the distributions of $\delta_m^+(h)$, $\delta_m^-(g)$ and $(\delta_m^+(h) , \delta_m^-(g))$ for any $h(u)$ and $g(u)$ do not depend on $F(x)$. Obviously $\delta_m^+(h)$ and $\delta_m^-(g)$ have the same distribution if $h(u) = g(1-u)$ for $0 \leq u \leq 1$.

In the case where $F(x)$ is a continuous distribution function and $h(u) = 1$ for $\alpha \leq u \leq \beta$ $(0 \leq \alpha < \beta \leq 1)$ and $h(u) = 0$ otherwise, the limiting distributions of $m^{1/2}\,\delta_m^+(h)$ and $m^{1/2}\,\delta_m(h)$ as $m \to \infty$ were found in 1949 by G. M. Maniya [293]. These results have been generalized by I. I. Gikhman [431]. In the case where $F(x)$ is a continuous distribution function and $h(u) = 0$ for $\alpha \leq u \leq \beta$ $(0 \leq \alpha < \beta \leq 1)$ and $h(u) = 1$ otherwise, the limiting distribution of $m^{1/2}\,\delta_m^+(h)$ as $m \to \infty$ was found in 1952 by Kh. L. Berlyand and I. D. Kvit [408]. In the case where $h(u) = u$ for $\alpha \leq u \leq \beta$ $(0 < \alpha < \beta \leq 1)$ and $h(u) = 0$ otherwise the limiting distribution of $m^{1/2}\,\delta_m^+(h)$ as $m \to \infty$ was found in 1953 by A. Rényi [314]. See also S. Malmquist [292] who considered some more examples of this nature. The joint distribution of $\delta_m^+(h)$ and $\delta_m^-(g)$ was found in 1971 by G. P. Steck [459] for arbitrary functions $h(u)$ and $g(u)$. See also S. G. Mohanty [302], [449], K. Sarkadi [320] and E. J. G. Pitman [448].

Now let us consider another statistic which was introduced in 1939 by N. V. Smirnov [328]. Let $(\xi_1, \xi_2, \ldots, \xi_m)$ be a sample of m mutually independent and identically distributed random variables with distribution function $F(x)$. Denote by $\xi_1^*, \xi_2^*, \ldots, \xi_m^*$ the elements of the sample arranged in increasing order of magnitude. Let $F_m(x)$ be the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$. For any real a define $\sigma_m^*(a)$ as the number of integers $k = 1, 2, \ldots, m$ for which

(39) $$\frac{k-1}{m} \leq F(\xi_k^*) + \frac{a}{m} \leq \frac{k}{m} .$$

It is easy to see that if $F(x)$ is a continuous distribution function, then the distribution of $\sigma_m^*(a)$ does not depend on $F(x)$. Furthermore, it follows by symmetry that $\sigma_m^*(a)$ and $\sigma_m^*(-a)$ have the same distribution.

Obviously, we have

(40) $$\mathbf{P}\{\delta_m^+ < \frac{a}{m}\} = \mathbf{P}\{\delta_m^- < \frac{a}{m}\} = \mathbf{P}\{\sigma_m^*(a) = 0\}$$

and

(41) $$\underset{\sim}{P}\{\delta_m < \frac{a}{m}\} = \underset{\sim}{P}\{\sigma_m^*(a) + \sigma_m^*(-a) = 0\}$$

for $a > 0$.

N. V. Smirnov [328] proved that if $F(x)$ is a continuous distribution function, then

(42) $$\lim_{m\to\infty} \underset{\sim}{P}\{\sigma_m^*(z\sqrt{m}) \leq w\sqrt{m}\} = 1-e^{-\frac{(2z+w)^2}{2}}$$

for $z \geq 0$ and $w \geq 0$, and

(43) $$\lim_{m\to\infty} \underset{\sim}{P}\{\sigma_m^*(z\sqrt{m}) + \sigma_m^*(-z\sqrt{m}) \leq w\sqrt{m}\} = 1-2\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{d^j}{dw^j} (w^j e^{-\frac{[2(j+1)z+w]^2}{2}})$$

for $z > 0$ and $w \geq 0$. If we put $w = 0$ in the right-hand side of (43) then we obtain $K(z)$ given by (13).

At the beginning of this section we dealt with the statistic α_m defined by (2). We observed that the distribution of α_m depends on $F(x)$. To eliminate this disadvantage, in 1936 N. V. Smirnov [325], [326] introduced the following modification of α_m

(44) $$\omega_m^2 = m\int_{-\infty}^{\infty} [F_m(x) - F(x)]^2 g(F(x))dF(x)$$

where $g(u)$ for $0 \leq u \leq 1$ is some preassigned weight function.

It can easily be seen that if $F(x)$ is a continuous distribution function, then the distribution of ω_m^2 does not depend on $F(x)$. If we can determine the distribution of (44), then we can replace the statistic α_m by ω_m^2 in the test described after formula (2).

Now let us suppose that $F_m(x)$ is the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$ where $\xi_1, \xi_2, \ldots, \xi_m$ are mutually independent and identically distributed random variables with distribution function $F(x)$. Then ω_m^2 is a random variable and our objective is to find the limiting distribution of ω_m^2 in the case when $F(x)$ is a continuous distribution function.

Let us introduce the notation

(45) $$H(z) = \lim_{m\to\infty} \mathrm{P}\{\omega_m^2 \leq z\}$$

and

(46) $$\psi(s) = \int_0^\infty e^{-sz} dH(z)$$

for $\mathrm{Re}(s) \geq 0$.

In the particular case when $g(u) = 1$ for $0 \leq u \leq 1$, N. V. Smirnov [325], [326] proved that

(47) $$\int_0^\infty e^{-sz} dH(z) = \left(\frac{\sinh \sqrt{2s}}{\sqrt{2s}}\right)^{-\frac{1}{2}}$$

for $\mathrm{Re}(s) > 0$. If we introduce the stochastic process $\{\eta(t), 0 \leq t \leq 1\}$ defined after formula (23) and if we use a result of M. D. Donsker [245], then we can conclude that in this case

$$H(z) = \underset{\sim}{P}\{ \int_0^1 [\eta(t)]^2 dt \leq z\} \quad . \tag{48}$$

The stochastic process $\{\eta(t)\ ,\ 0 \leq t \leq 1\}$ can be represented in the following way

$$\eta(t) = \sqrt{2} \sum_{k=1}^{\infty} \xi_k \frac{\sin k\pi t}{k\pi} \tag{49}$$

where $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ are mutually independent random variables having the same normal distribution function $\Phi(x)$ defined by (1). In (49) the right-hand side converges almost everywhere and represents a random variable. By (48) and (49) we obtain that

$$H(z) = \underset{\sim}{P}\{ \sum_{k=1}^{\infty} \frac{\xi_k^2}{k^2\pi^2} \leq z\} \tag{50}$$

and thus

$$\int_0^{\infty} e^{-sz} dH(z) = \prod_{k=1}^{\infty} (1+ \frac{2s}{k^2\pi^2})^{-\frac{1}{2}} = (\frac{\sinh \sqrt{2s}}{\sqrt{2s}})^{-\frac{1}{2}} \tag{51}$$

where we used that

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty} (1- \frac{z^2}{k^2}) \tag{52}$$

for any z . By inversion N. V. Smirnov [326] found that

$$(53) \qquad H(z) = 1 - \frac{2}{\pi} \sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{2k\pi} \frac{e^{-\frac{zx^2}{2}}}{\sqrt{-x \sin x}} dx$$

for $z > 0$, and T. W. Anderson and D. A. Darling [201] proved that

$$(54) \qquad H(z) = \frac{1}{\pi\sqrt{z}} \sum_{j=0}^{\infty} (-1)^j \binom{-\frac{1}{2}}{j} (4j+1)^{\frac{1}{2}} e^{-\frac{(4j+1)^2}{16z}} K_{\frac{1}{4}}\left(\frac{(4j+1)^2}{16z}\right)$$

for $z > 0$ where

$$(55) \qquad K_\nu(z) = \frac{\pi[I_{-\nu}(z) - I_\nu(z)]}{2 \sin \nu\pi}$$

for $\nu \neq 0, \pm 1, \pm 2, \ldots$ and

$$(56) \qquad I_\nu(z) = \sum_{j=0}^{\infty} \frac{(z/2)^{2j+\nu}}{j!\,\Gamma(j+\nu+1)}$$

for $\nu \neq -1, -2, \ldots$ and $I_\nu(z) = I_{-\nu}(z)$ for $\nu = -1, -2, \ldots$. The function $I_\nu(z)$ is called the modified Bessel function of the first kind of order ν . The paper of T. W. Anderson and D. A. Darling [201] contains a table for the limiting distribution $H(z)$.

N. V. Smirnov [326] also showed that if $g(u)$ has a continuous derivative for $0 < u < 1$ and we define $H(z)$ again by (45), then

$$(57) \qquad \psi(s) = \int_0^{\infty} e^{-sz} dH(z) = \prod_{k=1}^{\infty} \left(1 + \frac{2s}{\lambda_k}\right)^{-1/2}$$

for $\mathrm{Re}(s) \geq 0$ where $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$ are the proper values of the integral equation

$$f(x) = \lambda \int_0^1 k(x,y)f(y)dy \tag{58}$$

for $0 \leq x \leq 1$ where

$$k(x,y) = \begin{cases} \sqrt{g(x)g(y)}\, x(1-y), & \text{for } 0 \leq x \leq y \leq 1, \\ \sqrt{g(x)g(y)}\, y(1-x) & \text{for } 0 \leq y \leq x \leq 1. \end{cases} \tag{59}$$

By inversion he obtained that

$$H(z) = 1 - \frac{1}{\pi} \sum_{k=1}^{\infty} \int_{\lambda_{2k-1}}^{\lambda_{2k}} \frac{e^{-\frac{zx}{2}}}{\sqrt{-D(x)}} \frac{dx}{x} \tag{60}$$

for $z > 0$ where $D(s)$ is the Fredholm determinant of the kernel $k(x,y)$ defined by (59).

We have

$$D(s) = \prod_{k=1}^{\infty} \left(1 - \frac{s}{\lambda_k}\right) \tag{61}$$

for all s.

It should be noted that if we write $y(x) = f(x)/\sqrt{g(x)}$, then the integral equation (58) is equivalent to the second order differential equation

$$\frac{d^2y}{dx^2} + \lambda g(x)y = 0 \tag{62}$$

with the boundary conditions $y(0) = y(1) = 0$. (See also R. v. Mises [300] and [301 pp. 482-490].)

Finally, we note that in 1949 M. Kac [276] introduced a statistic for the comparison of an empirial and a theoretical distribution function in the case of random sample sizes. See also the author [327] and J. L. Allen and J. A. Beekman [198], [199].

Now we turn to the discussion of the second main problem of order statistics.

The second problem is as follows: We have m independent observations $x_1, x_2, \ldots, x_m$ on a random variable ξ. The distribution function $P\{\xi \leq x\} = F(x)$ is unknown. Also we have n independent observations $y_1, y_2, \ldots, y_n$ on a random variable η. The distribution function $P\{\eta \leq x\} = G(x)$ is unknown. The two sets of observations are also independent. It is to be decided whether or not the observations $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ are compatible with the hypothesis that $F(x) \equiv G(x)$.

If we want to solve this problem in a mathematical way, it is convenient to introduce the empirical distribution functions of the samples $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$. Let us define $F_m(x)$ as the number of observations $x_1, x_2, \ldots, x_m$ less than or equal to x divided by m, and $G_n(x)$ as the number of observations $y_1, y_2, \ldots, y_n$ less than or equal to x divided by n. Next we introduce a measure for the discrepancy of the two empirical distribution functions. In 1939

N. V. Smirnov [327] suggested the following measure

(63) $$\delta_{m,n} = \sup_{-\infty<x<\infty} |F_m(x)-G_n(x)| \; .$$

If $\delta_{m,n}$ is small, then it is reasonable to accept the hypothesis that $F(x) = G(x)$. If $\delta_{m,n}$ is large, then it is justified to reject the hypothesis that $F(x) = G(x)$. However, it remains to be decided that for how small $\delta_{m,n}$ should we accept the hypothesis and for how large $\delta_{m,n}$ should we reject the hypothesis. To give an adequate answer to the last question let us proceed in the following way:

Let us suppose that $\mathbf{P}\{\xi \leq x\} = F(x)$ where $F(x)$ is a given distribution function and let us make m independent observations on the random variable ξ . Then we obtain a sample $(\xi_1, \xi_2, \ldots, \xi_m)$ where $\xi_1, \xi_2, \ldots, \xi_m$ are mutually independent random variables having the same distribution function $F(x)$. Let us suppose also that $\mathbf{P}\{\eta \leq x\} = G(x)$ where $G(x)$ is a given distribution function and let us make n independent observations on the random variable η . Then we obtain a sample $(\eta_1, \eta_2, \ldots, \eta_n)$ where $\eta_1, \eta_2, \ldots, \eta_n$ are mutually independent random variables having the same distribution function $G(x)$. In this case $F_m(x)$, the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$, and $G_n(x)$, the empirical distribution function of the sample $(\eta_1, \eta_2, \ldots, \eta_n)$, are random variables for every x . By definition $F_m(x)$ is equal to the number of variables $\xi_1, \xi_2, \ldots, \xi_m$ less than or equal to x divided by m and $G_n(x)$ is equal to the number of variables $\eta_1, \eta_2, \ldots, \eta_n$ less than or equal to x divided by n . Let us suppose also that the two samples $(\xi_1, \xi_2, \ldots, \xi_m)$ and $(\eta_1, \eta_2, \ldots, \eta_n)$ are independent. Then $\{F_m(x)\}$ and $\{G_n(x)\}$ are independent and $\delta_{m,n}$ defined by (63) is a random variable whose distribution is completely determined by $F(x)$ and $G(x)$.

In principle we can determine the distribution of $\delta_{m,n}$ in the case when $F(x) \equiv G(x)$ is a given distribution function, or when it belongs to a class of distribution functions. If we know the distribution of $\delta_{m,n}$ in the case when $F(x) \equiv G(x)$, then we can decide that an actually calculated value of $\delta_{m,n}$ is compatible with this distribution or not and we can make our decision accordingly.

N. V. Smirnov [327] noticed that if $F(x) \equiv G(x)$ is a continuous distribution function, then the distribution of $\delta_{m,n}$ does not depend on $F(x) \equiv G(x)$. This observation makes it possible to give a simple method for testing the hypothesis that ξ and η have the same continuous distribution function. If we know the distribution of $\delta_{m,n}$, then for some ε values $(0 < \varepsilon < 1)$ we can find a $d_{m,n} = d_{m,n}(\varepsilon)$ such that

$$\mathop{P}\{\delta_{m,n} \leq d_{m,n}\} = 1-\varepsilon \ . \tag{64}$$

Then for any possible ε we can design a test in the following way: We observe $\delta_{m,n}$. If $\delta_{m,n} \leq d_{m,n}$, then we accept the hypothesis that ξ and η have the same continuous distribution function, and if $\delta_{m,n} > d_{m,n}$, then we reject the hypothesis. If we perform the test on the level ε , then if the hypothesis is correct we accept it with probability $1-\varepsilon$ and reject it with probability ε .

There are several other possible statistics which have the same property as (63), namely, if $F(x) \equiv G(x)$ is a continuous distribution function, then the distribution of the statistic does not depend on $F(x) \equiv G(x)$. In what follows we shall consider exclusively such statistics and our aim is to find the distributions of such statistics in the case when $F(x) \equiv G(x)$ is a continuous distribution function. If we know this distribution, then we can use the test mentioned above in checking the hypothesis that ξ and η have the same continuous distribution function.

In 1939 N. V. Smirnov [327] proved that if $F(x) \equiv G(x)$ is a continuous distribution function, then

(65) $$\lim_{\substack{m \to \infty \\ n \to \infty}} \mathrm{P}\left\{\sqrt{\frac{mn}{m+n}}\ \delta_{m,n} \leq z\right\} = K(z)$$

where $K(z)$ is defined by (13) or by (14).

If m and n are large, then we can use the approximation $\mathrm{P}\{\delta_{m,n} \leq d_{m,n}\} \sim K\left(d_{m,n}\sqrt{\frac{mn}{m+n}}\right)$ in (64). If m and n are small then it is convenient to know the exact distribution of $\delta_{m,n}$.

In the particular case when $n = m$, we obtain easily that

(66) $$\mathrm{P}\left\{\delta_{m,m} \leq \frac{c}{m}\right\} = \frac{1}{\binom{2m}{m}} \sum_k (-1)^k \binom{2m}{m+k(c+1)}$$

for $c = 0, 1, \ldots, m$. This result can easily be deduced from classical results for random walks as have been shown by B. V. Gnedenko and V. S. Korolyuk [263]. (See formula (37.6).) The probabilities (66) have been tabulated by F. J. Massey [297] for $m \leq 40$ and $c \leq 15$.

Mathematical methods for finding the distribution of $\delta_{m,n}$ where $m = 1,2,\ldots$ and $n = 1,2,\ldots$ have been given by V. S. Korolyuk [286], (for $n=mp$), J. H. B. Kemperman [277], J. Blackman [210], M. Depaix [243], G.P. Steck [459], S. G. Mohanty [302], K. Sarkadi [320] and E.J.G. Pitman [448]. Obviously, $\delta_{m,n}$ and $\delta_{n,m}$ have the same distribution.

If we introduce the following two statistics

(67) $$\delta^{+}_{m,n} = \sup_{-\infty<x<\infty} [F_m(x) - G_n(x)]$$

and

(68) $$\delta^{-}_{m,n} = \sup_{-\infty<x<\infty} [G_n(x) - F_m(x)]$$

which are also distribution-free statistics, then $\delta_{m,n}$ can also be expressed as follows:

(69) $$\delta_{m,n} = \max(\delta^{+}_{m,n} , \delta^{-}_{m,n}) .$$

This is equivalent to (63).

If $F(x) \equiv G(x)$ is a continuous distribution function, then $\delta^{+}_{m,n}$ and $\delta^{-}_{m,n}$ have the same distribution. The asymptotic distribution of $\delta^{+}_{m,n}$ has been found by N. V. Smirnov [327] who showed that

(70) $$\lim_{\substack{m\to\infty \\ n\to\infty}} P\{\sqrt{\frac{mn}{m+n}}\, \delta^{+}_{m,n} \leq z\} = 1-e^{-2z^2}$$

for $z \geq 0$.

B. V. Gnedenko and V. S. Korolyuk [263] found that

$$(71)\qquad \underset{\sim}{P}\{\delta^{+}_{m,m} \leq \frac{c}{m}\} = 1 - \frac{\binom{2m}{m+1+c}}{\binom{2m}{m}}$$

for $c = 0,1,\ldots, m$. For any $m = 1,2,\ldots$ and $n = 1,2,\ldots$ the distribution of $\delta^{+}_{m,n}$ has been determined by V. S. Korolyuk [286], and G. P. Steck [332].

For $x > 0$ and $y > 0$ we have

$$(72)\qquad \lim_{\substack{m \to \infty \\ n \to \infty}} \underset{\sim}{P}\{\sqrt{\frac{mn}{m+n}}\,\delta^{+}_{m,n} \leq x ,\ \sqrt{\frac{mn}{m+n}}\,\delta^{-}_{m,n} \leq y\} = K(x,y)$$

where $K(x,y)$ is given by (35) or by (36).

If $n = m$, then have

$$(73)\qquad \underset{\sim}{P}\{\delta^{+}_{m,m} < \frac{a}{m},\ \delta^{-}_{m,m} < \frac{b}{m}\} = \frac{1}{\binom{2m}{m}} \sum_{k} [\binom{2m}{m+k(a+b)} - \binom{2m}{m+a+k(a+b)}] = \frac{2^{2m+1}}{(a+b)\binom{2m}{m}} \sum_{k=0}^{a+b} (\cos\frac{k\pi}{a+b})^{2m} \sin^2 \frac{k\pi a}{a+b}$$

for $a = 1,2,\ldots, m$ and $b = 1,2,\ldots, m$. (See B. V. Gnedenko and E. L. Rvacheva [266], and the author [334].)

For any $m = 1,2,\ldots$ and $n = 1,2,\ldots$ the joint distribution of $\delta^{+}_{m,n}$ and $\delta^{-}_{m,n}$ has been determined by J. H. B. Kemperman [277], G. P. Steck [332], S.G. Mohanty [302], K. Sarkadi [320], and E.J.G. Pitman [448]. J. Blackman [210] considered the joint distribution of $\delta^+_{m,n}$ and $\delta^-_{m,n}$ in the case when n is an integral multiple of m.

In 1969 I. Vincze [348] determined the distributions and the asymtotic distributions of $\delta^+_{m,n}$ and $\delta_{m,m}$ in the case when $F(x) \equiv G(x)$ is arbitrary.

The limit theorems (65) and (72) can easily be proved by using the method of J. L. Doob [246]. Since the distributions of $\delta_{m,n}$, $\delta^+_{m,m}$, and $\delta^-_{m,n}$ do not depend on $F(x) \equiv G(x)$ if this is a continuous distribution function, therefore in finding the distributions of these random variables we may assume without loss of generality that

$$F(x) \equiv G(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases} \tag{74}$$

If we suppose that $\{\eta_m(t),\ 0 \leq t \leq 1\}$ and $\{\eta^*_n(t),\ 0 \leq t \leq 1\}$ are independent stochastic processes defined in the same way as (20) except that in the second process m is replaced by n, then we can write that

$$\delta^+_{m,n} = \max_{0\leq t\leq 1} \left[\frac{\eta_m(t)}{\sqrt{m}} - \frac{\eta^*_n(t)}{\sqrt{n}}\right] \tag{75}$$

and

$$\delta^-_{m,n} = \max_{0\leq t\leq 1} \left[\frac{\eta^*_n(t)}{\sqrt{n}} - \frac{\eta_m(t)}{\sqrt{m}}\right]. \tag{76}$$

Now let us suppose that $m \to \infty$ and $n \to \infty$ in such a way that $n/m \to p$ where p is a positive real number. Then the finite dimensional distributions of the process $\{\eta_m(t), 0 \leq t \leq 1\}$ converge to the corresponding finite dimensional distributions of a process $\{\eta(t), 0 \leq t \leq 1\}$ and the finite dimensional distributions of the process $\{\eta_n^*(t), 0 \leq t \leq 1\}$ converge to the corresponding finite dimensional distributions of a process $\{\eta^*(t), 0 \leq t \leq 1\}$ where $\{\eta(t), 0 \leq t \leq 1\}$ and $\{\eta^*(t), 0 \leq t \leq 1\}$ are independent Gaussian processes for which $\mathbf{E}\{\eta(t)\} = \mathbf{E}\{\eta^*(t)\} = 0$ and $\mathbf{Cov}\{\eta(u), \eta(t)\} = \mathbf{Cov}\{\eta^*(u), \eta^*(t)\} = \min(u,t) - ut$ for $0 \leq u \leq 1$ and $0 \leq t \leq 1$. If we suppose that $\{\eta(t), 0 \leq t \leq 1\}$ and $\{\eta^*(t), 0 \leq t \leq 1\}$ are separable stochastic processes, then by a theorem of M. D. Donsker [245] we can conclude that

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \mathbf{P}\left\{\sqrt{\frac{mn}{m+n}}\,\delta^+_{m,n} \leq x, \sqrt{\frac{mn}{m+n}}\,\delta^-_{m,n} \leq y\right\} = \tag{77}$$

$$\mathbf{P}\left\{\sup_{0\leq t\leq 1}\left[\frac{\sqrt{p}\,\eta(t)-\eta^*(t)}{\sqrt{1+p}}\right] \leq x, \sup_{0\leq t\leq 1}\left[\frac{\eta^*(t)-\sqrt{p}\,\eta(t)}{\sqrt{1+p}}\right] \leq y\right\}$$

for $x > 0$ and $y > 0$. The process

$$\left\{\frac{\sqrt{p}\,\eta(t)-\eta^*(t)}{\sqrt{1+p}}, \quad 0 \leq t \leq 1\right\} \tag{78}$$

is obviously a Gaussian process and it is easy to see that for any $p > 0$ it has the same finite dimensional distributions as the process $\{\eta(t), 0 \leq t \leq 1\}$. Accordingly we can conclude that if $m \to \infty$ and $n \to \infty$ in such a way that $n/m \to p$ where p is a positive real number, then

$$(79)\qquad \lim_{\substack{m\to\infty\\ n\to\infty}} \underset{\sim}{P}\left\{\sqrt{\frac{mn}{m+n}}\,\delta^{+}_{m,n} \leqq x\ ,\ \sqrt{\frac{mn}{m+n}}\,\delta^{-}_{m,n} \leqq y\right\} = K(x,y)$$

for $x > 0$ and $y > 0$ where

$$(80)\qquad K(x,y) = \underset{\sim}{P}\{-y \leqq \eta(t) \leqq x \quad \text{for}\quad 0 \leqq t \leqq 1\}\ .$$

In the particular case when $n = m$ the probability on the left-hand side of (79) can be obtained explicitly by (73). If we put $a = x\sqrt{m/2}$ and $b = y\sqrt{m/2}$ in (73) and let $m \to \infty$, then we obtain $K(x,y)$ given by (35) or by (36). This completes the proof of (72) in the case when $n/m \to p$ and p is a positive real number. Actually (72) is true even if n/m does not tend to a limit as $m \to \infty$ and $n \to \infty$. (See Problem 40.11.) If $x = z$ and $y = z$ in (72), then we obtain (65).

We note that if we use the representation (25), then by (80) it follows that

$$(81)\qquad \underset{\sim}{P}\{-(1+u)y \leqq \xi(u) \leqq (1+u)x \quad \text{for}\quad 0 \leqq u < \infty\} = K(x,y)$$

for $x > 0$ and $y > 0$ where $\{\xi(u)\ ,\ 0 \leqq u < \infty\}$ is a separable Brownian motion process for which $\underset{\sim}{E}\{\xi(u)\} = 0$ and $\underset{\sim}{\mathrm{Cov}}\{\xi(u)\ ,\ \xi(t)\} = \min(u,t)$ for $0 \leqq u < \infty$ and $0 \leqq t < \infty$.

In analogy with (37) and (38) we can intoroduce the statistics

$$\delta^{+}_{m,n}(h) = \sup_{-\infty<x<\infty} [F_m(x) - G_n(x)]\, h(G_n(x))$$

and

$$\delta^{-}_{m,n}(h) = \sup_{-\infty<x<\infty} [G_n(x) - F_m(x)]\, h(G_n(x))$$

where $h(u)$ for $0 \leqq u \leqq 1$ is some preassigned weight function.

Furthermore, let $\delta_{m,n}(h) = \max(\delta^{+}_{m,n}(h), \delta^{-}_{m,n}(h))$. If $F(x) \equiv G(x)$ is a continuous distribution function then $\delta^{+}_{m,n}(h)$, $\delta^{-}_{m,n}(h)$, and $\delta_{m,n}(h)$ are distribution-free statistics.

In 1950 I. D. Kvit [441] found the asymptotic distributions of $\delta^{+}_{m,n}(h)$ and $\delta^{-}_{m,n}(h)$ in the case where $h(u) = 1$ for $\alpha \leqq u \leqq \beta$ $(0 \leqq \alpha < \beta \leqq 1)$ and $h(u) = 0$ otherwise. In 1952 E. L. Rvacheva [454] determined the distributions of $\delta^{+}_{m,n}(h)$ and $\delta_{m,n}(h)$ in the case where $n = m$ and either $h(u) = 1$ for $\alpha \leqq u \leqq \beta$ and $h(u) = 0$ otherwise, or $h(u) = 0$ for $\alpha \leqq u \leqq \beta$ and $h(u) = 1$ otherwise $(0 \leqq \alpha < \beta \leqq 1)$. She has also found the asymptotic distributions of these statistics as $m \to \infty$. Also in 1952 I. I. Gikhman [431] found as a particular case of a somewhat more general result the limit of the joint distribution of $\delta^{+}_{m,n}(h)m^{1/2}\,n^{1/2}/(m+n)^{1/2}$ and $\delta^{-}_{m,n}(h)m^{1/2}\,n^{1/2}/(m+n)^{1/2}$ as $m \to \infty$ and $n \to \infty$ in the case where $h(u) = 1$ for $\alpha \leqq u \leqq \beta$ and $h(u) = 0$ otherwise $(0 \leqq \alpha < \beta \leqq 1)$.

For any $h(u)$ and $g(u)$ the joint distribution of $\delta^{+}_{m,n}(h)$ and $\delta^{-}_{m,n}(g)$ can be obtained by the results of G. P. Steck [332] , S.G. Mohanty [302] , K. Sarkadi [320] , and E.J.G. Pitman [448] .

In 1939 N. V. Smirnov [327] introduced also a more general statistic than $\delta^{+}_{m,n}$ and $\delta^{-}_{m,n}$. Let $\xi_1, \xi_2, \ldots, \xi_m$ be mutually independent random variables having the same distribution function $F(x)$. Let $\eta_1, \eta_2, \ldots, \eta_n$ be mutually independent random variables having the same distribution function $G(x)$. Denote by $F_m(x)$ the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$ and denote by $G_n(x)$ the empirical distribution function of the sample $(\eta_1, \eta_2, \ldots, \eta_n)$. Denote by $\eta_1^*, \eta_2^*, \ldots, \eta_n^*$ the random variables $\eta_1, \eta_2, \ldots, \eta_n$, arranged in increasing order of magnitude.

Now let us introduce the statistic $\sigma_{m,n}(a)$ for any real a defined as the number of subscripts $r = 1,2,\ldots, n$ for which

(82) $$G_n(\eta_r^* - 0) \leqq F_m(\eta_r^*) + \frac{a}{n} < G_n(\eta_r^*) .$$

If we suppose that $F(x)$ and $G(x)$ are two identical continuous distribution functions, and the two samples $(\xi_1, \xi_2, \ldots, \xi_m)$ and $(\eta_1, \eta_2, \ldots, \eta_n)$ are independent, then we can easily see that the distribution of the random variable $\sigma_{m,n}(a)$ does not depend on $F(x) \equiv G(x)$.

We have the obvious relations

(83) $$\underset{\sim}{P}\{\delta^+_{m,n} \leqq \frac{a}{n}\} = \underset{\sim}{P}\{\delta^-_{m,n} \leqq \frac{a}{n}\} = \underset{\sim}{P}\{\sigma_{m,n}(a) = 0\}$$

for $a \geqq 0$ and

(84) $$\underset{\sim}{P}\{\delta_{m,n} \leqq \frac{a}{n}\} = \underset{\sim}{P}\{\sigma_{m,n}(a) + \sigma_{m,n}(-a) = 0\}$$

for $a > 0$.

N. V. Smirnov [327] proved that if $m \to \infty$ and $n \to \infty$ in such a way that $n/m \to p$ where p is a positive real number, then

$$(85) \qquad \lim_{\substack{m\to\infty \\ n\to\infty}} \mathbf{P}\{\sigma_{m,n}\left(z\sqrt{\frac{(m+n)n}{m}}\right) \leqq w\sqrt{\frac{mn}{m+n}}\} = 1-e^{-\frac{(2z+w)^2}{2}}$$

for $z \geqq 0$ and $w \geqq 0$. The limiting distribution (85) is identical with (42).

N. V. Smirnov [327] also proved that if $m \to \infty$ and $n \to \infty$ in such a way that $n/m \to p$ where p is a positive real number, then

$$(86) \qquad \begin{aligned} &\lim_{\substack{m\to\infty \\ n\to\infty}} \mathbf{P}\{\sigma_{m,n}\left(z\sqrt{\frac{(m+n)n}{m}}\right) + \sigma_{m,n}\left(-z\sqrt{\frac{(m+n)n}{m}}\right) \leqq w\sqrt{\frac{mn}{m+n}}\} = \\ &= 1-2\sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\frac{d^j}{dw^j}\left(w^j e^{-\frac{[2(j+1)z+w]^2}{2}}\right) \end{aligned}$$

for $z > 0$ and $w \geqq 0$. The limiting distribution (86) is identical with (43).

We can introduce an analogue of the statistic (44) in the following way

$$(87) \qquad \omega^2_{mn} = \frac{mn}{m+n}\int_{-\infty}^{\infty}[F_m(x)-G_n(x)]^2 g\left(\frac{mF_m(x)+nG_n(x)}{m+n}\right)d\left(\frac{mF_m(x)+nG_n(x)}{m+n}\right)$$

where $g(u)$ for $0 \leqq u \leqq 1$ is some preassigned weight function.

If we suppose that $F(x)$ and $G(x)$ are two identical continuous distribution functions, then we can easily see that the distribution of the random variable $\omega^2_{m,n}$ does not depend on $F(x) \equiv G(x)$, that is, $\omega^2_{m,n}$ is a distribution-free statistic.

The statistic (87) in the case when $g(u) = 1$ for $0 \leq u \leq 1$ was proposed in 1951 by E. L. Lehmann [290]. In 1952 M. Rosenblatt [318] proved that if $g(u) = 1$ for $0 \leq u \leq 1$,then

$$\lim_{\substack{m\to\infty \\ n\to\infty}} P\{\omega^2_{m,n} \leq z\} = H(z) \tag{88}$$

where $H(z)$ is given by (53). (See also M. Fisz [256].) In his proof M. Rosenblatt [318] assumed that $n/m \to p$ where p is a positive real number. For $g(u) \equiv 1$ the expectation and the variance of $\omega^2_{m,n}$ has been found by T. W. Anderson [200], In 1957 D. A. Darling [237] mentioned that if $m \to \infty$ and $n \to \infty$ in such a way that $0 < a \leq \frac{n}{m} \leq b < \infty$, then for a general $g(u)$ the statistic $\omega^2_{m,n}$ has the same limiting distribution as (44) whenever $m \to \infty$.

Distribution-free statistics analogues to (63) and (87) can be introduced for the comparison of more than two samples. In this respect we refer to V. Ozols [304], L. C. Chang and M. Fisz [217], [218], I. I. Gikman [258], J. Kiefer [280], [281], M. Fisz [255], [257], and H. T. David [241].

In what follows we shall study several statistics in detail. These statistics can be used for the comparison of a theoretical and an empirical distribution function or for the comparison of two empirical distribution functions. Our aim is to find the exact distributions of these statistics for finite samples.

<u>The Comparison of a Theoretical and an Empirical Distribution Function</u>. Let $\xi_1, \xi_2, \ldots, \xi_m$ be mutually independent random variables having a common distribution function $\underset{\sim}{P}\{\xi_r \leq x\} = F(x)$ $(r = 1,2,\ldots, m)$. Let $F_m(x)$ be the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$, that is, $F_m(x)$ is defined as the number of variables $\xi_1, \xi_2, \ldots, \xi_m$ less than or equal to x divided by m .

Denote by $\xi_1^*, \xi_2^*, \ldots, \xi_m^*$ the random variables $\xi_1, \xi_2, \ldots, \xi_m$ arranged in increasing order of magnitude. The random variable ξ_r^* is called the r-th order statistic of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$.

First we shall consider such statistics which depend on the derivations

$$\delta_m(r) = F_m(\xi_r^*) - F(\xi_r^*) \tag{89}$$

for $r = 1,2,\ldots, m$.

If $F(x)$ is a continuous distribution function, then we can easily see that the joint distribution of the random variables $\delta_m(r)$ $(r = 1,2,\ldots, m)$ does not depend on $F(x)$. In this case in finding the distribution of any random variable depending on the deviations $\delta_m(r)$ $(r = 1,2,\ldots, m)$, we may assume without loss of generality that $F(x)$ is the distribution function of a random variable which has a uniform distribution over the interval $(0,1)$, that is, $F(x)$ is given by (17). Then $F(\xi_r^*) = \xi_r^*$ for $r = 1,2,\ldots, m$, and $F_m(\xi_r^*) = r/m$ for $r = 1,2,\ldots, m$ with probability one. In this particular case, (89) can be expressed in the following equivalent form

$$\delta_m(r) = \frac{r}{m} - \xi_r^* , \tag{90}$$

for $r = 1,2,\ldots, m$.

First, let us consider the statistic

$$\delta_m^+ = \sup_{-\infty<x<\infty} [F_m(x) - F(x)] \tag{91}$$

which was introduced in 1939 by N. V. Smirnov [328]. Equivalently we can write that

$$\delta_m^+ = \max_{1\leq r\leq m} \delta_m(r) \tag{92}$$

where $\delta_m(r)$ is given by (89).

If $F(x)$ is a continuous distribution function, then the distribution of δ_m^+ does not depend on $F(x)$ and in finding the distribution of δ_m^+ we may assume without loss of generality that $F(x)$ is given by (17). Then we can replace $\delta_m(r)$ in (92) by (90). For a continuous $F(x)$ the distribution of δ_m^+ was found in 1944 by N. V. Smirnov [329]. This distribution is given by the next two theorems.

Theorem 1. *If* $F(x)$ *is a continuous distribution function, then*

$$(93) \qquad \mathbf{P}\{\delta_m^+ \leq \frac{k}{m}\} = 1 - \sum_{j=1}^{m-k} \frac{k}{m-j} \binom{m}{j+k} \left(\frac{j}{m}\right)^{j+k} \left(1 - \frac{j}{m}\right)^{m-j-k}$$

for $k = 1,2,\ldots, m$.

Proof. Suppose that $F(x)$ is given by (17). Denote by ν_r ($r = 1,2,\ldots, m$) the number of variables $\xi_1, \xi_2, \ldots, \xi_m$ falling in the interval $(\frac{r-1}{m}, \frac{r}{m}]$, and define $N_r = \nu_1 + \nu_2 + \ldots + \nu_r$ for $r = 1,2,\ldots, m$ and $N_0 = 0$. Then

$$(94) \qquad \mathbf{P}\{N_r = i\} = \binom{m}{i} \left(\frac{r}{m}\right)^i \left(1 - \frac{r}{m}\right)^{m-i}$$

for $r = 1,2,\ldots, m$ and $N_m = m$.

Since in this case $\delta_m(r) = \frac{r}{m} - \xi_r^* < \frac{k}{m}$ if and only if $N_{r-k} < r$, provided that $r = k, k+1, \ldots, m$, by (92) we can write that

$$(95) \qquad \mathbf{P}\{\delta_m^+ < \frac{k}{m}\} = \mathbf{P}\{ \max_{1 \leq r \leq m} \left(\frac{N_r}{m} - \frac{r-1}{m}\right) \leq \frac{k}{m}\} .$$

Accordingly,

$$(96) \qquad \mathbf{P}\{\delta_m^+ < \frac{k}{m}\} = \mathbf{P}\{ \max_{1 \leq r \leq m} (N_r - r) < k \Big\} = 1 - \sum_{j=1}^{m-k} \frac{k}{m-j} \mathbf{P}\{N_j = j+k\}$$

for $k = 1,2,\ldots, m$. For in this case $\nu_1, \nu_2, \ldots, \nu_m$ are interchangeable random variables taking on nonnegative integers only, and the last equality follows from Theorem 20.1. Since δ_m^+ has a continuous distribution function, this proves (93).

By using the following auxiliary theorem we can find the complete distribution of δ_m^+ .

Lemma 1. *Let us suppose that* n *random points are distributed independently and uniformly on the interval* $(0,t)$. *Let* $\chi(u)$ $(0 \leq u \leq t)$ *be* c *times the number of points in the interval* $(0,u]$ *where* c *is a positive constant. Then*

$$(97) \qquad P\{\chi(u) \leq u \text{ for } 0 \leq u \leq t\} = \begin{cases} 1-\frac{nc}{t} & \text{for } 0 \leq nc \leq t , \\ 0 & \text{for } nc \geq t . \end{cases}$$

Proof. If $nc \geq t$, then (97) is evidently true. Let $nc \leq t$. Denote by ν_r $(r = 1,2,\ldots, n)$ the number of random points in the interval $((r-1)c, rc]$. Set $N_r = \nu_1+\ldots+\nu_r$ for $r = 1,2,\ldots, n$. Now $\nu_1, \nu_2,\ldots, \nu_n$ are interchangeable random variables taking on nonnegative integers. We have $N_n \leq n$, and

$$(98) \qquad P\{N_n = j\} = \binom{n}{j}\left(\frac{nc}{t}\right)^j\left(1-\frac{nc}{t}\right)^{n-j}$$

for $j = 0,1,\ldots, n$. Thus $E\{N_n\} = n^2c/t$. By (20.8) it follows that

$$(99) \qquad \begin{aligned} P\{\chi(u) \leq u \text{ for } 0 \leq u \leq t\} &= P\{N_r < r \text{ for } r = 1,2,\ldots,n\} = \\ &= E\{[1-\frac{N_n}{n}]^+\} = E\{1-\frac{N_n}{n}\} = 1-\frac{nc}{t} \end{aligned}$$

for $0 \leq nc \leq t$ which was to be proved.

We note that (97) can also be expressed as follows

$$(100) \qquad P\{\chi(u) \leq u \text{ for } 0 \leq u \leq t\} = \begin{cases} 1-\frac{\chi(t)}{t} & \text{for } 0 \leq \chi(t) \leq t , \\ 0 & \text{for } \chi(t) \geq t . \end{cases}$$

Lemma 1 can easily be proved also by mathematical induction on n. See the author [62].

Theorem 2. If $F(x)$ is a continuous distribution function, then

$$\mathbf{P}\{\delta_m^+ \leq x\} = 1 - \sum_{mx \leq j \leq m} \frac{mx}{m-j+mx}\binom{m}{j}\left(\frac{j}{m} - x\right)^j\left(1+x-\frac{j}{m}\right)^{m-j} \tag{101}$$

for $0 < x \leq 1$.

Proof. We shall determine the probability $\mathbf{P}\{\delta_m^+ > x\}$. Suppose that $F(x)$ is given by (17). If $\delta_m^+ > x$, then for some u $(0 < u < 1)$ the empirical distribution function $F_m(u)$ intersects the line $u+x$ $(0 < u < 1)$. Suppose that the last intersection occurs at $u = v$. Then $F_m(v) = \frac{j}{m}$ for some j $(mx \leq j \leq m)$ and $v = (j-mx)/m$. In this case there are j elements of the sample in the interval $(0,v]$ and $m-j$ elements in the interval $(v,1]$. This event has probability

$$\binom{m}{j}v^j(1-v)^{m-j}. \tag{102}$$

Furthermore, if the last intersection occurs at $u = v$, then $F_m(u) \leq u+x$ for $v \leq u \leq 1$ or $F_m(u) - F_m(v) \leq u-v$ for $v \leq u \leq 1$. Since $F_m(1) - F_m(v) = 1-(v+x)$, by Lemma 1 the latter event has probability

$$\frac{x}{1-v}. \tag{103}$$

Thus by (102) and (103)

$$\mathbf{P}\{\delta_m^+ > x\} = \sum_{0 \leq v \leq 1-x} \frac{x}{1-v}\binom{m}{j}v^j(1-v)^{m-j} \tag{104}$$

where $v = (j-mx)/m$ and $mx \leq j \leq m$. Formula (101) follows from (104).

In the particular case when $x = k/m$, formula (101) reduces to (93).

See also N. V. Smirnov [329], A. Wald and J. Wolfowitz [349],

Z. W. Birnbaum and F. H. Tingey [209], B. L. van der Waerden [343], A. P. Dempster [242], M. Dwass [250], and the author [338], [339].

Let us consider two more statistics depending on the deviations $\delta_m(1), \delta_m(2), \ldots, \delta_m(m)$ defined by (92).

Denote by γ_m^* the number of nonnegative elements among $\delta_m(r)$ $(r = 1,2,\ldots, m)$.

Define ρ_m^* as the largest r for which $\delta_m(r)$ $(r = 1,2,\ldots, m)$ attains its maximum.

If we assume that $F(x)$ is a continuous distribution function, then the distributions of the random variables γ_m^* and ρ_m^* do not depend on $F(x)$, and consequently in finding the distributions of γ_m^* and ρ_m^* we may assume without loss of generality that $F(x)$ is given by (17).

Theorem 3. *If* $F(x)$ *is a continuous distribution function, then*

$$(105) \qquad \mathbf{P}\{\gamma_m^* = j\} = \frac{1}{m} \sum_{i=1}^{j} \frac{1}{i} \binom{m}{i-1} \left(\frac{i}{m}\right)^{i-1} \left(1 - \frac{i}{m}\right)^{m-i}$$

for $j = 1,2,\ldots, m$.

Proof. Let us suppose that $F(x)$ is given by (17), and let us use the same notation as in the proof of Theorem 1. Since $\delta_m(r) \geqq 0$ if and only if $\xi_r^* \leqq \frac{r}{m}$, that is, $N_r \geqq r$, therefore γ_m^* is equal to the number of subscripts $r = 1,2,\ldots, m$ for which $N_r \geqq r$. Now $N_m = m$ and by (26.6) we obtain that

$$(106)\qquad \underset{\sim}{P}\{\gamma_m^* = j\} = \begin{cases} \sum_{i=1}^{j} \frac{1}{i(m-i)} \underset{\sim}{P}\{N_i = i-1\} & \text{for } j = 1,2,\ldots, m-1 , \\ 1 - \sum_{i=1}^{m-1} \frac{1}{i} \underset{\sim}{P}\{N_i = i-1\} & \text{for } j = m . \end{cases}$$

Hence by (94) we get (105).

The distribution of γ_m^* was found in 1958 by P. Cheng [221]. See also the author [338].

Theorem 4. If $F(x)$ is a continuous distribution function, then

$$(107)\qquad \underset{\sim}{P}\{\rho_m^* = j\} = \underset{\sim}{P}\{\gamma_m^* = j\}$$

for $j = 1,2,\ldots, m$ where the right-hand side of (107) is given by (105).

Proof. Let us suppose again that $F(x)$ is given by (17) and let us use the same notation as in the proof of Theorem 1. The random variable γ_m^* is equal to the number of subscripts $r = 1,2,\ldots, m$ for which $\delta_m(r) \geqq 0$, that is, $N_r - r \geqq 0$, and ρ_m^* is the largest subscript $r = 1,2,\ldots, m$ for which $N_r - r$ attains its maximum. By Theorem 22.1 the position of the last maximum in the sequence $N_r - r$ $(r = 0,1,\ldots, m)$ has the same distribution as the number of nonnegative elements in the sequence $N_r - r$ $(r = 1,2,\ldots, m)$. This proves (107).

We note that the random variables $\delta_m(r)$ $(r = 1,2,\ldots, m)$ are continuous, and with probability 1 there is only a single maximum in the sequence $\delta_m(r)$ $(r = 1,2,\ldots, m)$.

The distribution of ρ_m^* was found in 1958 by Z. W. Birnbaum and R. Pyke [208]. See also the author [338].

We mention briefly two more theorems.

Theorem 5. *If $F(x)$ is a continuous distribution function, then the random variable*

$$\frac{\rho_m^*}{m} - \delta_m^+ \tag{108}$$

has a uniform distribution over the interval $(0,1)$, *that is,*

$$\mathbf{P}\left\{ \frac{\rho_m^*}{m} - \delta_m^+ \leq x \right\} = \begin{cases} 0 & \text{if } x < 0 , \\ x & \text{if } 0 \leq x \leq 1 , \\ 1 & \text{if } x > 1 . \end{cases} \tag{109}$$

This theorem was found in 1958 by Z. W. Birnbaum and R. Pyke [208]. For other proofs see M. Dwass [249], N. H. Kuiper [289] and the author [338]. Theorem 5 can easily be proved by using Lemma 1.

Theorem 6. *Let $F(x)$ be a continuous distribution function, and let*

$$G_k(x) = \mathbf{P}\left\{ \frac{\rho_m^*}{m} - \delta_m^+ \leq \frac{x}{m} \text{ and } \rho_m^* = k \right\} \tag{110}$$

for $k = 1,2,\ldots, m$ and all x . If $x \leq 0$, then $G_k(x) = 0$. If $x \geq k$, then $G_k(x) = \mathbf{P}\{\rho_m^* = k\}$ is given by (107). If $0 < x < k$, then

$$\frac{dG_k(x)}{dx} = \binom{m}{k} \frac{(k-x)(m-x)^{m-k-1}}{m^m} \left[kx^{k-1} - \sum_{j=1}^{[x]} \binom{k}{j} j^{j-1}(x-j)^{k-j}\right] . \tag{111}$$

Probability (110) was found in 1958 by Z. W. Birnbaum and R. Pyke [208]. See also reference [338]. Formula (111) can also be proved by using Lemma 1 .

As a generalization of (91) let us introduce the following statistic

$$\delta_m^+(\alpha,\beta,\gamma) = \sup_{\alpha\leq F(u)\leq\beta} [F_m(u) - \gamma F(u)] \tag{112}$$

for $0 \leq \alpha < \beta < 1$ and $\gamma \geq 1$. Furthermore, let us define

$$\mu_m^+(\alpha,\beta,\gamma) = \sup_{\alpha\leq F(u)\leq\beta} \left[\frac{F_m(u) - \gamma F(u)}{F(u)}\right] \tag{113}$$

for $0 \leq \alpha < \beta \leq 1$ and $\gamma \geq 1$.

It is easy to see that if $F(x)$ is a continuous distribution function, then the distributions of $\delta_m^+(\alpha,\beta,\gamma)$ and $\mu_m^+(\alpha,\beta,\gamma)$ do not depend on $F(x)$. Thus if we want to find the distributions of $\delta_m^+(\alpha,\beta,\gamma)$ and $\mu_m^+(\alpha,\beta,\gamma)$ in this case, then we may assume without loss of generality that $F(x)$ is given by (17). In this particular case let $F_m(x) = \chi_m(x)$ for $0 \leq x \leq 1$. Here $\{\chi_m(u), 0 \leq u \leq 1\}$ is a stochastic process which has interchangeable increments. The increments are nonnegative integral multiple of $1/m$ and $\chi_m(1) = 1$. In this case

$$\underset{\sim}{P}\{\delta_m^+(\alpha,\beta,\gamma) > x\} = \underset{\sim}{P}\{\sup_{\alpha\leq u\leq\beta} [\chi_m(u) - \gamma u] > x\} \tag{114}$$

and

$$\underset{\sim}{P}\{\mu_m^+(\alpha,\beta,\gamma) > x\} = \underset{\sim}{P}\{\sup_{\alpha\leq u\leq\beta} [\chi_m(u) - (x+\gamma)u] > 0\} \tag{115}$$

for $x \geq 0$.

We note that

$$\underset{\sim}{P}\{\chi_m(u) = \tfrac{j}{m}\} = \binom{m}{j} u^j (1-u)^{m-j} \tag{116}$$

for $0 \leq j \leq m$ and $0 \leq u \leq 1$, and

$$(117)\qquad \underset{\sim}{P}\{\chi_m(u) = \frac{j}{m}, \ \chi_m(v) = \frac{k}{m}\} = \frac{m!}{j!(k-j)!(m-k)!} u^j (v-u)^{k-j} (1-v)^{m-k}$$

for $0 \leq j \leq k \leq m$ and $0 \leq u \leq v \leq 1$.

Theorem 7. If $F(x)$ is a continuous distribution function, then

$$\underset{\sim}{P}\{\delta_m^+(\alpha,\beta,\gamma) > x\} = \sum_{k>m(x+\beta\gamma)} \underset{\sim}{P}\{\chi_m(\beta) = \frac{k}{m}\} +$$

$$(118)\qquad + \sum_{m(x+\alpha\gamma)\leq j\leq k\leq m(x+\beta\gamma)} \sum \left[\frac{m(x+\beta\gamma)-k}{m(x+\beta\gamma)-j}\right] \underset{\sim}{P}\{\chi_m(\frac{j-mx}{m}) = \frac{j}{m}, \ \chi_m(\beta) = \frac{k}{m}\}$$

for $x \geq 0$, and if, in particular, $\beta = 1$, then for $x \geq 0$

$$(119)\qquad \underset{\sim}{P}\{\delta_m^+(\alpha,1,\gamma) > x\} = \sum_{m(x+\alpha\gamma)\leq j\leq m} \left[\frac{m(x+\gamma)-m}{m(x+\gamma)-j}\right] \underset{\sim}{P}\{\chi_m(\frac{j-mx}{m\gamma}) = \frac{j}{m}\}$$

for $x \geq 0$.

Proof. If $0 \leq \alpha \leq \beta \leq 1$ and $\gamma \geq 1$ and $x \geq 0$, then (118) can be obtained by (114). In (114)

$$\underset{\sim}{P}\{\sup_{\alpha\leq u\leq\beta} [\chi_m(u)-\gamma u] > x\} = \underset{\sim}{P}\{\chi_m(\beta)-\gamma\beta > x\} +$$

$$(120)\qquad + \sum_{\alpha\leq y\leq z\leq\beta} \sum \left(\frac{\beta-z}{\beta-y}\right) \underset{\sim}{P}\{\chi_m(y) = \gamma y + x, \ \chi_m(\beta) = \gamma z + x\}.$$

To prove (120) we observe that the event on the left-hand side of (120) can occur in two mutually exclusive ways: either $\chi_m(\beta)-\gamma\beta > x$ or $\chi_m(\beta)-\gamma\beta \leq x$ and $\chi_m(u)-\gamma u > x$ for some $u \in [\alpha, \beta]$. The first event has probability $\underset{\sim}{P}\{\chi_m(\beta)-\gamma\beta > x\}$. To find the probability of the second event let us suppose that $\sup\{u : \chi_m(u)-\gamma u > x$ and $\alpha \leq u \leq \beta\} = y$. Then necessarily $\chi_m(y)-\gamma y = x$ and $\chi_m(u)-\gamma u \leq x$ for $y \leq u \leq \beta$, or equivalently $\chi_m(u)-\chi_m(y) \leq \gamma(u-y)$ for $y \leq u \leq \beta$. If we suppose that $\chi_m(\beta) = \gamma z + x \leq \gamma\beta + x$ and apply Lemma 1 to $[\chi_m(u) - \chi_m(y)]/\gamma$ where $y \leq u \leq \beta$, then we obtain that

$$(121) \quad \underset{\sim}{P}\{\chi_m(u)-\chi_m(y) \leq \gamma(u-y) \text{ for } y \leq u \leq \beta \mid \chi_m(\beta)-\chi_m(y) = \gamma(z-y)\} = \frac{\beta-z}{\beta-y}$$

for $y \leq z$ and 0 if $y > z$. If we multiply (121) by $\underset{\sim}{P}\{\chi_m(y) = \gamma y+x$, $\chi_m(\beta) = \gamma z+x\}$ and add for all possible y and z satisfying the inequalities $\alpha \leq y \leq z \leq \beta$, then we get the probability of the second event. This proves (120). In (120) $\underset{\sim}{P}\{\chi_m(y) = \gamma y+x$, $\chi_m(\beta) = \gamma z+x\} = 0$ except if $y = (j-mx)/m\gamma$ and $z = (k-mx)/m$ where $0 \leq j \leq k \leq m$. Thus we obtain (118). If $\beta = 1$, then (118) reduces to (119) because $\chi_m(1) = 1$.

<u>Theorem 8.</u> <u>If</u> $F(x)$ <u>is a continuous distribution function, then</u>

$$(122) \quad \begin{aligned} \underset{\sim}{P}\{\mu_m^+(\alpha,\beta,\gamma) > x\} = {} & \sum_{k>m(x+\gamma)\beta} \underset{\sim}{P}\{\chi_m(\beta) = \frac{k}{m}\} + \\ & \sum_{m(x+\gamma)\alpha \leq j} \ \sum_{\leq k \leq m(x+\gamma)\beta} \left[\frac{m(x+\gamma)\beta-k}{m(x+\gamma)\beta-j}\right] \underset{\sim}{P}\{\chi_m\left(\frac{j}{m(x+\gamma)}\right) = \frac{j}{m}, \ \chi_m(\beta) = \frac{k}{m}\} \end{aligned}$$

<u>for</u> $x \geq 0$, <u>and if, in particular,</u> $\beta = 1$, <u>then for</u> $x \geq 0$

$$(123) \quad \underset{\sim}{P}\{\mu_m^+(\alpha,1,\gamma) > x\} = \sum_{m(x+\gamma)\alpha \leq j \leq m} \left[\frac{m(x+\gamma)-m}{m(x+\gamma)-j}\right] \underset{\sim}{P}\{\chi_m\left(\frac{j}{m(x+\gamma)}\right) = \frac{j}{m}\}$$

<u>for</u> $x \geq 0$.

<u>Proof</u>. Now (122) can be obtained by (115) for $0 \leq \alpha < \beta \leq 1$ and $\gamma \geq 1$ and $x \geq 0$. If we compare (114) and (115), then we can conclude that if in (118) we replace x by 0, and then we replace γ by $\gamma+x$, then we obtain (122). If $\beta = 1$ in (122) and if we take into consideration that $\chi_m(\beta) = 1$, then we obtain (123).

The probabilities occurring in formulas (118), (119), (122) and (123) are determined by (116) and (117).

We note that if $\alpha = 0$ and $x = 0$, then (123) further reduces to

(124) $$\underset{\sim}{P}\{\mu_m^+(0,1,\gamma) > 0\} = 1/\gamma$$

where $\gamma \geq 1$. For by Lemma 1

(125) $$\underset{\sim}{P}\{\mu_m^+(0,1,\gamma) > 0\} = \underset{\sim}{P}\{\sup_{0 \leq u \leq 1} [\chi_m(u) - \gamma u] > 0\} = 1/\gamma$$

whenever $\gamma \geq 1$.

In various particular cases several authors determined the distributions of $\delta_m^+(\alpha,\beta,\gamma)$ and $\mu_m^+(\alpha,\beta,\gamma)$. The distribution for $\delta_m^+(0,1,1)$ was found by <u>N. V. Smirnov</u> [329] and by <u>Z. W. Birnbaum</u> and <u>F. H. Tingey</u> [209], for $\delta_m^+(\alpha,1,1)$ by <u>N. V. Smirnov</u> [331], and for $\delta_m^+(0,1,\gamma)$ by <u>A. P. Dempster</u> [242], and <u>M. Dwass</u> [250]. The distribution for $\mu_m^+(0,\beta,0)$ was found by <u>L. C. Chang</u> [215], and for $\mu_m^+(\alpha,1,1)$ by <u>G. Ishii</u> [275], and <u>N. V. Smirnov</u> [331]. The probability (124) was found by <u>H. E. Daniels</u> [235], <u>H. Robbins</u> [317], <u>L. C. Chang</u> [215], and <u>D. G. Chapman</u> [220]. Theorem 7 and Theorem 8 were found by the author [336].

Next we shall determine the distribution of the statistic $\sigma_m^*(a)$ defined by (39). Actually, we shall consider some slight modifications of $\sigma_m^*(a)$.

First, for any real a let $\sigma_m(a)$ denote the number of intersections of $F(x)$ with $F_m(x) + \frac{a}{m}$ for $-\infty \leqq x \leqq \infty$. In other words, $\sigma_m(a) = k$ if the set

(126) $$S_a = \{x : F(x) = F_m(x) + \frac{a}{m} \text{ and } -\infty \leqq x \leqq \infty \}$$

is the union of k separated intervals or points.

Second, for any real a let $\tau_m(a)$ denote the number of jumps of $F_m(x) + \frac{a}{m}$ over $F(x)$ for $-\infty < x < \infty$. In other words, $\tau_m(a) = k$ if and only if

(127) $$F_m(\xi_r^* - 0) + \frac{a}{m} < F(\xi_r^*) \leqq F_m(\xi_r^*) + \frac{a}{m}$$

holds for precisely k subscript $r = 1,2,\ldots, m$.

Third, for any real a let $\tau_m^*(a)$ denote the number of subscripts $r = 1,2,\ldots, m$ for which

(128) $$\frac{a+r-1}{m} \leqq F(\xi_r^*) \leqq \frac{a+r}{m} .$$

It is easy to see that if $F(x)$ is a continuous distribution function, then the distributions of the statistics $\sigma_m(a)$, $\tau_m(a)$ and $\tau_m^*(a)$ do not depend on $F(x)$.

In this case we have $\tau_m(a) = \sigma_m(a)$ if $a \neq 0$ and $\tau_m(0) = \sigma_m(0)-1$.

These relations can be seen immediately if we take into consideration that if $F(x)$ is a continuous distribution function, then the intersections and the jumps (if any) alternate as x varies from $-\infty$ to ∞.

Furthermore, we have $\mathop{P}\limits_{\sim}\{\tau_m^*(a) = \tau_m(a)\} = 1$ for any a. For if $F(x)$ is a continuous distribution function, then $F(\xi_r^* - 0) = \frac{r-1}{m}$ and $F(\xi_r^*) = \frac{r}{m}$ for $r = 1,2,\ldots, m$ with probability one and the event $F(\xi_r^*) = \frac{a+r-1}{m}$ has probability zero for $r = 1,2,\ldots, m$ and for any a. By the above substitutions (127) becomes (128). Obviously $\tau_m^*(a) = \sigma_m^*(-a)$ defined by (39).

In what follows we assume that $F(x)$ is a continuous distribution function and we shall determine the distribution of the random variable $\sigma_m(a)$. It follows by symmetry that $\sigma_m(a)$ and $\sigma_m(-a)$ have the same distribution. If we know the distribution of $\sigma_m(a)$, the distributions of $\tau_m(a)$ and $\tau_m^*(a)$ can be obtained immediately by the above relations.

The distribution of $\sigma_m(a)$ for $0 < a \leq m$ has been given without proof by D. A. Darling [238] and for $a = 0,1,2,\ldots, m$ it has been given by W. Nef [303]. Some generalizations have been given by the author [341].

Theorem 9. *If* $F(x)$ *is a continuous distribution function and* $a \geq 0$, *then we have*

$$\mathop{P}\limits_{\sim}\{\sigma_m(a) > k\} = \frac{(a+k)}{m^m} \sum_{k\leq j\leq m-a} \frac{m!}{(j-k)!(m-j)!} (a+j)^{j-k-1}(m-a-j)^{m-j} = \tag{129}$$

$$= \frac{m!}{(m-k)!m^k} - \frac{(a+k)}{m^m} \sum_{m-a<j\leq m} \frac{m!}{(j-k)!(m-j)!} (a+j)^{j-k-1}(m-a-j)^{m-j}$$

for $0 \leq k \leq m-a$.

Proof. Let us assume that $F(x)$ is given by (17). For $0 \leq u \leq m$ denote by $\nu_m(u)$ the number of variables $\xi_1, \xi_2, \ldots, \xi_m$ falling in the interval $(0, u/m]$. Then $F_m(x) = \nu_m(mx)/m$ for $0 \leq x \leq 1$ and $\sigma_m(a)$ can be interpreted as the number of distinct points in the set

$$(130) \qquad S_a = \{x : \nu_m(mx) = mx-a \text{ and } 0 \leq x \leq 1\} ,$$

or equivalently, as the number of integers $j = 0,1,\ldots, m$ for which $\nu_m(a+j) = j$. Accordingly, we have

$$(131) \qquad \underset{\sim}{P}\{\sigma_m(a) = k\} = \underset{\sim}{P}\{\nu_m(a+j) = j \text{ for } k \text{ values } j = 0,1,\ldots,m\} .$$

Here $\{\nu_m(u) , 0 \leq u \leq m\}$ is a stochastic process with interchangeable increments for which $\nu_m(m) = m$,

$$(132) \qquad \underset{\sim}{P}\{\nu_m(u) = i\} = \binom{m}{i}\left(\frac{u}{m}\right)^i\left(1- \frac{u}{m}\right)^{m-i}$$

for $0 \leq i \leq m$ and $0 \leq u \leq m$, and

$$(133) \qquad \underset{\sim}{P}\{\nu_m(u) = i|\nu_m(t) = j\} = \binom{j}{i}\left(\frac{u}{t}\right)^i\left(1- \frac{u}{t}\right)^{j-i}$$

for $0 \leq i \leq j \leq m$ and $0 \leq u \leq t \leq m$. Furthermore, for $0 < t \leq m$ and $0 \leq r \leq m$ we have

$$(134) \qquad \underset{\sim}{P}\{\nu_m(u) < u \text{ for } 0 < u \leq t|\nu_m(t) = r\} = \begin{cases} 1- \frac{r}{t} & \text{if } 0 \leq r \leq t , \\ 0 & \text{if } t \leq r \leq m , \end{cases}$$

which follows from Lemma 1.

If $0 \leq k \leq m-a$, then we can write that

$$
\begin{aligned}
&\underset{\sim}{P}\{\sigma_m(a) > k\} = \\
(135)\quad &= \sum_{k \leq j \leq m-a} \underset{\sim}{P}\{\nu_m(a+j) = j\}\, \underset{\sim}{P}\{\nu_m(a+i) = i \text{ for } k \text{ values} \\
&\quad i = 0,1,\ldots,j-1 \mid \nu_m(a+j) = j\} = \\
&= \sum_{k \leq j \leq m-a} \underset{\sim}{P}\{\nu_m(a+j) = j\}\, \underset{\sim}{P}\{\nu_m(i) = i \text{ for } k \text{ values} \\
&\quad i = 1,2,\ldots,j \mid \nu_m(a+j) = j\} .
\end{aligned}
$$

For the event $\sigma_m(a) > k$ occurs if and only if $\nu_m(a+i) = i$ for more than k values $i = 0,1,\ldots, m$. This event can occur in several mutually exclusive ways: the $(k+1)$st largest $i = 0,1,\ldots, m$ for which $\nu_m(a+i) = i$ is $i = j$ where $k \leq j \leq m-a$. The last equality in (135) follows by symmetry.

Let us introduce the notation

$$
(136)\qquad q_k(s) = \underset{\sim}{P}\{\nu_m(i) = i \text{ for } k \text{ values } i = 1,2,\ldots,s \mid \nu_m(s) = s\}
$$

for $1 \leq k \leq s \leq m$. Obviously $q_k(s)$ is independent of m whenever $s \leq m$. By using this notation in (135) we can write that

$$
\begin{aligned}
&\underset{\sim}{P}\{\nu_m(i) = i \text{ for } k \text{ values } i = 1,2,\ldots,j \mid \nu_m(a+j) = j\} = \\
(137)\quad &= \sum_{s=k}^{j} q_k(s) \underset{\sim}{P}\{\nu_m(s) = s \mid \nu_m(a+j) = j\} .
\end{aligned}
$$

For the event $\{\nu_m(i) = i$ for k values $i = 1,2,\ldots,j\}$ can occur in such a way that $\nu_m(s) = s$ for some s $(k \leq s \leq j)$ and $\nu_m(i) = i$ for exactly k values $i = 1,2,\ldots, j$.

If we put (137) into (135), then we obtain that

$$P\{\sigma_m(a) > k\} = \sum_{k\leq j\leq m-a} \sum_{s=k}^{j} q_k(s) P\{\nu_m(s) = s,\ \nu_m(a+j) = j\} \tag{138}$$

for $0 \leq k \leq m-a$.

It remains only to find $q_k(s)$ for $1 \leq k \leq s$. We shall prove that

$$q_k(s) = \frac{s!k}{(s-k)!s^{k+1}} \tag{139}$$

for $1 \leq k \leq s$. If we put (139) into (138) and use Abel's identity

$$k \sum_{s=k}^{j} \frac{s^{s-k+1}(a+j-s)^{j-s}}{(s-k)!(j-s)!} = \frac{(a+j)^{j-k}}{(j-k)!} , \tag{140}$$

then we obtain the first expression in (129). The second expression in (129) follows again by Abel's identity

$$(a+k) \sum_{j=k}^{m} \frac{(a+j)^{j-k-1}(m-a-j)^{m-j}}{(j-k)!(m-j)!} = \frac{m^{m-k}}{(m-k)!} . \tag{141}$$

The probabilities $q_k(s)$ for $1 \leq k \leq s$ can be obtained by the following recurrence relations:

$$q_k(s) = \sum_{1\leq j\leq s-k+1} P\{\nu_m(j) = j \mid \nu_m(s) = s\} q_1(j) q_{k-1}(s-j) \tag{142}$$

for $2 \leq k \leq s$ and

$$q_1(s) = 1 - \sum_{1\leq j< s} P\{\nu_m(j)=j \mid \nu_m(s) = s\} q_1(j) \tag{143}$$

for $s \geq 1$.

Let us introduce the following notation

$$Q_k(s) = \frac{s^s}{s!} q_k(s) \tag{144}$$

for $1 \leq k \leq s$. Then by using (133) we can express (142) and (143) in the following equivalent forms:

$$Q_k(s) = \sum_{1 \leq j \leq s-k+1} Q_1(j) Q_{k-1}(s-j) \tag{145}$$

for $2 \leq k \leq s$ and

$$Q_1(s) = \frac{s^s}{s!} - \sum_{1 \leq j < s} Q_1(j) \frac{(s-j)^{s-j}}{(s-j)!} \tag{146}$$

for $s \geq 1$. From (145) and (146) we can find $Q_k(s)$ for $1 \leq k \leq s$ by using generating functions.

It will be convenient to derive first the generating functions which we need in solving (145) and (146).

By using Rouché's theorem we can prove that if $|z| < 1/e$, then the equation

$$we^{-w} = z \tag{147}$$

has a single root $w = \rho(z)$ in the circle $|w| < 1$, and by Lagrange's expansion we obtain that

$$e^{a\rho(z)}[\rho(z)]^k = z^k + (a+k) \sum_{j=k+1}^{\infty} \frac{(a+j)^{j-k-1}}{(j-k)!} z^j , \tag{148}$$

and

(149) $$\frac{e^{a\rho(z)}[\rho(z)]^k}{1-\rho(z)} = \sum_{j=k}^{\infty} \frac{(a+j)^{j-k}}{(j-k)!} z^j$$

for $|z| < 1/e$, $k = 0,1,2,\ldots$ and for any a . In particular, we have

(150) $$\frac{\rho(z)}{1-\rho(z)} = z\rho'(z) = \sum_{j=1}^{\infty} \frac{j^j z^j}{j!}$$

for $|z| < 1/e$. We note that $\rho(z) \to 1$ if $z \to 1/e$.

The equation (147) has been investigated first by L. Euler [21]. (See also G. Pólya [50].)

If we form the generating function of (146), then by (150) we obtain that

(151) $$\sum_{s=1}^{\infty} Q_1(s)z^s = \frac{\rho(z)}{1-\rho(z)} [1- \sum_{s=1}^{\infty} Q_1(s)z^s]$$

and hence

(152) $$\sum_{s=1}^{\infty} Q_1(s)z^s = \rho(z)$$

for $|z| < 1/e$. By (145) we obtain that

(153) $$\sum_{s=k}^{\infty} Q_k(s)z^s = (\sum_{s=1}^{\infty} Q_1(s)z^s)^k = [\rho(z)]^k$$

for $|z| < 1/e$ and $k = 1,2,\ldots$. Thus (139) follows from (144) and (148). This completes the proof of Theorem 9.

NOTE. If we form the coefficient of z^m in the product of

$e^{a\rho(z)}[\rho(z)]^k$ and $e^{b\rho(z)}$, then by (148) we obtain that

$$(154)\qquad (a+k)b\sum_{j=k}^{m}\frac{(a+j)^{j-k-1}(b+m-j)^{m-j-1}}{(j-k)!(m-j)!} = (a+b+k)\frac{(a+b+m)^{m-k-1}}{(m-k)!}$$

for $0 \leq k \leq m$ and for any a and b.

If we form the coefficient of z^m in the product of $e^{a\rho(z)}[\rho(z)]^k$ and $e^{b\rho(z)}/[1-\rho(z)]$, then by (148) and (149) we obtain Abel's identity

$$(155)\qquad (a+k)\sum_{j=k}^{m}\frac{(a+j)^{j-k-1}(b+m-j)^{m-j}}{(j-k)!(m-j)!} = \frac{(a+b+m)^{m-k}}{(m-k)!}$$

for $0 \leq k \leq m$ and for any a and b.

The Comparison of Two Empirical Distribution Functions. Let $\xi_1, \xi_2, \ldots, \xi_m$ be mutually independent random variables having a common distribution function $P\{\xi_r \leq x\} = F(x)$ $(r = 1,2,\ldots, m)$. Denote by $F_m(x)$ the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$. Let $\eta_1, \eta_2, \ldots, \eta_n$ be also mutually independent random variables having a common distribution function $P\{\eta_r \leq x\} = G(x)$ $(r = 1,2,\ldots, n)$. Denote by $G_n(x)$ the empirical distribution function of the sample $(\eta_1, \eta_2, \ldots, \eta_n)$. Let us suppose also that $(\xi_1, \xi_2, \ldots, \xi_m)$ and $(\eta_1, \eta_2, \ldots, \eta_n)$ are independent.

Denote by $\eta_1^*, \eta_2^*, \ldots, \eta_n^*$ the random variables arranged in increasing order of magnitude.

For the purpose of testing the hypothesis that $F(x) \equiv G(x)$ we can introduce several statistics depending on the deviations

$$F_m(\eta_r^*) - G_n(\eta_r^* - 0) \quad (r = 1,2,\ldots, n) \,. \tag{156}$$

Let

$$\delta_{m,n}^{+} = \max_{1 \leq r \leq n} [F_m(\eta_r^*) - G_n(\eta_r^* - 0)] \tag{157}$$

which is in agreement with (67).

Denote by $\gamma_{m,n}(a)$ the number of subscripts $r = 1,2,\ldots, n$ for which

$$F_m(\eta_r^*) < G_n(\eta_r^*) - \frac{a}{n} \tag{158}$$

where a is a real number.

Denote by $\rho_{m,n}$ the smallest $r = 1,2,\ldots, n$ for which

$$F_m(\eta_r^*) - G_n(\eta_r^* - 0) \tag{159}$$

attains its maximum.

For any real number a let us define $\sigma_{m,n}(a)$ as the number of subscripts $r = 1,2,\ldots, n$ for which

$$G_n(\eta_r^* - 0) \leqq F_m(\eta_r^*) + \frac{a}{n} < G_n(\eta_r^*) \,. \tag{160}$$

If we suppose that $F(x)$ and $G(x)$ are two identical continuous distribution functions, then we can easily see that the distributions of $\delta_{m,n}^{+}$, $\gamma_{m,n}(a)$, $\tau_{m,n}$, and $\sigma_{m,n}(a)$ do not depend on $F(x) \equiv G(x)$. Accordingly $\delta_{m,n}^{+}$, $\gamma_{m,n}(a)$, $\tau_{m,n}$, and $\sigma_{m,n}(a)$ are distribution-free statistics. In what follows we shall determine the distributions of these statistics in the case when $F(x)$ and $G(x)$ are two identical continuous distribution functions and $n = mp$ where p is a positive integer.

To find the distributions of the above statistics let us introduce the following notation.

For $r = 1,2,\ldots, n+1$ let us define ν_r as p times the number of variables $\xi_1, \xi_2,\ldots, \xi_m$ falling in the interval $(\eta^*_{r-1}, \eta^*_r]$ where $\eta^*_0 = -\infty$ and $\eta^*_{n+1} = \infty$. Here p is a positive constant.

If $F(x)$ and $G(x)$ are two identical continuous distribution functions, then $\nu_1, \nu_2,\ldots, \nu_{n+1}$ are interchangeable random variables taking on nonnegative integral multiples of p. If we set $N_r = \nu_1 + \nu_2 + \ldots + \nu_r$ for $r = 1,2,\ldots, n+1$, then we have $N_{n+1} = mp$,

$$(161) \qquad \underset{\sim}{P}\{N_i = sp\} = \frac{\binom{i+s-1}{s}\binom{m+n-i-s}{m-s}}{\binom{m+n}{m}}$$

for $1 \leq i \leq n$ and $0 \leq s \leq m$,

$$(162) \qquad \underset{\sim}{P}\{N_i = sp \mid N_{i+j} = (s+t)p\} = \frac{\binom{i+s-1}{s}\binom{j+t-1}{t}}{\binom{i+j+s+t-1}{s+t}}$$

for $1 \leq i < i+j \leq n$ and $0 \leq s \leq s+t \leq m$, and

$$P\{N_i = sp\ ,\ N_{i+j} = (s+t)p \,|\, N_{i+j+k} = (s+t+u)p\} = \tag{163}$$

$$= \frac{\binom{i+s-1}{s}\binom{j+t-1}{t}\binom{k+u-1}{u}}{\binom{i+j+k+s+t+u-1}{s+t+u}}$$

for $1 \leq i < i+j < i+j+k \leq n$ and $0 \leq s \leq s+t \leq s+t+u \leq m$.

By using the above notation we can write that

$$F_m(\eta_r^*) = \frac{N_r}{mp} \tag{164}$$

for $r = 1,2,\ldots, n$, and obviously

$$G_n(\eta_r^*) = \frac{r}{n} \quad \text{and} \quad G_n(\eta_r^* - 0) = \frac{r-1}{n} \tag{165}$$

for $r = 1,2,\ldots, n$ with probability one.

Thus we obtain easily that

$$\delta_{m,n}^+ = \max_{1 \leq r \leq n} \left(\frac{N_r}{mp} - \frac{r-1}{n}\right) . \tag{166}$$

The variable $\gamma_{m,n}(a)$ is equal to the number of subscripts $r = 1,2,\ldots, n$ for which

$$\frac{N_r}{mp} < \frac{r-a}{n} \ , \tag{167}$$

the variable $\rho_{m,n}$ is the smallest $r = 1,2,\ldots, n$ for which

$$\frac{N_r}{mp} - \frac{r-1}{n} \tag{168}$$

attains its maximum, and finally, the variable $\sigma_{m,n}(a)$ is equal to the number of subscripts $r = 1,2,\ldots, n$ for which

$$\frac{r-1}{n} \leqq \frac{N_r}{mp} + \frac{a}{n} < \frac{r}{n} . \tag{169}$$

In the particular case when $n = mp$ and p is a positive integer we can determine the distributions of the above statistics by using the combinatorial methods developed in Section 20, and in Section 26. For if p is a positive integer, then $\nu_1, \nu_2,\ldots, \nu_{n+1}$ are interchangeable random variables taking on nonnegative integers only.

<u>The distribution of</u> $\delta^+_{m,n}$. If, in particular, $n = mp$, then by (166) it follows that

$$\delta^+_{m,n} = \frac{1}{n} \max_{1\leqq r\leqq n} (N_r - r+1) . \tag{170}$$

<u>Theorem 10. If</u> $n = mp$ <u>where</u> p <u>is a positive integer, and</u> $c = 0,1,\ldots, n$, <u>then</u>

$$\begin{aligned} &\underset{\sim}{P}\{\delta^+_{m,n} \leqq \frac{c}{n}\} = 1 - \\ &- \frac{1}{\binom{m+n}{m}} \sum_{\frac{c+1}{p}\leqq s\leqq m} \frac{c+1}{n+c+1-sp} \binom{sp+s-c-1}{s}\binom{m+n+c-sp-s}{m-s} , \end{aligned} \tag{171}$$

<u>and, in particular,</u>

$$\underset{\sim}{P}\{\delta^+_{m,m} \leqq \frac{c}{m}\} = 1- \frac{\binom{2m}{m+1+c}}{\binom{2m}{m}} \tag{172}$$

<u>for</u> $c = 0,1,\ldots, m$.

Proof. If $n = mp$, then by (170)

$$(173) \qquad \mathbf{P}\{\delta^{+}_{m,n} \leq \frac{c}{n}\} = \mathbf{P}\{N_r < r+c \quad \text{for} \quad r = 1,2,\ldots, n+1\} .$$

If we take into consideration that in this case $N_{n+1} = mp = n$, then by Theorem 20.1 or by Theorem 26.6 we obtain that

$$(174) \qquad \mathbf{P}\{\delta^{+}_{m,n} \leq \frac{c}{n}\} = 1 - \sum_{j=1}^{n} \frac{c+1}{n+1-j} \mathbf{P}\{N_j = j+c\}$$

for $c = 0,1,\ldots, n$. By (161) we obtain (171). If, in particular, $p = 1$, then (171) reduces to (172).

The result (171) can also be interpreted in the following way: Let us combine the two samples $(\xi_1, \xi_2,\ldots, \xi_m)$ and $(\eta_1, \eta_2,\ldots, \eta_n)$, where now $n = mp$, and let us arrange the $m+n$ variables in increasing order of magnitude. Let us define $\chi_i = p$ if the i-th ordered variable in the combined sample belongs to $(\xi_1, \xi_2,\ldots, \xi_m)$ and $\chi_i = -1$ if the i-th ordered variable in the combined sample belongs to $(\eta_1, \eta_2,\ldots, \eta_n)$. Now let us suppose that a particle performs a one-dimensional random walk on the x-axis. The particle starts at $x = 0$ and takes $m+n$ steps. At the i-th step it moves either p unit distance to the right if $\chi_i = p$ or a unit distance to the left if $\chi_i = -1$.

Now all the $\binom{m+n}{m}$ paths are equally probable and the event $\{\delta^{+}_{m,n} \leq \frac{c}{n}\}$ can be interpreted as the event that the particle never reaches the point $x = c+1$ during the $m+n$ steps.

If $n = m$ then by using the above interpretation we can easily find (172) directly by using the method of reflection.

The distribution of the random variable $\delta^+_{m,n}$ for $n = m$ was found in 1951 by B. V. Gnedenko and V. S. Korolyuk [263], and for $n = mp$ where p is a positive integer in 1955 by V. S. Korolyuk [286]. See also the author [335].

If we suppose that $c = [nx]$ where $0 < x < 1$ and $n = mp$, then by (171) we obtain that

$$\lim_{p\to\infty} \underset{\sim}{P}\{\delta^+_{m,n} \leqq \frac{c}{n}\} = \underset{\sim}{P}\{\delta^+_m \leqq x\} =$$

(175)

$$= 1 - \sum_{mx \leqq j \leqq m} \frac{mx}{m+mx-j} \binom{m}{j} \left(\frac{j}{m} - x\right)^j \left(1+x-\frac{j}{m}\right)^{m-j}$$

which is in agreement with (101).

If we suppose that $c = [z\sqrt{2m}]$ where $0 \leqq z < \infty$, then by (172) we obtain that

$$\lim_{m\to\infty} \underset{\sim}{P}\{\sqrt{\frac{m}{2}}\, \delta^+_{m,m} \leqq z\} = 1 - e^{-2z^2} \tag{176}$$

which is a particular case of (70).

If we suppose that $c = [z\sqrt{p(p+1)m}\,]$ where $0 \leqq z < \infty$, then by (171) we obtain that

$$\lim_{m\to\infty} \underset{\sim}{P}\left\{\sqrt{\frac{pm}{p+1}}\,\delta^{+}_{m,mp} \le z\right\} =$$

(177)

$$= 1 - \frac{z}{\sqrt{2\pi}} \int_0^1 \frac{e^{-\frac{z^2}{2u(1-u)}}}{(1-u)^{3/2}u^{1/2}}\,du = 1-e^{-2z^2}$$

which is another particular case of (70).

We can prove (176) and (177) easily if we use Stirling's formula (35.28) or A. De Moivre's approximation of the Bernoulli distribution.

The distribution of $\gamma_{m,n}(a)$. Let $n = mp$ where p is a positive integer. In this case $\gamma_{m,n}(a) = \gamma_{m,n}([a])$ for any real a where $[a]$ is the greatest integer $\le a$. Furthermore, in this case $\gamma_{m,n}(a)$ and $n-\gamma_{m,n}(-a)$ have the same distribution for $a = 0, \pm 1, \pm 2, \ldots$, that is,

(178) $$\underset{\sim}{P}\{\gamma_{m,n}(a) = j\} = \underset{\sim}{P}\{\gamma_{m,n}(-a) = n-j\}$$

holds for $j = 0,1,\ldots, n$ and $a = 0, \pm 1, \pm 2, \ldots$. This follows from (167) and from the following relations

(179)
$$\underset{\sim}{P}\{\gamma_{m,n}(a) = j\} = \underset{\sim}{P}\left\{\frac{N_r}{mp} < \frac{r-a}{n} \text{ for } j \text{ subscripts } r = 1,2,\ldots, n\right\} =$$
$$= \underset{\sim}{P}\left\{\frac{N_{n+1} - N_r}{mp} > \frac{n-r+a}{n} \text{ for } j \text{ subscripts } r = 1,2,\ldots, n\right\} =$$
$$= \underset{\sim}{P}\left\{\frac{N_s}{mp} > \frac{s+a-1}{n} \text{ for } j \text{ subscripts } s = 1,2,\ldots, n\right\} =$$
$$= \underset{\sim}{P}\left\{\frac{N_s}{mp} < \frac{s+a}{n} \text{ for } n-j \text{ subscripts } s = 1,2,\ldots, n\right\} =$$
$$= \underset{\sim}{P}\{\gamma_{m,n}(-a) = n-j\}$$

for $j = 0,1,\dots, n$ and $a = 0, \pm 1, \pm 2, \dots$.

We note that by (166) and (179) we can conclude also that

(180) $$\mathbf{P}\{\gamma_{m,n}(a) = 0\} = \mathbf{P}\{\delta^{+}_{m,n} \leq \frac{a}{n}\}$$

for $a = 0,1,\dots, n$ and for all m and n .

If, in particular, $n = mp$ then by (167) it follows that $\gamma_{m,n}(a)$ is equal to the number of subscripts $r = 1,2,\dots, n$ for which

(181) $$N_r < r-a .$$

Now let us find the distribution of $\gamma_{m,n}(a)$ for $a = 0,1,\dots, n$ whenever $n = mp$ and p is a positive integer.

Theorem 11. Let $n = mp$ where p is a positive integer. We have

(182) $$\mathbf{P}\{\gamma_{m,n}(0) = j\} = \frac{1}{n+1}$$

for $j = 0,1,2,\dots, n$. If $a = 1,2,\dots, n$, then

(183) $$\mathbf{P}\{\gamma_{m,n}(a) = 0\} = 1- \frac{1}{\binom{m+n}{m}} \sum_{\frac{a+1}{p} \leq s \leq m} \frac{a+1}{n+a+1-sp} \binom{sp+s-a-1}{s}\binom{m+n+a-sp-s}{m-s}$$

and

$$
(184)\quad \underset{\sim}{P}\{\gamma_{m,n}(a) = j\} = \frac{1}{\binom{m+n}{m}} \sum_{s=0}^{[\frac{j-1}{p}]} \left(1-\frac{sp}{j}\right) \cdot
$$

$$
\cdot \left[\sum_{t=0}^{[\frac{n-j-a}{p}]} \frac{a}{a+tp} \binom{j+s-1}{s}\binom{a+tp+t-1}{t}\binom{m+n-j-s-a-tp-t}{m-s-t} - \right.
$$

$$
\left. - \sum_{t=0}^{[\frac{n-j-a-1}{p}]} \frac{a+1}{a+1+tp} \binom{j+s-1}{s}\binom{a+tp+t}{t}\binom{m+n-j-s-a-tp-t-1}{m-s-t} \right]
$$

for $j = 1,2,\ldots, n-a$.

If, in particular, $p = 1$, _then_ (183) _reduces to_

$$
(185)\qquad \underset{\sim}{P}\{\gamma_{m,m}(a) = 0\} = 1 - \frac{\binom{2m}{m+1+a}}{\binom{2m}{m}}
$$

for $a = 1,2,\ldots, m$, _and_ (184) _becomes_

$$
(186)\qquad \underset{\sim}{P}\{\gamma_{m,m}(a) = j\} = \frac{1}{\binom{2m}{m}} \sum_{i=j}^{m-a} \frac{a}{(i+1)(m-i)} \binom{2i}{i}\binom{2m-2i}{m+a-i}
$$

for $a = 1,2,\ldots, m-1$ _and_ $j = 1,2,\ldots, m-a$.

Proof. If $n = mp$, then by (181)

$$
(187)\qquad \underset{\sim}{P}\{\gamma_{m,n}(a) = j\} = \underset{\sim}{P}\{N_r < r-a \text{ for } j \text{ subscripts } r = 1,2,\ldots, n\}
$$

for $j = 0,1,2,\ldots, n$ and $a = 0,1,2,\ldots, n$.

If $a = 0$ and we take into consideration that $N_{n+1} = n$, then by

(26.5) we obtain (182).

If $a = 1,2,\ldots, n$ and $j = 0$, then (180) holds and (183) follows from (171), and (185) from (172).

If $a = 1,2,\ldots, n$ and $j = 1,2,\ldots, n$, then by (26.52) we obtain that

$$\underset{\sim}{P}\{\gamma_{m,n}(a) = j\} = \sum_{\ell=0}^{j} \left(1-\frac{\ell}{j}\right)\Big[\sum_{i=j+a}^{n} \frac{a}{(i-j)} \underset{\sim}{P}\{N_j = \ell,\ N_i - N_j = i-j-a\} - \tag{188}$$
$$- \sum_{i=j+a+1}^{n} \frac{a+1}{(i-j)} \underset{\sim}{P}\{N_j = \ell,\ N_i - N_j = i-j-a-1\}\Big]$$

and the probabilities on the right-hand side of (188) can be obtained by (161) and (162). This proves (184). If, in particular, $p = 1$, then (184) reduces to (186). Formula (186) can be proved directly by using the random walk interpretation mentioned after Theorem 10.

The distribution of the random variable $\gamma_{m,n}(0)$ for $n = m$ was found in 1952 by B. V. Gnedenko and V. S. Mihalevich [264] and for $n = mp$ where p is a positive integer also in 1952 by B. V. Gnedenko and V. S. Mihalevich [265]. See also the author [335].

The distribution of the random variable $\gamma_{m,n}(a)$ in the particular case of $n = m$ was found in 1952 by V. S. Mihalevich [298]. For $n = mp$ where p is a postiive integer the distribution of $\gamma_{m,n}(a)$ was found in 1969 by the author [339].

We note that if $0 \leq x \leq 1$ and $y > 0$, then we have

$$(189) \qquad \lim_{\substack{m\to\infty \\ n\to\infty}} \mathbf{P}\{\gamma_{m,n}(y\sqrt{\frac{mn}{m+n}}) \leq nx\} = 1 - \frac{y}{\sqrt{2\pi}} \int_x^1 \frac{(u-x)e^{-\frac{y^2}{2(1-u)}}}{[u(1-u)]^{3/2}}\, du \; .$$

In the particular case when $n = m$, the distribution of $\gamma_{m,m}(a)$ is given by (185) and (186) and we can easily show that in this case (189) is true. On the other hand, we can show that the limiting distribution (189) does not depend on the manner in which $m \to \infty$ and $n \to \infty$. Thus it follows that (189) is valid in the general case too. See I. I. Gikhman [431] and I. Vincze [346].

The distribution of $\rho_{m,n}$. Let $n = mp$ where p is a positive integer. Then by (168) it follows that $\rho_{m,n}$ is the smallest $r = 1,2,\ldots,n$ for which

$$(190) \qquad N_r - r$$

attains its maximum.

In what follows we shall determine the joint distribution of $\delta^+_{m,n}$ and $\rho_{m,n}$.

Theorem 12. *Let* $n = mp$ *where* p *is a positive integer. If* $k = 0,1,\ldots, n$, $j = 1,2,\ldots, n-k+1$ *and* $j+k = tp+1$ *where* $t = 0,1,2,\ldots,$ *then we have*

$$\mathbf{P}\{\delta^{+}_{m,n} = \frac{k}{n},\ \rho_{m,n} = j\} = \frac{1}{\binom{m+n}{m}} \left[\binom{j+t-\alpha(p)}{j-1} - \sum_{2\leq sp\leq j-1} \frac{1}{(sp-1)} \binom{sp+s-2}{s} \binom{j+t-sp-s-1}{t-s}\right] \tag{191}$$

$$\cdot \left[\binom{m+n-j-t}{m-t} - \sum_{2\leq sp\leq n+1-j} \frac{k+1}{n+2-j-sp} \binom{sp+s-2}{s} \binom{m+n-j-t-sp-s+1}{m-t-s}\right]$$

<u>where</u> $\alpha(p) = 3$ <u>if</u> $p = 1$ <u>and</u> $\alpha(p) = 2$ <u>if</u> $p > 1$.

<u>If, in particular</u>, $p = 1$, <u>then we have</u>

$$\mathbf{P}\{\delta^{+}_{m,m} = \frac{k}{m},\ \rho_{m,m} = j\} = \frac{k(k+1)}{(k+2j-2)(m+2-j)} \binom{k+2j-2}{j-1} \binom{2m+2-2j-k}{m+1-j} \Big/ \binom{2m}{m} \tag{192}$$

<u>for</u> $k = 1,2,\ldots, m$ <u>and</u> $j = 1,2,\ldots, m+1-k$, <u>and</u>

$$\mathbf{P}\{\delta^{+}_{m,m} = 0\ ,\ \rho_{mm} = 1\} = \frac{1}{m+1} . \tag{193}$$

<u>Proof</u>. We can write that

$$\mathbf{P}\{\delta^{+}_{m,n} = \frac{k}{n}\ ,\ \rho_{m,n} = j\} = \tag{194}$$

$$= \mathbf{P}\{N_r - r+1 < N_j - j+1 = k \text{ for } 1 \leq r < j \text{ and } N_r - r+1 \leq N_j - j+1$$

$$\text{for } j \leq r \leq n\} = \mathbf{P}\{N_j - N_r > j-r \text{ for } 1 \leq r < j \text{ and } N_j = j+k-1\} \cdot$$

$$\cdot\ \mathbf{P}\{N_r - N_j \leq r-j \text{ for } j \leq r \leq n \mid N_j = j+k-1\} .$$

By Theorem 26.4 the first factor on the right-hand side of (194) is $\mathbf{P}\{N_1 = k\}$ if $j = 1$ and

$$\mathbf{P}\{N_1 > 1\ ,\ N_j = j+k-1\} - \sum_{i=2}^{j-1} \frac{1}{(i-1)} \mathbf{P}\{N_1 = 0\ ,\ N_i = i\ ,\ N_j = j+k-1\} \tag{195}$$

if $j = 2,\ldots, n+1-k$, and 0 if $j > n+1-k$.

We note that if $k = 1$, then (195) reduces to

(196) $$\frac{1}{(j-1)} \underset{\sim}{P}\{N_1 = 0 , N_j = j\}$$

which follows from (26.6) .

If we take into consideration that $N_{n+1} = n$, then by (26.37) we obtain that the second factor on the right-hand side of (194) is

(197) $$1 - \sum_{i=1}^{n-j} \frac{k+1}{n+1-j-i} \underset{\sim}{P}\{N_{j+i} - N_j = i+1 | N_j = j+k-1\}$$

for $j = 1,2,\ldots, n+1-k$.

We can easily see that (194) is 0 unless $k = 0,1,2,\ldots, n$, $j = 1,2,\ldots, n+1-k$ and $j+k = tp+1$ where $t = 0,1,2,\ldots$. If we use (161), (162), and (163), then by (194), (195) and (196) we obtain (191).

We note that if, in particular, $k = 1$ and $j = 2,3,\ldots, n$ where $j = tp$ and $t = 0,1,2,\ldots,$ then

(198) $$\underset{\sim}{P}\{\delta^+_{m,n} = \frac{1}{n} , \rho_{m,n} = j\} = \frac{1}{(j-1)} \frac{\binom{j+t-2}{t}}{\binom{m+n}{m}} \cdot$$
$$\cdot \left[\binom{m+n-j-t}{m-t} - \sum_{2\leq sp\leq n+1-j} \frac{2}{n+2-j-sp} \binom{sp+s-2}{s}\binom{m+n-j-t-sp-s+1}{m-t-s}\right] .$$

This result can be obtained by (196).

If, in particular, $k = 0$ and $j = 1$, then we have

$$\underset{\sim}{P}\{\delta^{+}_{m,n} = 0 \, , \, \rho_{m,n} = 1\} =$$

(199)

$$= \frac{1}{\binom{m+n}{m}} \left[\binom{m+n-1}{m} - \sum_{2 \leq sp \leq n} \frac{1}{n+1-sp} \binom{sp+s-2}{s} \binom{m+n-sp-s}{m-s} \right] .$$

Formulas (192) and (193) can be proved directly by using the random walk interpretation mentioned after Theorem 10.

The joint distribution of the random variables $\delta^{+}_{m,n}$ and $\rho_{m,n}$ in the particular case of $n = m$ was found in 1957 by I. Vincze [344], [345]. Theorem 12 was found in 1969 by the author [339].

Now we shall find the asymptotic distributions of the random variables $\delta^{+}_{m,n}$ and $\rho_{m,n}$ in the case when $n = mp$ and $p \to \infty$.

Theorem 13. *If* $n = mp$, *then*

$$\lim_{p\to\infty} \underset{\sim}{P}\left\{ \frac{\rho_{m,n}}{n} \leq x \right\} = x \tag{200}$$

for $0 \leq x \leq 1$ *and*

$$\lim_{p\to\infty} \underset{\sim}{P}\left\{ \delta^{+}_{m,n} \leq x \, , \, \frac{\rho_{m,n}}{n} \leq y \right\} = \sum_{k=1}^{[m(x+y)]} [G_k(my) - G_k(k-mx)] \tag{201}$$

for $0 \leq x \leq x+y \leq 1$ *where* $G_k(x)$ *is defined in Theorem 6.*

Proof. Without loss of generality we may assume that $F(x) \equiv G(x)$ is given by (74), that is, it is the distribution function of a random variable which has a uniform distribution over the interval $(0, 1)$.

Since by a theorem of V. Glivenko [259] it follows that

(202) $$\underset{\sim}{P}\{\lim_{n\to\infty} \sup_{-\infty<x<\infty} |G_n(x)-G(x)| = 0\} = 1 ,$$

we can conclude that if $n = mp$, then

(203) $$\underset{\sim}{P}\{\lim_{p\to\infty} \delta^+_{m,n} = \delta^+_m\} = 1$$

where

(204) $$\delta^+_m = \max_{1\leq r\leq m} (\frac{r}{m} - \xi^*_r)$$

defined by (90) and (92). Furthermore,

(205) $$\underset{\sim}{P}\{\lim_{p\to\infty} \frac{N_{\rho_{m,n}}}{p} = \rho^*_m\} = 1$$

where ρ^*_m is defined after formula (104).

Since by (170)

(206) $$\delta^+_{m,n} = \frac{1}{n} [N_{\rho_{m,n}} - \rho_{m,n} + 1] ,$$

it follows that

(207) $$\underset{\sim}{P}\{\lim_{p\to\infty} \frac{\rho_{m,n}}{n} = \frac{\rho^*_m}{m} - \delta^+_m\} = 1 .$$

Accordingly, we have

(208) $$\lim_{p\to\infty} \underset{\sim}{P}\{ \frac{\rho_{m,n}}{n} \leq x\} = \underset{\sim}{P}\{ \frac{\rho^*_m}{m} - \delta^+_m \leq x\}$$

and the right-hand side is given by Theorem 5. This proves (200).

By (203) and (207) we have

$$(209)\qquad \lim_{p\to\infty} \mathrm{P}\{\delta^{+}_{m,n} \leq x\ ,\ \frac{\rho_{m,n}}{n} \leq y\} = \mathrm{P}\{\delta^{+}_{m} \leq x\ ,\ \frac{\rho^{*}_{m}}{m} - \delta^{+}_{m} \leq y\}$$

and the right-hand side can be obtained by Theorem 6. Thus we get (201).

We note that if we do not make the assumption that $n = mp$, then (200) and (201) hold unchangeably whenever $n \to \infty$.

Finally, we note that I. Vincze [344] proved that

$$(210)\qquad \lim_{m\to\infty} \mathrm{P}\{\sqrt{\frac{m}{2}}\,\delta^{+}_{m,m} \leq x\ ,\ \frac{\rho_{m,m}}{m} \leq y\} = \sqrt{\frac{2}{\pi}}\int_0^x\int_0^y \frac{u^2}{[v(1-v)]^{3/2}}\, e^{-\frac{u^2}{2v(1-v)}}\,du\,dv$$

for $0 \leq x$ and $0 \leq y \leq 1$, and

$$(211)\qquad \lim_{m\to\infty} \mathrm{P}\{\sqrt{\frac{m^3}{2}}\,\frac{\delta^{+}_{m,m}}{\rho_{m,m}} \leq x\} = \frac{3}{2^{3/4}}\int_0^x e^{\frac{u^2}{4}}\sqrt{u}\, W_{-\frac{7}{4},\frac{1}{4}}\left(\frac{u^2}{2}\right)du$$

for $x > 0$ where

$$(212)\qquad W_{k,m}(z) = \frac{e^{-\frac{z}{2}} z^{k}}{\Gamma(m+\frac{1}{2}-k)}\int_0^\infty e^{-t}\, t^{m-\frac{1}{2}-k}\left(1+\frac{t}{z}\right)^{m-\frac{1}{2}+k} dt$$

is the Whittaker function defined for $\mathrm{Re}(m-k+\frac{1}{2}) > 0$. (See E. T. Whittaker [67].)

The distribution of $\sigma_{m,n}(a)$. If $n = mp$, where p is a positive integer, then $\sigma_{m,n}(a)$ can be interpreted as the number of subscripts $r = 1,2,\ldots, n$ for which

$$N_r = r - [a] - 1 \tag{213}$$

where $[a]$ is the greatest integer $\leq a$. This follows from (169).

Thus if $n = mp$, then we have

$$\underset{\sim}{P}\{\sigma_{m,n}(a) = k\} = \underset{\sim}{P}\{\sigma_{m,n}([a]) = k\} \tag{214}$$

for all a and $k = 0,1,\ldots, m$.

Furthermore, we have also

$$\underset{\sim}{P}\{\sigma_{m,n}(a) = k\} = \underset{\sim}{P}\{\sigma_{m,n}(-[a+1]) = k\} \tag{215}$$

for all a and $k = 0,1,\ldots, m$. For $N_{n+1} = mp$ and thus

$$\begin{aligned} \underset{\sim}{P}\{\sigma_{m,n}(a) = k\} &= \underset{\sim}{P}\{N_r = r-[a]-1 \text{ for } k \text{ subscripts } r = 1,2,\ldots, n\} = \\ &= \underset{\sim}{P}\{N_{n+1} - N_r = n+1-r+[a] \text{ for } k \text{ subscripts } r = 1,2,\ldots, n\} = \\ &= \underset{\sim}{P}\{N_i = i+[a] \text{ for } k \text{ subscripts } i = 1,2,\ldots, n\} \end{aligned} \tag{216}$$

which proves (215).

Accordingly, if $n = mp$ and if we know the distribution of $\sigma_{m,n}(a)$ for $a = 0,1,2,\ldots$, then by (214) and (215) we can find the distribution of $\sigma_{m,n}(a)$ for all a. Obviously, $\sigma_{m,n}(a) = 0$ if $a \geq n$. If $a = 0,1,\ldots, n$, then $\sigma_{m,n}(a)$ is a discrete random variable with possible values $k = 0,1,\ldots, [(mp-a)/p]$. Thus it is sufficient to determine the distribution of $\sigma_{m,n}(a)$ for $a = 0,1,\ldots, n$.

<u>Theorem 14. If</u> $n = mp$ <u>where</u> p <u>is a positive integer and</u> $a = 0,1,\ldots, mp$, <u>then we have</u>

$$\begin{aligned}(217)\quad \underset{\sim}{P}\{\sigma_{m,n}(a) \leq k\} &= 1-\frac{p^k}{\binom{mp+m}{m}}\sum_{\frac{a}{p}<j\leq m-k}\frac{k(p+1)+a+1}{(m-j)(p+1)+a+1}\binom{jp+j-a-1}{j}\binom{(m-j)(p+1)+a+1}{m-j-k} = \\ &= 1-\frac{p^k\binom{mp+m}{m-k}}{\binom{mp+m}{m}}+\frac{p^k}{\binom{mp+m}{m}}\sum_{0\leq j\leq\frac{a}{p}}\frac{k(p+1)+a+1}{(m-j)(p+1)+a+1}\binom{jp+j-a-1}{j}\binom{(m-j)(p+1)+a+1}{m-j-k}\end{aligned}$$

<u>for</u> $0 \leq k < (mp-a)/p$. <u>If, in particular,</u> $a = 0$, <u>then (217) reduces to</u>

$$(218)\qquad \underset{\sim}{P}\{\sigma_{m,n}(0) \leq k\} = 1-\frac{p^{k+1}\binom{mp+m}{m-k-1}}{\binom{mp+m}{m}}$$

<u>for</u> $0 \leq k < m$.

<u>Proof.</u> We shall determine the probability

$$(219)\qquad p_k(m,a) = \underset{\sim}{P}\{\sigma_{m,n}(a) > k\}$$

for $0 \leq k < (mp-a)/p$ and $a = 0,1,\ldots, mp$. By (216) we have

$$(220)\qquad p_k(m,a) = \underset{\sim}{P}\{N_i = i+a \text{ for more than } k \text{ subscripts } i = 1,2,\ldots, mp\} .$$

As we shall see $p_k(m,a)$ can be expressed by the following probabilities:

$$(221)\qquad q_k(s) = \underset{\sim}{P}\{N_i = i \text{ for } k \text{ subscripts } i = 1,2,\ldots, sp \mid N_{sp} = sp\}$$

for $1 \leq k \leq s \leq m$ and

$$(222) \qquad r_k(s,a) = \underset{\sim}{P}\{N_i = i+a \text{ for at least } k \text{ subscripts } i = 1,2,\ldots, sp-a-1 \mid N_{sp-a} = sp\}$$

for $0 \leq k < (sp-a)/p \leq (mp-a)/p$. Obviously, $r_o(s,a) = 1$ for $0 \leq a < sp \leq mp$.

We shall need the following result: If $0 \leq r \leq j \leq n+1$ and $\underset{\sim}{P}\{N_j = r\} > 0$, then

$$(223) \qquad \underset{\sim}{P}\{N_i < i \text{ for } i = 1,2,\ldots, j \mid N_j = r\} = 1 - \frac{r}{j} .$$

This result follows from Lemma 20.2. It can easily be proved by mathematical induction.

Now we can write that

$$(224) \qquad p_k(m,a) = \sum_{k+\frac{a}{p}<s\leq m} \frac{a+1}{(m-s)p+a+1} \underset{\sim}{P}\{N_{sp-a} = sp\} r_k(s,a)$$

for $0 \leq k < (mp-a)/p$ and $a = 0,1,\ldots, n$. For the event $\{N_i = i+a$ for more than k subscripts $i = 1,2,\ldots, n\}$ can occur in such a way that for some s where $a+kp < sp \leq mp$ we have $N_{sp-a} = sp$, further $N_i = i+a$ for at least k subscripts $i = 1,2,\ldots, sp-a-1$ and $N_i < i+a$ for $sp - a < i \leq n$. By using (223) and the fact that $N_{n+1} = n$, we obtain that

$$(225) \qquad \underset{\sim}{P}\{N_i < i+a \text{ for } sp-a < i \leq n \mid N_{sp-a} = sp\} = \frac{a+1}{n-sp+a+1}$$

for $a \leq sp \leq mp$. Hence (224) follows.

Furthermore, we have

$$r_k(s,a) = \sum_{k \leq u < s - \frac{a}{p}} \underset{\sim}{P}\{N_{up} = up \mid N_{sp-a} = sp\} q_k(u) \tag{226}$$

for $1 \leq k < (sp-a)/p$ and $a = 0,1,\ldots, sp$.

It follows immediately from the definition of $q_k(s)$ that

$$q_k(s) = \sum_{1 \leq u \leq s} \underset{\sim}{P}\{N_{up} = up \mid N_{sp} = sp\} q_1(u) q_{k-1}(s-u) \tag{227}$$

for $2 \leq k \leq s$ and

$$q_1(s) = 1 - \sum_{1 \leq u \leq s} \underset{\sim}{P}\{N_{up} = up \mid N_{sp} = sp\} q_1(u) \tag{228}$$

for $s \geq 1$.

In the above formulas we have

$$\underset{\sim}{P}\{N_{up} = up \mid N_{sp-a} = sp\} = \frac{\binom{up+u-1}{u}\binom{(s-u)(p+1)-a-1}{s-u}}{\binom{sp+s-a-1}{s}} \tag{229}$$

which follows from (162).

Accordingly, the problem of finding $p_k(m,a)$ can be reduced to the problem of finding $r_k(s,a)$ for $(a+kp)/p < s \leq m$, $q_k(s)$ for $k \leq s \leq m$ and $q_1(s)$ for $1 \leq s \leq m$. These probabilities can be determined by (224), (226), (227) and (228).

It will be convenient to use the following notation. Let

(230) $$P_k(m,a) = \binom{mp+m}{m} p_k(m,a) ,$$

(231) $$R_k(s,a) = \binom{sp+s-a-1}{s} r_k(s,a) ,$$

and

(232) $$Q_k(s) = \binom{sp+s-1}{s} q_k(s) .$$

It is easy to see that $Q_k(s)$ and $R_k(s,a)$ are independent of m whenever $1 \leq s \leq m$.

By using the above notation, equations (224), (226), (227) and (228) can also be expressed in the following way

(233) $$P_k(m,a) = \sum_{k+\frac{a}{p} < s \leq m} \frac{a+1}{(m-s)p+a+1} \binom{(m-s)(p+1)+a}{m-s} R_k(s,a)$$

for $0 \leq k < (mp-a)/p$ and $a = 0,1,\ldots, mp$,

(234) $$R_k(s,a) = \sum_{k \leq u < s-\frac{a}{p}} \binom{(s-u)(p+1)-a-1}{s-u} Q_k(u)$$

for $1 \leq k < (sp-a)/p$ and $a = 0,1,\ldots, sp$,

(235) $$Q_k(s) = \sum_{1 \leq u \leq s} Q_1(u) Q_{k-1}(s-u)$$

for $2 \leq k \leq s$ and

(236) $$Q_1(s) = \binom{sp+s-1}{s} - \sum_{1 \leq u \leq s} \binom{(s-u)(p+1)-1}{s-u} Q_1(u)$$

for $s \geq 1$.

To obtain $P_k(m,a)$ for $0 \leqq k < (mp-a)/p$ we should determine first $Q_k(s)$ for $1 \leqq k \leqq s$ and then $R_k(s,a)$ for $1 \leqq k < (sp-a)/p$ where $1 \leqq s \leqq m$. We shall determine these quantities by using generating functions.

First we shall derive some generating functions which we shall need in what follows.

By using Rouché's theorem we can prove that if $|z| < p^p/(p+1)^{p+1}$, then the equation

$$1 - w + zw^{p+1} = 0 \tag{237}$$

has a single root $w = \gamma(z)$ in the circle $|w - 1| < 1/p$ and if $g(w)$ is a regular function of w in this circle, then by Lagrange's expansion we obtain that

$$g(\gamma(z)) = g(1) + \sum_{r=1}^{\infty} \frac{z^r}{r!} \left[\frac{d^{r-1} g'(1+x)(1+x)^{rp+r}}{dx^{r-1}}\right]_{x=0} . \tag{238}$$

It follows immediately from (238) that

$$g(\gamma(z))\gamma'(z) = \sum_{r=0}^{\infty} \frac{z^r}{r!} \left[\frac{d^r g(1+x)(1+x)^{(r+1)(p+1)}}{dx^r}\right]_{x=0} . \tag{239}$$

If k is a nonnegative integer and a is any real or complex number, then by (238) we obtain that

$$[\gamma(z)]^a[\gamma(z)-1]^k = z^k + \sum_{r=k+1}^{\infty} \frac{kp+k+a}{r-k} \binom{rp+r+a-1}{r-k-1} z^r \tag{240}$$

and by (239) we obtain that

$$(241)\qquad [\gamma(z)]^a[\gamma(z)-1]^k\gamma'(z) = \sum_{r=k}^{\infty} \binom{(r+1)(p+1)+a}{r-k} z^r$$

for $|z| < p^p/(p+1)^{p+1}$.

We note further that

$$(242)\qquad \log\gamma(z) = \lim_{a\to 0} \frac{[\gamma(z)]^a-1}{a} = \sum_{r=1}^{\infty} \binom{rp+r}{r} \frac{z^r}{rp+r} ,$$

and hence

$$(243)\qquad \frac{p[\gamma(z)-1]}{1-p[\gamma(z)-1]} = \frac{pz\gamma'(z)}{\gamma(z)} = pz\,\frac{d\log\gamma(z)}{dz} = \sum_{r=1}^{\infty} \binom{rp+r-1}{r} z^r$$

for $|z| < p^p/(p+1)^{p+1}$. If $z \to p^p/(p+1)^{p+1}$, then $\gamma(z) \to (p+1)/p$.

Now let us find $Q_k(s)$ for $1 \leq k \leq s$. If we form the generating function of (236), then we obtain that

$$(244)\qquad \sum_{s=1}^{\infty} Q_1(s)z^s = \frac{\sum_{s=1}^{\infty} \binom{sp+s-1}{s} z^s}{1+\sum_{s=1}^{\infty} \binom{sp+s-1}{s} z^s} = p[\gamma(z)-1]$$

for $|z| < p^p/(p+1)^{p+1}$. Formula (244) follows from (243). If we form the generating function of (235), then we obtain that

$$(245)\qquad \sum_{s=k}^{\infty} Q_k(s)z^s = \Big(\sum_{s=1}^{\infty} Q_1(s)z^s\Big)^k = p^k[\gamma(z)-1]^k$$

for $k = 1,2,\ldots,$ and $|z| < p^p/(p+1)^{p+1}$. Thus by (240) we get

$$Q_k(s) = \binom{sp+s}{s-k} \frac{kp^k}{s} \tag{246}$$

for $1 \leq k \leq s$.

Next we shall prove that

$$\begin{aligned} R_k(s,a) &= kp^k \sum_{\frac{a}{p} < j \leq s-k} \frac{1}{(s-j)} \binom{(s-j)(p+1)}{s-j-k} \binom{jp+j-a-1}{j} = \\ &= p^k \binom{sp+s-a-1}{s-k} - kp^k \sum_{0 \leq j \leq \frac{a}{p}} \frac{1}{(s-j)} \binom{(s-j)(p+1)}{s-j-k} \binom{jp+j-a-1}{j} \end{aligned} \tag{247}$$

for $1 \leq k < (sp-a)/p$.

If we form the generating function of (234), then we obtain that

$$\sum_{k+\frac{a}{p} < s} R_k(s,a) z^s = \left(\sum_{u=k}^{\infty} Q_k(u) z^u \right) \left(\sum_{\frac{a}{p} < j} \binom{jp+j-a-1}{j} z^j \right) \tag{248}$$

for $|z| < p^p/(p+1)^{p+1}$. The first expression for $R_k(s,a)$ in (247) follows from (246) and (248).

If we take into consideration that by (241)

$$\sum_{j=0}^{\infty} \binom{jp+j-a-1}{j} z^j = [\gamma(z)]^{-a-p-2} \gamma'(z) \tag{249}$$

for $|z| < p^p/(p+1)^{p+1}$, then it follows from (248) that

$$\sum_{k+\frac{a}{p}<s} R_k(s,a)z^s = p^k[\gamma(z)]^{-a-p-2}[\gamma(z)-1]^k\gamma'(z) -$$

(250)
$$- \Big(\sum_{u=k}^{\infty} Q_k(u)z^u\Big)\Big(\sum_{0\leq j\leq \frac{a}{p}} \binom{jp+j-a-1}{j} z^j\Big)$$

for $|z| < p^p/(p+1)^{p+1}$. If we use (241) and (246), and form the coefficient of z^s in (250), then we obtain the second expression for $R_k(s,a)$ in (247).

We note that by definition

(251)
$$R_0(s,a) = \binom{sp+s-a-1}{s}$$

for $0 \leq a < sp$.

Finally, we are in a position to prove that

$$P_k(m,a) = p^k \sum_{\frac{a}{p}<j\leq m-k} \frac{k(p+1)+a+1}{(m-j)(p+1)+a+1}\binom{jp+j-a-1}{j}\binom{(m-j)(p+1)+a+1}{m-j-k} =$$

(252)
$$= p^k\binom{mp+m}{m-k} - p^k \sum_{0\leq j\leq \frac{a}{p}} \frac{k(p+1)+a+1}{(m-j)(p+1)+a+1}\binom{jp+j-a-1}{j}\binom{(m-j)(p+1)+a+1}{m-j-k}$$

for $0 \leq k < (mp-a)/p$.

Let us form the generating function of (233). Then we obtain that

(253)
$$\sum_{k+\frac{a}{p}<m} P_k(m,a)z^m = \Big(\sum_{k+\frac{a}{p}<s} R_k(s,a)z^s\Big)\Big(\sum_{s=0}^{\infty} \frac{a+1}{sp+a+1}\binom{sp+s+a}{s} z^s\Big)$$

for $|z| < p^p/(p+1)^{p+1}$. Here by (245) and (248)

$$(254)\qquad \sum_{k+\frac{a}{p}<s} R_k(s,a)z^s = p^k[\gamma(z)-1]^k \sum_{\frac{a}{p}<j} \binom{jp+j-a-1}{j} z^j$$

for $k = 1,2,\ldots$. If $k = 0$, then (254) is trivially true. Furthermore, by (240) we have

$$(255)\qquad \sum_{s=0}^{\infty} \frac{a+1}{sp+a+1} \binom{sp+s+a}{s} z^s = 1 + \sum_{s=1}^{\infty} \frac{a+1}{s} \binom{sp+s+a}{s-1} z^s = [\gamma(z)]^{a+1} .$$

Thus by (253), (254) and (255) we get

$$(256)\qquad \sum_{k+\frac{a}{p}<m} P_k(m,a)z^m = p^k[\gamma(z)]^{a+1}[\gamma(z)-1]^k \sum_{\frac{a}{p}<j} \binom{jp+j-a-1}{j} z^j$$

for $|z| < p^p/(p+1)^{p+1}$. If we make use of (240) and form the coefficient of z^m in (256), then we obtain the first expression for $P_k(m,a)$ in (252).

If in (256) we write

$$(257)\qquad \sum_{\frac{a}{p}<j} \binom{jp+j-a-1}{j} z^j = [\gamma(z)]^{-a-p-2}\gamma'(z) - \sum_{0\leq j\leq \frac{a}{p}} \binom{jp+j-a-1}{j} z^j ,$$

which follows from (249), then we can obtain the second expression for $P_k(m,a)$ in (252).

By (252) and (230) we obtain $\underset{\sim}{P}\{\sigma_{m,n}(a) \leq k\} = 1-p_k(m,a)$ for $0 \leq k < (mp-a)/p$ and $a = 0,1,\ldots, mp$. **This completes the proof of Theorem 14.**

In the particular case when $n = m$, the distribution of $\sigma_{m,m}(a)$ was found in 1952 by V. S. Mihalevich [298]. He showed that if $a = 0,1,\ldots, m$, then

$$(258) \qquad P\{\sigma_{m,m}(a) \leq k\} = P\{\delta^{+}_{m,m} \leq \frac{k+a}{m}\} = 1 - \frac{\binom{2m}{m+k+a+1}}{\binom{2m}{m}}$$

for $k = 0,1,2,\ldots, m-a$. The distribution of $\sigma_{m,n}(a)$ for $n = mp$ where p is a positive integer was found in 1970 by the author [340].

Theorem 15. If $a \geq 0$ and $n = mp$, then

$$(259) \qquad \begin{aligned} \lim_{p\to\infty} P\{\sigma_{m,n}(ap) \leq k\} &= 1 - \frac{(a+k)m!}{m^m} \sum_{a<j\leq m-k} \frac{(j-a)^j(m-j+a)^{m-j-k-1}}{j!(m-j-k)!} = \\ &1 - \frac{m!}{(m-k)!m^k} + \frac{(a+k)m!}{m^m} \sum_{0\leq j\leq a} \frac{(j-a)^j(m-j+a)^{m-j-k-1}}{j!(m-j-k)!} \end{aligned}$$

for $0 \leq k < m-a$. If, in particular, $a = 0$, then (259) reduces to

$$(260) \qquad \lim_{p\to\infty} P\{\sigma_{m,mp}(0) \leq k\} = 1 - \frac{m!}{(m-k-1)!m^{k+1}}$$

for $0 \leq k < m$.

Proof. Since

$$(261) \qquad P\{\sigma_{m,n}(ap) \leq k\} = P\{\sigma_{m,n}([ap]) \leq k\}$$

if $n = mp$, $a \geq 0$ and $k = 0,1,2,\ldots,$ the limit relations can be obtained immediately from (217) and (218) if we replace a by $[ap]$ and let $p \to \infty$.

By (217) we can easily determine the limiting distribution of $\sigma_{m,mp}(ap)$ as $p \to \infty$ when a is a nonnegative real number.

Note. By using Theorem 15 we can provide a new proof for Theorem 9. In Theorem 9 the random variable $\sigma_m(a)$ denotes the number of intersections of $F(x) \equiv G(x)$ with $F_m(x) + \frac{a}{m}$ for $-\infty < x < \infty$. More precisely, $\sigma_m(a) = k$ if the set $S_a = \{x : G(x) = F_m(x) + \frac{a}{m}$ and $-\infty \leq x \leq \infty\}$ is the union of k separated intervals or points. Since by a theorem of V. Glivenko [259] we have

$$(262) \qquad \underset{\sim}{P}\{\lim_{n\to\infty} \sup_{-\infty<x<\infty} |G_n(x) - G(x)| = 0\} = 1 ,$$

we can conclude that

$$(263) \qquad \underset{\sim}{P}\{\sigma_m(a) \leq k\} = \lim_{p\to\infty} \underset{\sim}{P}\{\sigma_{m,mp}(ap) \leq k\}$$

for $0 \leq k < m-a$ and $0 < a < m$. This proves (129) for $a > 0$. If $a = 0$, then we have

$$(264) \qquad \underset{\sim}{P}\{\sigma_m(0) \leq k\} = \lim_{a\to 0} \underset{\sim}{P}\{\sigma_m(a) \leq k-1\}$$

for $1 \leq k \leq m$. For if we suppose that $a > 0$ and let $a \to 0$ in S_a, then we obtain every interval or point in S_0 except one which contains $x = \infty$. Formula (264) implies that

$$(265) \qquad \underset{\sim}{P}\{\sigma_m(0) \leq k\} = 1 - \frac{m!}{(m-k)!m^k}$$

for $1 \leq k \leq m$. This proves (129) for $a = 0$.

Finally, we shall determine the asymptotic distribution of $\sigma_{m,mp}(y\sqrt{mp(p+1)})$ in the case when $m \to \infty$.

Theorem 16. If $x \geqq 0$ and $y \geqq 0$, then

$$\lim_{m\to\infty} \underset{\sim}{P}\{\sigma_{m,mp}(y\sqrt{mp(p+1)}) \leqq x\sqrt{mp/(p+1)}\} = \tag{266}$$
$$= 1 - e^{-\frac{(x+2y)^2}{2}}$$

for any $p = 1,2,\ldots$.

Proof. Now

$$\underset{\sim}{P}\{\sigma_{m,mp}(a) \leqq k\} = \underset{\sim}{P}\{\sigma_{m,mp}([a]) \leqq k\} \tag{267}$$

is given explicitly by (217). If in the first formula on the right-hand side of (217) we put $a = [y\sqrt{mp(p+1)}]$ and $k = [x\sqrt{mp/(p+1)}]$, $j = mu$ and let $m \to \infty$, then we obtain that

$$\lim_{m\to\infty} \underset{\sim}{P}\{\sigma_{m,mp}(a) \leqq k\} = 1 - \frac{(x+y)}{\sqrt{2\pi}} \int_0^1 \frac{e^{-\frac{1}{2}[\frac{(x+y)^2}{1-u} + \frac{y^2}{u}]}}{(1-u)^{3/2}\, u^{1/2}}\, du \tag{268}$$

for $x \geqq 0$ and $y \geqq 0$. If we evaluate the integral on the right-hand side of (268), then we obtain (266) which was to be proved.

Theorem 16 is a particular case of a limit theorem of N. V. Smirnov [327]. According to the result of Smirnov, (266) is valid for any non-negative real p. Smirnov's result is given by formula (85) in this section.

40. Problems

40.1. Two players, A and B, play a series of games. In each game, independently of the others, either A wins a counter from B with probability p or B wins a counter from A with probability q where $p > 0$, $q > 0$ and $p + q = 1$. The series ends if either A wins a total number of a counters from B, or B wins a total number of b counters from A. Denote by $p_n(a,b)$ the probability that A wins the series in exactly n games. Determine the generating function of the sequence $\{p_n(a,b)$, $n = 1,2,\ldots\}$. (See P. S. Laplace [39 p. 228].)

40.2. Two players, A and B, play a series of games. In each game, independently of the others, either A wins a counter from B with probability p, or B wins a counter from A with probability q where $p > 0$, $q > 0$ and $p + q = 1$. The series ends if A wins a total number of a counters from B. Denote by ρ the duration of the games. Determine the generating function of ρ. (See P. S. Laplace [39 p. 229].)

40.3. Prove that

$$\frac{1}{\sqrt{(n+\frac{1}{2})\pi}} < \binom{2n}{n} < \frac{1}{\sqrt{n\pi}}$$

for $n = 1,2,\ldots$.

40.4. Let

$$s_k = \sum_{i=1}^{n} \omega_i^k$$

for $k = 0,1,\ldots$ and $a_0 = 1$ and

$$a_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \omega_{i_1} \omega_{i_2} \cdots \omega_{i_k}$$

for $k = 1,2,\ldots, n$. Prove that

$$(-1)^k a_k = \sum_{i_1+2i_2+\ldots+ki_k=k} \frac{(-1)^{i_1+\ldots+i_k} s_1^{i_1} s_2^{i_2} \ldots s_k^{i_k}}{i_1! i_2! \ldots i_k! \, 2^{i_2} 3^{i_3} \ldots k^{i_k}}$$

for $k = 1,2,\ldots, n$.

40.5. Let $\xi_1, \xi_2, \ldots, \xi_m$ be mutually independent real random variables having the same distribution function $F(x)$. Denote by $F_m(x)$ the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$, that is, $F_m(x)$ is equal to the number of variables $\le x$ divided by m . Prove that if $F(x)$ is a continuous distribution function, then

$$\delta_m^+ = \sup_{-\infty<x<\infty} [F_m(x) - F(x)] \quad \text{and} \quad \delta_m^- = \sup_{-\infty<x<\infty} [F(x) - F_m(x)]$$

have the same distribution function.

40.6. Let $(\xi_1, \xi_2, \ldots, \xi_m)$ and $(\eta_1, \eta_2, \ldots, \eta_n)$ be independent sequences of mutually independent real random variables with distribution functions $F(x)$ and $G(x)$ respectively. Denote by $F_m(x)$ and $G_n(x)$ the empirical distribution functions of the samples $(\xi_1, \xi_2, \ldots, \xi_m)$ and $(\eta_1, \eta_2, \ldots, \eta_n)$ respectively. Prove that if $F(x)$ and $G(x)$ are identical continuous distribution functions, then

$$\delta^+_{m,n} = \sup_{-\infty<x<\infty} [F_m(x) - G_n(x)] \quad \text{and} \quad \delta^-_{m,n} = \sup_{-\infty<x<\infty} [G_n(x) - F_m(x)]$$

have the same distribution function.

40.7. Find the distribution and the moments of the random variables $\xi_1^*, \xi_2^*, \ldots, \xi_m^*$ in the solution of Problém 40.5 in the case where $F(x) = x$ for $0 \leqq x \leqq 1$.

40.8. Find the distribution and the moments of the random variables $N_1, N_2, \ldots, N_n$ in the solution of Problem 40.6.

40.9. Let $\xi_1, \xi_2, \ldots, \xi_m$ be mutually independent real random variables having the same distribution function $F(x)$. Denote by $F_m(x)$ the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$. Determine the joint distribution function of

$$\delta^+_m = \sup_{-\infty<x<\infty}[F_m(x) - F(x)] \quad \text{and} \quad \delta^-_m = \sup_{-\infty<x<\infty} [F(x) - F_m(x)]$$

in the case where $F(x)$ is a continuous distribution function. (See K. Sarkadi [320], S. G. Mohanty [302], G. D. Steck [459], and E. J. G. Pitman [448].)

40.10. Let $\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots, \eta_n$ be real random variables. Denote by $F_m(x)$ the empirical distirbution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$ and by $G_n(x)$ the empirical distribution function of the sample $(\eta_1, \eta_2, \ldots, \eta_n)$. Determine the joint distribution function of

$$\delta^+_{m,n} = \sup_{-\infty<x<\infty} [F_m(x) - G_n(x)] \quad \text{and} \quad \delta^-_{m,n} = \sup_{-\infty<x<\infty} [G_n(x) - F_m(x)]$$

in the case where $\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots, \eta_n$ are mutually independent random variables having the same continuous distribution function.

40.11. Prove that (39.79) holdsif $m \to \infty$ and $n \to \infty$ in an arbitrary way.

40.12. Let $\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots, \eta_n$ be mutually independent random variables having the same continuous distribution function. Denote by $F_m(x)$ the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$ and by $G_n(x)$ the empirical distribution function of the sample $(\eta_1, \eta_2, \ldots, \eta_n)$. Define

$$\delta^+_{m,n}(\alpha,\beta) = \sup_{\alpha \leq G_n(x) \leq \beta} [F_m(x) - G_n(x)]$$

for $0 \leq \alpha < \beta \leq 1$. Find the asymptotic distribution of $\delta^+_{m,n}(0,\alpha)$, $\delta^+_{m,n}(\alpha,\beta)$, $\delta^+_{m,n}(\beta,1)$ as $m \to \infty$ and $n \to \infty$. (See E. L. Rvacheva [454].)

40.13. Consider Problem 40.12 and determine the limit

$$\lim_{\substack{m \to \infty \\ n \to \infty}} \mathbf{P}\{\delta^+_{m,n}(\alpha,\beta) \leq 0\}$$

for $0 < \alpha < \beta < 1$. (See B. V. Gnedenko [260] and I. I. Gikhman [431].)

40.14. Let $\xi_1, \xi_2, \ldots, \xi_m$ be mutually independent random variables having the same continuous distribution function $F(x)$. Denote by $F_m(x)$ the empirical distribution function of the sample $(\xi_1, \xi_2, \ldots, \xi_m)$. Define

$$\delta_m^+(\alpha,\beta) = \sup_{\alpha \leq F(x) \leq \beta} [F_m(x) - F(x)]$$

where $0 \leq \alpha < \beta \leq 1$. Find the limiting distribution of $\sqrt{m}\,\delta_m^+(\alpha,\beta)$ as $m \to \infty$. (See G. M. Maniya [293].)

40.15. Under the assumptions of Problem 40.14 let

$$\mu_m^+(\alpha,\beta) = \sup_{\alpha \leq F(x) \leq \beta} \left[\frac{F_m(x) - F(x)}{F(x)}\right]$$

for $0 < \alpha < \beta \leq 1$. Find the limiting distribution of $\sqrt{m}\,\mu_m^+(\alpha,1)$ as $m \to \infty$. (See A. Rényi [314].)

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