CHAPTER III

POSITIVE PARTIAL SUMS

22. An Equivalence Theorem. First we shall prove a useful basic theorem which we shall use not only in this chapter but in the subsequent chapters too. We shall formulate this theorem a little more generally than we need in this chapter. This theorem was found in 1953 by E. S. Andersen [2] and a simple proof for it was given in 1959 by W. Feller [23].

Theorem 1. Let $\xi_1, \xi_2, \ldots, \xi_n$ be interchangeable real random variables. Define $\zeta_r = \xi_1 + \xi_2 + \ldots + \xi_r$ for $r = 1,2,\ldots, n$ and $\zeta_0 = 0$. Denote by Δ_n the number of positive partial sums $\zeta_1, \zeta_2, \ldots, \zeta_n$ and by ρ_n the subscript of the first maximal element in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$. Denote by Δ_n^* the number of nonnegative partial sums $\zeta_1, \zeta_2, \ldots, \zeta_n$, and by ρ_n^* the subscript of the last maximal element in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$.

We have

$$(1) \qquad \mathbf{P}\{\Delta_n = j \,,\, \zeta_n \leq x\} = \mathbf{P}\{\rho_n = j \,,\, \zeta_n \leq x\}$$

and

$$(2) \qquad \mathbf{P}\{\Delta_n^* = j \,,\, \zeta_n \leq x\} = \mathbf{P}\{\rho_n^* = j \,,\, \zeta_n \leq x\}$$

for $j = 0,1,\ldots, n$ and all x.

Proof. It is sufficient to prove one of the two relations (1) and (2) because each one implies the other. For if we apply (1) to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n$, then we obtain (2), and if we apply (2) to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n$, then we obtain (1). This can easily be seen if we take into consideration that for the sequence $-\xi_1, -\xi_2, \ldots, -\xi_n$ the number of positive partial sums is $n-\Delta_n^*$, and the number of nonnegative partial sums is $n-\Delta_n$, and for the sequence $-\xi_n, -\xi_{n-1}, \ldots, -\xi_1$ the subscript of the first maximal partial sum is $n-\rho_n^*$, and the subscript of the last maximal partial sum is $n-\rho_n$.

Now we shall prove (1). If $n=1$, then $\Delta_1 = \rho_1$ and thus (1) holds. We shall prove by mathematical induction that (1) holds for all $n = 1,2,\ldots$. Let us suppose that for n $(n = 2,3,\ldots)$, the vector random variables $(\Delta_{n-1}, \zeta_{n-1})$ and $(\rho_{n-1}, \zeta_{n-1})$ have the same distribution. This implies that $(\Delta_{n-1}, \zeta_{n-1}, \zeta_n)$ and $(\rho_{n-1}, \zeta_{n-1}, \zeta_n)$ have also the same distribution. For Δ_{n-1} and ρ_{n-1} depend only on $\xi_1, \xi_2, \ldots, \xi_{n-1}$ and these random variables are conditionally interchangeable given ζ_{n-1} and ζ_n. Hence it follows that (Δ_{n-1}, ζ_n) and (ρ_{n-1}, ζ_n) have also the same distribution.

Let $x \leqq 0$ and $j = 0,1,\ldots, n-1$. Then we have

$$\underset{\sim}{P}\{\Delta_n = j \ , \ \zeta_n \leqq x\} = \underset{\sim}{P}\{\Delta_{n-1} = j \ , \ \zeta_n \leqq x\} . \tag{3}$$

For if $\zeta_n \leqq 0$, then the n-th partial sum cannot be positive and therefore $\Delta_n = \Delta_{n-1}$. Furthermore, we have

(4) $$P\{\rho_n = j\ ,\ \zeta_n \leq x\} = P\{\rho_{n-1} = j\ ,\zeta_n \leq x\}\ .$$

For if $\zeta_n \leq 0$, then the first maximum cannot occur at the n-th place (being $\zeta_0 = 0$) and therefore $\rho_n = \rho_{n-1}$. By the induction hypothesis, the right-hand sides of (3) and (4) are equal and hence

(5) $$P\{\Delta_n = j\ ,\ \zeta_n \leq x\} = P\{\rho_n = j\ ,\ \zeta_n \leq x\}$$

for $x \leq 0$ and $j = 0,1,2,\ldots, n$. If $j = n$, then both sides of (5) are evidently 0 .

Let $x \geq 0$ and $j = 1,2,\ldots, n$. Then we have

(6) $$P\{\Delta_n = j\ ,\ \zeta_n > x\} = P\{\Delta_{n-1} = j-1\ ,\ \zeta_n > x\}\ .$$

For if $\zeta_n > 0$, then $\Delta_n = \Delta_{n-1}+1$. Furthermore, we have

(7) $$P\{\rho_n = j\ ,\ \zeta_n > x\} = P\{\rho_{n-1} = j-1\ ,\ \zeta_n > x\}\ .$$

For $n-\rho_n$ can be interpreted as the subscript of the last maximal element in the partial sums of $-\xi_n, -\xi_{n-1},\ldots, -\xi_1$. If $-\zeta_n < 0$, then the last maximum cannot occur at the n-th place and thus $n-\rho_n = n-1-\rho_{n-1}$, that is, $\rho_n = \rho_{n-1}+1$. By the induction hypothesis the right-hand sides of (6) and (7) are equal, and hence

(8) $$P\{\Delta_n = j\ ,\ \zeta_n > x\} = P\{\rho_n = j\ ,\ \zeta_n > x\}$$

for $x \geq 0$ and $j = 0,1,2,\ldots, n$. If $j = 0$, then both sides of (8) are evidently 0 .

Now (5) and (8) imply that (Δ_n, ζ_n) and (ρ_n, ζ_n) have identical distributions. This completes the proof of (1), and (2) follows from (1).

In 1961 A. Brandt [14] generalized Theorem 1. We shall prove this generalization in the following version.

Theorem 2. Let $\xi_1, \xi_2, \ldots, \xi_n$ be interchangeable random variables. Define $\zeta_r = \xi_1 + \xi_2 + \ldots + \xi_r$ for $r = 1, 2, \ldots, n$ and $\zeta_0 = 0$. Denote by $\Delta_n(c)$ the number of partial sums greater than c in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$, and by $\Delta_n^*(c)$ the number of partial sums greater than or equal to c in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$. Denote by $\rho_n(c)$ the smallest subscript $r = 0, 1, \ldots, n$ for which $\zeta_r \geqq \max(\zeta_0, \zeta_1, \ldots, \zeta_n) - c$ and by $\rho_n^*(c)$ the largest subscript $r = 0, 1, \ldots, n$ for which $\zeta_r \geqq \max(\zeta_0, \zeta_1, \ldots, \zeta_n) - c$.

If $c \geqq 0$, then we have

$$(9) \qquad \mathbf{P}\{\Delta_n(c) = j, \ \zeta_n \leqq x\} = \mathbf{P}\{\rho_n(c) = j, \ \zeta_n \leqq x\}$$

and

$$(10) \qquad \mathbf{P}\{\Delta_n^*(-c) = j, \ \zeta_n \leqq x\} = \mathbf{P}\{\rho_n^*(c) = j, \ \zeta_n \leqq x\}$$

for $j = 0, 1, \ldots, n$ and all x.

Proof. If $c = 0$, then Theorem 2 reduces to Theorem 1. It is sufficient to prove one of the two relations (9) and (10) because each one implies the other. For if we apply (9) to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n$, then we obtain (10), and if we apply (10) to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n$, then we obtain (9). This can easily be seen if we take into consideration that for the sequence $-\xi_1, -\xi_2, \ldots, -\xi_n$ the number of partial sums greater than c is $n-\Delta_n^*(-c)$, and the number of partial sums greater than or equal to c is $n-\Delta_n(-c)$, and for the sequence $-\xi_n, -\xi_{n-1}, \ldots, -\xi_1$ the subscript of the first partial sum greater than or equal to the maximal partial sum minus c is $n-\rho_n^*(c)$, and the subscript of the last partial sum greater than or equal to the maximal partial sum minus c is $n-\rho_n(c)$.

Now we shall prove (9). If $n = 1$, then $\Delta_1(c) = \rho_1(c)$ for $c \geqq 0$ and thus (9) holds. We shall prove by mathematical induction that (9) holds for all $n = 1,2,\ldots$. Let us suppose that for n $(n = 2,3,\ldots)$ the vector random variables $(\Delta_{n-1}(c), \zeta_{n-1})$ and $(\rho_{n-1}(c), \zeta_{n-1})$ have the same distribution. This implies that $(\Delta_{n-1}(c), \zeta_{n-1}, \zeta_n)$ and $(\rho_{n-1}(c), \zeta_{n-1}, \zeta_n)$ have also the same distribution. For $\Delta_{n-1}(c)$ and $\rho_{n-1}(c)$ depend only on $\xi_1, \xi_2, \ldots, \xi_{n-1}$ and these random variables are conditionally interchangeable given ζ_{n-1} and ζ_n. Hence it follows that $(\Delta_{n-1}(c), \zeta_n)$ and $(\rho_{n-1}(c), \zeta_n)$ have also the same distribution.

Let $x \leqq c$ and $j = 0,1,\ldots, n-1$. Then we have

$$\underset{\sim}{P}\{\Delta_n(c) = j, \zeta_n \leqq x\} = \underset{\sim}{P}\{\Delta_{n-1}(c) = j, \zeta_n \leqq x\}. \tag{11}$$

For if $\zeta_n \leqq x \leqq c$, then the n-th partial sum cannot be greater than c and therefore $\Delta_n(c) = \Delta_{n-1}(c)$. Furthermore, we have

$$\underset{\sim}{P}\{\rho_n(c) = j, \zeta_n \leqq x\} = \underset{\sim}{P}\{\rho_{n-1}(c) = j, \zeta_n \leqq x\}. \tag{12}$$

For $\zeta_0 = 0$ and if $\zeta_n \leq x \leq c$ then $\zeta_n - c \leq 0$. Thus the smallest subscript $r = 0,1,\ldots, n$ for which $\zeta_r \geq \max(\zeta_0, \zeta_1,\ldots, \zeta_n)-c$ cannot be $r = n$. Therefore $\rho_n(c) = \rho_{n-1}(c)$.

By the induction hypothesis, the right-hand sides of (11) and (12) are equal and hence

$$\underset{\sim}{P}\{\Delta_n(c) = j\ ,\ \zeta_n \leq x\} = \underset{\sim}{P}\{\rho_n(c) = j\ ,\ \zeta_n \leq x\} \tag{13}$$

for $x \leq c$ and $j = 0,1,\ldots, n$. If $j = n$, then both sides of (13) are evidently 0.

Let $x \geq c$ and $j = 1,2,\ldots, n$. Then we have

$$\underset{\sim}{P}\{\Delta_n(c) = j\ ,\ \zeta_n > x\} = \underset{\sim}{P}\{\Delta_{n-1}(c) = j-1\ ,\ \zeta_n > x\}\ . \tag{14}$$

For if $\zeta_n > x \geq c$, then $\Delta_n(c) = \Delta_{n-1}(c)+1$. Furthermore, we have

$$\underset{\sim}{P}\{\rho_n(c) = j\ ,\ \zeta_n > x\} = \underset{\sim}{P}\{\rho_{n-1}(c) = j\ ,\ \zeta_n > x\}\ . \tag{15}$$

For $n-\rho_n(c)$ can be interpreted as the largest subscript $r = 0,1,\ldots, n$ for which $\bar{\zeta}_r \geq \max(\bar{\zeta}_0, \bar{\zeta}_1,\ldots, \bar{\zeta}_n)-c$ where $\bar{\zeta}_r = -\xi_n-\xi_{n-1}-\ldots-\xi_{n-r+1}$ for $r = 1,2,\ldots, n-1$ and $\bar{\zeta}_0 = 0$. If $\bar{\zeta}_n = -\zeta_n < -c$, then this largest subscript cannot be $r = n$, and thus $n-\rho_n(c) = n-1-\rho_{n-1}(c)$, that is, $\rho_n(c) = \rho_{n-1}(c)+1$.

By the induction hypothesis, the right-hand sides of (14) and (15) are equal and hence

$$\underset{\sim}{P}\{\Delta_n(c) = j,\ \zeta_n > x\} = \underset{\sim}{P}\{\rho_n(c) = j,\ \zeta_n > x\} \tag{16}$$

for $x \geqq c$ and $j = 0,1,\ldots, n$. If $j = 0$, then both sides of (16) are evidently 0.

Now (13) and (16) imply that $(\Delta_n(c), \zeta_n)$ and $(\rho_n(c), \zeta_n)$ have identical distributions. This completes the proof of (9), and (10) follows from (9).

Finally, we note that in 1961 E. S. Andersen [6] generalized Theorem 1 in another way.

23. <u>The Distribution of the Number of Positive Partial Sums.</u> Now let us suppose that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of mutually independent and identically distributed real random variables. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1, 2, \ldots$ and $\zeta_0 = 0$.

Let us denote by Δ_n the number of <u>positive</u> partial sums among $\zeta_1, \zeta_2, \ldots, \zeta_n$ and by Δ_n^* the number of <u>nonnegative</u> partial sums among $\zeta_1, \zeta_2, \ldots, \zeta_n$. Let $\Delta_0 = \Delta_0^* = 0$.

Denote by ρ_n the subscript of the <u>first</u> maximal element in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$ and by ρ_n^* the subscript of the <u>last</u> maximal element in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$.

For any event A let us denote by $\delta(A)$ the indicator variable of A, that is, $\delta(A) = 1$ if A occurs and $\delta(A) = 0$ if A does not occur.

Let us introduce the following notation:

$$V_{nk}(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_n = k)\} \tag{1}$$

and

$$V_{nk}^*(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_n^* = k)\} \tag{2}$$

for $\mathrm{Re}(s) = 0$ and $0 \leq k \leq n$.

The joint distribution of the random variables ζ_n and Δ_n is uniquely determined by $V_{nk}(s)$ for $k = 0, 1, \ldots, n$ and the joint distribution of the random variables ζ_n and Δ_n^* is uniquely determined by $V^*_{nk}(s)$ for $k = 0,1,\ldots, n$. Our next aim is to find (1) and (2). The solutions of these problems were given in 1953 by E. S. Andersen [3],[5], in 1961 by G. Baxter [10] and in 1962 by D. A. Darling [20].

First, we shall show that if we know $V_{n0}(s)$ and $V_{nn}(s)$ for $n = 0,1,2,\ldots,$ then $V_{nk}(s)$ can be obtained immediately for $0 \leqq k \leqq n$, and similarly if we know $V^*_{n0}(s)$ and $V^*_{nn}(s)$ for $n = 0,1,2,\ldots,$ then $V^*_{nk}(s)$ can be obtained immediately for $0 \leqq k \leqq n$.

Theorem 1. *We have*

$$V_{nk}(s) = V_{kk}(s)V_{n-k,0}(s) \tag{3}$$

and

$$V^*_{nk}(s) = V^*_{kk}(s)V^*_{n-k,0}(s) \tag{4}$$

for $Re(s) = 0$ *and* $0 \leqq k \leqq n$.

Proof. The case of $n = 0$ is trivially true. Let $n \geqq 1$. By Theorem 22.1 we can write that

$$V_{nk}(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\rho_n = k)\} \tag{5}$$

for $Re(s) = 0$ and $0 \leqq k \leqq n$.

Let us define $\overline{\rho}_{n-k}$ as the subscript of the first maximal element in the sequence $\overline{\zeta}_i = \zeta_{k+i} - \zeta_k$ $(i = 0,1,\ldots, n-k)$. Then we can write that

$$\delta(\rho_n = k) = \delta(\rho_k = k)\delta(\overline{\rho}_{n-k} = 0) \tag{6}$$.

For $\rho_n = k$ if and only if $\zeta_i < \zeta_k$ for $0 \leq i < k$ and $\zeta_i \leq \zeta_k$ for $i \leq k \leq n$. Hence it follows that

$$(7) \qquad e^{-s\zeta_n}\delta(\rho_n = k) = [e^{-s\zeta_k}\delta(\rho_k = k)][e^{-s\overline{\zeta}_{n-k}}\delta(\overline{\rho}_{n-k} = 0)] .$$

The two factors in brackets on the right-hand side of (7) are independent, and the second factor has the same distribution as $e^{-s\zeta_{n-k}}\delta(\rho_{n-k} = 0)$. Thus if we form the expectation of (7), then we obtain (3).

We can prove (4) in a similar way. We can also obtain (4) from (3), if we apply (3) to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n$. Denote by $\overline{\Delta}_n$ the number of positive elements in the sequence $-\zeta_0, -\zeta_1, \ldots, -\zeta_n$ $(n = 0,1,2,\ldots)$. Obviously $\overline{\Delta}_n = n-\Delta_n^*$ for $n = 0,1,\ldots$. Now by (3) we can write that

$$(8) \qquad \begin{aligned} V_{nk}^*(s) &= \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_n^* = k)\} = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\overline{\Delta}_n = n-k)\} = \\ &= \underset{\sim}{E}\{e^{-s\zeta_{n-k}}\delta(\overline{\Delta}_{n-k} = n-k)\}\underset{\sim}{E}\{e^{-s\zeta_k}\delta(\overline{\Delta}_k = 0)\} = \\ &= \underset{\sim}{E}\{e^{-s\zeta_{n-k}}\delta(\Delta_{n-k}^* = 0)\}\underset{\sim}{E}\{e^{-s\zeta_k}\delta(\Delta_k^* = k)\} = V_{n-k,0}^*(s)V_{kk}^*(s) \end{aligned}$$

for $0 \leq k \leq n$ and $\mathrm{Re}(s) = 0$ which is in agreement with (4).

By (3) and (4) we can write that

$$(9) \qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n} V_{nk}(s)\rho^n\omega^k = \Big(\sum_{n=0}^{\infty} V_{nn}(s)(\rho\omega)^n\Big)\Big(\sum_{n=0}^{\infty} V_{n0}(s)\rho^n\Big)$$

and

(10) $$\sum_{n=0}^{\infty} \sum_{k=0}^{n} V_{nk}^{*}(s)\rho^n \omega^k = \left(\sum_{n=0}^{\infty} V_{nn}^{*}(s)(\rho\omega)^n\right)\left(\sum_{n=0}^{\infty} V_{n0}^{*}(s)\rho^n\right)$$

for $|\rho| < 1$, $|\rho\omega| < 1$ and $\operatorname{Re}(s) = 0$.

Let us denote by $\phi(s)$ the Laplace-Stieltjes transform of ξ_n $(n = 1,2,\ldots)$, that is,

(11) $$\phi(s) = \underset{\sim}{E}\{e^{-s\xi_n}\}$$

for $\operatorname{Re}(s) = 0$.

If we put $\omega = 1$ in (9) and in (10), then we obtain that

(12) $$\left(\sum_{n=0}^{\infty} V_{nn}(s)\rho^n\right)\left(\sum_{n=0}^{\infty} V_{n0}(s)\rho^n\right) = \frac{1}{1-\rho\phi(s)}$$

and

(13) $$\left(\sum_{n=0}^{\infty} V_{nn}^{*}(s)\rho^n\right)\left(\sum_{n=0}^{\infty} V_{n0}^{*}(s)\rho^n\right) = \frac{1}{1-\rho\phi(s)}$$

for $|\rho| < 1$ and $\operatorname{Re}(s) = 0$. For if $\omega = 1$, then the left-hand sides of (9) and (10) both reduce to

(14) $$\sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-s\zeta_n}\}\rho^n = \sum_{n=0}^{\infty} [\phi(s)]^n \rho^n = \frac{1}{1-\rho\phi(s)}$$

whenever $|\rho| < 1$ and $\operatorname{Re}(s) = 0$.

Accordingly, if we know $V_{nn}(s)$ and $V_{nn}^{*}(s)$ for $n = 0,1,2,\ldots$ and $\operatorname{Re}(s) = 0$, then by using the above results we can obtain $V_{nk}(s)$ for $0 \leq k \leq n$ and $\operatorname{Re}(s) = 0$. Thus the whole problem is reduced to finding $V_{nn}(s)$ and $V_{nn}^{*}(s)$ for $n = 0,1,2,\ldots$. This is our next aim.

24. The Determination of $V_{nn}(s)$ and $V^*_{nn}(s)$. First we recall that

$$V_{nn}(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_n = n)\} \tag{1}$$

and

$$V_{n0}(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_n = 0)\} \tag{2}$$

for $n = 0,1,2,\ldots$ and $\text{Re}(s) = 0$ where Δ_n denotes the number of positive partial sums among $\zeta_1, \zeta_2, \ldots, \zeta_n$ and $\Delta_0 = 0$. Furthermore,

$$V^*_{nn}(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta^*_n = n)\} \tag{3}$$

and

$$V^*_{n0}(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta^*_n = 0)\} \tag{4}$$

for $n = 0,1,2,\ldots$ and $\text{Re}(s) = 0$ where Δ^*_n denotes the number of non-negative partial sums among $\zeta_1, \zeta_2, \ldots, \zeta_n$ and $\Delta^*_0 = 0$.

Theorem 1. We have

$$\sum_{n=0}^{\infty} V_{nn}(s)\rho^n = \exp\left\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n > 0)\}\right\} \tag{5}$$

for $\text{Re}(s) \geq 0$ and $|\rho| < 1$,

$$\sum_{n=0}^{\infty} V_{n0}(s)\rho^n = \exp\left\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n \leq 0)\}\right\} \tag{6}$$

for $\text{Re}(s) \leq 0$ and $|\rho| < 1$,

(7) $$\sum_{n=0}^{\infty} V_{nn}^{*}(s)\rho^{n} = \exp\{\sum_{n=1}^{\infty} \frac{\rho^{n}}{n} \underset{\sim}{E}\{e^{-s\zeta_{n}}\delta(\zeta_{n} \geqq 0)\}\}$$

for $\operatorname{Re}(s) \geqq 0$ and $|\rho| < 1$, and

(8) $$\sum_{n=0}^{\infty} V_{n0}^{*}(s)\rho^{n} = \exp\{\sum_{n=1}^{\infty} \frac{\rho^{n}}{n} \underset{\sim}{E}\{e^{-s\zeta_{n}}\delta(\zeta_{n} < 0)\}\}$$

for $\operatorname{Re}(s) \leqq 0$ and $|\rho| < 1$.

Proof. We note that $V_{nn}(s)$ and $V_{nn}^{*}(s)$ exist for $\operatorname{Re}(s) \geqq 0$ and $|V_{nn}(s)| \leqq 1$ and $|V_{nn}^{*}(s)| \leqq 1$ for $\operatorname{Re}(s) \geqq 0$. Similarly, $V_{n0}(s)$ and $V_{n0}^{*}(s)$ exist for $\operatorname{Re}(s) \leqq 0$ and $|V_{n0}(s)| \leqq 1$ and $|V_{n0}^{*}(s)| \leqq 1$ for $\operatorname{Re}(s) \leqq 0$.

In what follows we shall prove first (5), and then we shall show that (6), (7), and (8) follow easily from (5), (23.12) and (23.13).

In Section 2 we introduced $\underset{\sim}{R}$, a space of functions $\Phi(s)$ defined for $\operatorname{Re}(s) = 0$ on the complex plane. In Section 3 we introduced a linear transformation $\underset{\sim}{T}$ defined for $\Phi(s) \in \underset{\sim}{R}$. We used the notation $\Phi^{+}(s) = \underset{\sim}{T}\{\Phi(s)\}$ for $\operatorname{Re}(s) \geqq 0$.

Now let us define another linear transformation $\underset{\sim}{S}$ by assuming that

(9) $$\underset{\sim}{S}\{\Phi(s)\} = \Phi^{+}(s) - \Phi^{+}(\infty)$$

for $\operatorname{Re}(s) \geqq 0$ and $\Phi(s) \in \underset{\sim}{R}$. In other words, if

(10) $$\Phi(s) = \underset{\sim}{E}\{\zeta e^{-s\eta}\}$$

for $\operatorname{Re}(s) = 0$ where ζ is a complex (or real) random variable for which

$\underset{\sim}{E}\{|\zeta|\} < \infty$ and η is a real random variable, then

$$\underset{\sim}{S}\{\Phi(s)\} = \underset{\sim}{E}\{\zeta e^{-s\eta}\delta(\eta > 0)\} \tag{11}$$

for $\operatorname{Re}(s) \geq 0$.

We can deduce a recurrence relation for $V_{nn}(s)$ $(n = 0,1,2,\ldots)$ if we use the transformation $\underset{\sim}{S}$. For the sake of brevity let us write

$$V_n(s) = V_{nn}(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_n = n)\} \tag{12}$$

for $n = 0,1,2,\ldots$ and $\operatorname{Re}(s) \geq 0$. We have $V_0(s) \equiv 1$ and

$$V_n(s) = \underset{\sim}{S}\{\phi(s)V_{n-1}(s)\} \tag{13}$$

for $n = 1,2,\ldots$. For

$$V_n(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_n = n)\} = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_{n-1} = n-1)\delta(\zeta_n > 0)\} =$$

$$= \underset{\sim}{S}\{\underset{\sim}{E}\{e^{-s\zeta_n}\delta(\Delta_{n-1} = n-1)\}\} = \underset{\sim}{S}\{\underset{\sim}{E}\{e^{-s\xi_n}\}\underset{\sim}{E}\{e^{-s\zeta_{n-1}}\delta(\Delta_{n-1} = n-1)\} = \tag{14}$$

$$= \underset{\sim}{S}\{\phi(s)V_{n-1}(s)\}$$

for $n = 1,2,\ldots$ and $\operatorname{Re}(s) \geq 0$.

Let

$$U(s,\rho) = e^{-\underset{\sim}{S}\{\log[1-\rho\phi(s)]\}} = e^{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{S}\{[\phi(s)]^n\}} \tag{15}$$

for $\operatorname{Re}(s) \geq 0$ and $|\rho| < 1$, and let us expand $U(s,\rho)$ in a power series as follows

$$U(s,\rho) = \sum_{n=0}^{\infty} U_n(s)\rho^n . \tag{16}$$

This series is convergent if $|\rho| < 1$ and $\mathrm{Re}(s) \geqq 0$. We can easily see that $U_0(s) \equiv 1$ and thus $\underset{\sim}{S}\{U_0(s)\} \equiv 0$, furthermore $U_n(s) \in \underset{\sim}{R}$ and $\underset{\sim}{S}\{U_n(s)\} = U_n(s)$ for $n = 1,2,\ldots$. Accordingly,

$$\underset{\sim}{S}\{U(s,\rho)\} = U(s,\rho) - 1 \tag{17}$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho| < 1$. On the other hand

$$\underset{\sim}{S}\{[1-\rho\phi(s)]U(s,\rho)\} = \underset{\sim}{S}\{\exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} [\underset{\sim}{S}\{[\phi(s)]^n\} - [\phi(s)]^n]\}\} = 0 \tag{18}$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho| < 1$. By (17) and (18) it follows that

$$U(s,\rho) - \rho\underset{\sim}{S}\{\phi(s)U(s,\rho)\} = 1 \tag{19}$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho| < 1$. If we put (16) into (19) and form the coefficient of ρ^n for $n = 0,1,\ldots$, then we obtain that $U_0(s) \equiv 1$ and

$$U_n(s) = \underset{\sim}{S}\{\phi(s)U_{n-1}(s)\} \tag{20}$$

for $n = 1,2,\ldots$ and $\mathrm{Re}(s) \geqq 0$. Thus we can conclude that the sequence $U_n(s)$ $(n = 0,1,\ldots)$ satisfies the same recurrence relation and the same initial condition as the sequence $V_n(s)$ $(n = 0,1,2,\ldots)$ and therefore it follows that $V_n(s) = U_n(s)$ for $n = 0,1,2,\ldots$. Accordingly, we proved that

$$\sum_{n=0}^{\infty} V_{nn}(s)\rho^n = e^{-\underset{\sim}{S}\{\log[1-\rho\phi(s)]\}} = e^{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{S}\{[\phi(s)]^n\}} \tag{21}$$

for $\operatorname{Re}(s) \geqq 0$ and $|\rho| < 1$. In (21) we can write

(22) $$\underset{\sim}{S}\{[\phi(s)]^n\} = \underset{\sim}{S}\{\underset{\sim}{E}\{e^{-s\zeta_n}\}\} = \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n > 0)\}$$

for $\operatorname{Re}(s) \geqq 0$ and thus we obtain (5) which was to be proved. We note that

(23) $$\underset{\sim}{S}\{\log[1-\rho\phi(s)]\} = \underset{\sim}{T}\{\log[1-\rho\phi(s)]\} + \sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{P}\{\zeta_n \leqq 0\}$$

for $\operatorname{Re}(s) \geqq 0$ and $|\rho| < 1$ and thus (5) can also be expressed in the following equivalent form

(24) $$\sum_{n=0}^{\infty} V_{nn}(s)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(s)]\} - \sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{P}\{\zeta_n \leqq 0\}}$$

where $\operatorname{Re}(s) \geqq 0$ and $|\rho| < 1$.

Formula (6) follows from (5) and (23.12). If $\operatorname{Re}(s) = 0$ and $|\rho| < 1$, then we can write that

(25) $$\frac{1}{1-\rho\phi(s)} = e^{-\log[1-\rho\phi(s)]} = e^{\sum_{n=1}^{\infty} \frac{\rho^n}{n} \underset{\sim}{E}\{e^{-s\zeta_n}\}}$$

and thus (23.12) and (5) imply (6) for $\operatorname{Re}(s) = 0$ and $|\rho| < 1$. Since the left-hand side of (6) is a regular function of s in the domain $\operatorname{Re}(s) < 0$ and continuous for $\operatorname{Re}(s) \leqq 0$, it follows that (6) remains valid for $\operatorname{Re}(s) \leqq 0$ too.

If $\operatorname{Re}(s) = 0$ and $|\rho| < 1$, then by (24) and (23.12) we can write that

$$(26) \qquad \sum_{n=0}^{\infty} V_{n0}(s)\rho^n = e^{-\log[1-\rho\phi(s)]+\mathbf{T}\{\log[1-\rho\phi(s)]\}+\sum_{n=1}^{\infty} \frac{\rho^n}{n}\mathbf{P}\{\zeta_n \leq 0\}} .$$

If we apply (6) to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n, \ldots$ and replace s by $-s$ then we obtain (7) for $\mathrm{Re}(s) \geq 0$, and if we apply (5) to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n, \ldots$ and replace s by $-s$, then we obtain (8) for $\mathrm{Re}(s) \leq 0$.

We can write down also that

$$(27) \qquad \sum_{n=0}^{\infty} V^{*}_{nn}(s)\rho^n = e^{-\mathbf{T}\{\log[1-\rho\phi(s)]\} - \sum_{n=1}^{\infty} \frac{\rho^n}{n}\mathbf{P}\{\zeta_n < 0\}}$$

for $\mathrm{Re}(s) \geq 0$ and $|\rho| < 1$, and

$$(28) \qquad \sum_{n=0}^{\infty} V^{*}_{n0}(s)\rho^n = e^{-\log[1-\rho\phi(s)]+\mathbf{T}\{\log[1-\rho\phi(s)]\} + \sum_{n=1}^{\infty} \frac{\rho^n}{n}\mathbf{P}\{\zeta_n < 0\}}$$

for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$. These formulas can be seen simply by using the fact that the ratio of (7) to (5), and the ratio of (6) to (8) are

$$(29) \qquad \exp\left\{\sum_{n=1}^{\infty} \frac{\rho^n}{n}\mathbf{P}\{\zeta_n = 0\}\right\} .$$

Now we are in the position that we can express the generating functions of $V_{nk}(s)$ $(0 \leq k \leq n)$ and $V^{*}_{nk}(s)$ $(0 \leq k \leq n)$ in a closed formula.

<u>Theorem 2</u>. <u>If</u> $\mathrm{Re}(s) = 0$, $|\rho| < 1$ <u>and</u> $|\rho\omega| < 1$, <u>then we have</u>

$$(30)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} V_{nk}(s)\rho^n\omega^k = \frac{e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s)]\}}}{1-\rho\phi(s)} \cdot$$

$$\cdot \exp\left\{ \sum_{n=1}^{\infty} \frac{\rho^n(1-\omega^n)}{n} \underset{\sim}{P}\{\zeta_n \leqq 0\}\right\} ,$$

and

$$(31)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} V^*_{nk}(s)\rho^n\omega^k = \frac{e^{-\underset{\sim}{T}\{\log[1-\rho\omega\phi(s)]\} + \underset{\sim}{T}\{\log[1-\rho\phi(s)]\}}}{1-\rho\phi(s)} \cdot$$

$$\cdot \exp\left\{ \sum_{n=1}^{\infty} \frac{\rho^n(1-\omega^n)}{n} \underset{\sim}{P}\{\zeta_n < 0\}\right\} .$$

Proof. By (23.9) we can express (30) as the product of (24) with ρ replaced by $\rho\omega$, and (26). If instead of (24) and (26) we use (5) and (6), then we obtain that

$$(32)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} V_{nk}(s)\rho^n\omega^k = \exp\left\{ \sum_{n=1}^{\infty} \left[\frac{(\rho\omega)^n}{n} \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n>0)\} + \frac{\rho^n}{n} \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n\leqq 0)\}\right]\right\}$$

for $Re(s) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

By (23.10) we can express (31) as the product of (27) with ρ replaced by $\rho\omega$, and (28). If instead of (27) and (28) we use (7) and (8), then we obtain that

$$(33)\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} V^*_{nk}(s)\rho^n\omega^k = \exp\left\{ \sum_{n=1}^{\infty} \left[\frac{(\rho\omega)^n}{n} \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n\geqq 0)\} + \frac{\rho^n}{n} \underset{\sim}{E}\{e^{-s\zeta_n}\delta(\zeta_n<0)\}\right]\right\}$$

for $Re(s) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

Note. We would like to mention here a natural generalization of the problems discussed in the previous sections of this chapter. The solution of this more general problem, however, will be given only in the next chapter. Let us consider again a sequence of mutually independent and identically distributed real random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ and define $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$.

Denote by $\theta_n(x)$ the number of partial sums $\zeta_0, \zeta_1, \ldots, \zeta_n$ which are $\leq x$ where $-\infty < x < \infty$. In the previous section we studied the distributions of $\Delta_n = n+1 - \theta_n(0)$ and $\Delta_n^* = n+1 - \theta_n(-0)$. As a generalization of the previous results we can ask what is the joint distribution of ζ_n and $\theta_n(x)$ for $n = 0,1,2,\ldots$ and $-\infty < x < \infty$.

If we denote by $\eta_{n0}, \eta_{n1}, \ldots, \eta_{nn}$ the partial sums $\zeta_0, \zeta_1, \ldots, \zeta_n$ arranged in increasing order of magnitude, then we can prove the following identity found by J. G. Wendel [42].

Theorem 3. We have

$$(34) \quad \sum_{n=0}^{\infty} \rho^n \int_{-\infty}^{\infty} e^{-sx} d_x E\{e^{-v\zeta_n} \omega^{\theta_n(x)}\} = -(1-\omega) \sum_{n=0}^{\infty} \sum_{k=0}^{n} E\{e^{-s\eta_{nk} - v\zeta_n}\} \rho^n \omega^k$$

for $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$.

Proof. If we suppose that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are numerical (non-random) quantities and if we define $\theta_n(x)$ and $\eta_{n0}, \eta_{n1}, \ldots, \eta_{nn}$ $(n = 0,1,2,\ldots)$ in exactly the same way as above, then we have the following idnetity

$$\int_{-\infty}^{\infty} e^{-sx} d_x \omega^{\theta_n(x)} = -(1-\omega) \sum_{k=0}^{n} e^{-s\eta_{nk}} \omega^k \tag{35}$$

for any s and ω. This follows from the fact that $\theta_n(x)$ is a step function for which $\theta_n(x) = 0$ if $x < \eta_{n0}$, $\theta_n(x) = k$ if $\eta_{n,k-1} \leqq x < \eta_{n,k}$ $(k = 1,2,\ldots, n)$ and $\theta_n(x) = n+1$ if $x \geqq \eta_{nn}$. We can easily see that (34) is valid regardless of whether the quantities $\eta_{n0}, \eta_{n1},\ldots, \eta_{nn}$ are distinct or not.

If $\xi_1, \xi_2,\ldots, \xi_n,\ldots$ are random variables, then the relation (35) is valid for almost all realizations of the sequence. If we form the expectation of (35), then we obtain that

$$\int_{-\infty}^{\infty} e^{-sx} d_x \underset{\sim}{E}\{\omega^{\theta_n(x)}\} = -(1-\omega) \sum_{k=0}^{n} \underset{\sim}{E}\{e^{-s\eta_{nk}}\}\omega^k \tag{36}$$

for $\text{Re}(s) = 0$ and $n = 0,1,2,\ldots$. If we multiply (35) by $e^{-v\zeta_n}$ and if we form the expectation of the product, then we obtain that

$$\int_{-\infty}^{\infty} e^{-sx} d_x \underset{\sim}{E}\{e^{-v\zeta_n}\omega^{\theta_n(x)}\} = -(1-\omega) \sum_{k=0}^{n} \underset{\sim}{E}\{e^{-s\eta_{nk}-v\zeta_n}\}\omega^k \tag{37}$$

for $\text{Re}(s) = 0$, $\text{Re}(v) = 0$ and $n = 0,1,2,\ldots$. If $|\rho| < 1$ and if we multiply (37) by ρ^n and add for $n = 0,1,2,\ldots,$ then we obtain (34) which was to be proved.

In the next chapter we shall determine the generating function

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \underset{\sim}{E}\{e^{-s\eta_{nk}-v\zeta_n}\}\rho^n\omega^k \tag{38}$$

for $\operatorname{Re}(s) = 0$, $\operatorname{Re}(v) = 0$, $|\rho| < 1$ and $|\rho\omega| < 1$. This makes it possible to find the joint distribution of ζ_n and $\Theta_n(x)$ for $n = 1,2,\ldots$ and $-\infty < x < \infty$.

25. Some Particular Results. By using Theorem 24.2 we can find the probabilities

(1) $$P\{\Delta_n = k, \zeta_n \leq x\}$$

and

(2) $$P\{\Delta_n^* = k, \zeta_n \leq x\}$$

for $0 \leq k \leq n$ and $-\infty < x < \infty$. In what follows we shall determine (1) in some particular cases. Probability (2) can be obtained in an analogous way, or by (1) if we apply it to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n, \ldots$.

First let us consider the distribution of Δ_n for $n = 0,1,2,\ldots$. By Theorem 23.1 we have

(3) $$P\{\Delta_n = k\} = P\{\Delta_k = k\} P\{\Delta_{n-k} = 0\}$$

for $0 \leq k \leq n$. By Theorem 24.1 we have

(4) $$\sum_{n=0}^{\infty} P\{\Delta_n = n\} \rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} P\{\zeta_n > 0\}\}$$

for $|\rho| < 1$ and

(5) $$\sum_{n=0}^{\infty} P\{\Delta_n = 0\} \rho^n = \exp\{\sum_{n=1}^{\infty} \frac{\rho^n}{n} P\{\zeta_n \leq 0\}\}$$

for $|\rho| < 1$. By (3) it follows that the product of (4) and (5) is necessarily $1/(1-\rho)$ and thus (4) implies (5) and conversely (5) implies (4). We note that (4) is equivalent to (19.12) and (5) is equivalent to (19.10).

If we use the notation

(6) $$a_n = \underset{\sim}{P}\{\zeta_n > 0\}$$

for $n = 1,2,\ldots,$ then by (4) we obtain that

(7) $$\underset{\sim}{P}\{\Delta_n = n\} = \sum_{i_1+2i_2+\ldots+ni_n=n} \frac{a_1^{i_1} a_2^{i_2} \ldots a_n^{i_n}}{i_1!i_2!\ldots i_n!\, 1^{i_1} 2^{i_2}\ldots n^{i_n}}$$

for $n = 1,2,\ldots$ where $i_1, i_2,\ldots$ are nonnegative integers, and by (5) we obtain that

(8) $$\underset{\sim}{P}\{\Delta_n = 0\} = \sum_{i_1+2i_2+\ldots+ni_n=n} \frac{(1-a_1)^{i_1}(1-a_2)^{i_2}\ldots(1-a_n)^{i_n}}{i_1!i_2!\ldots i_n!\,1^{i_1} 2^{i_2}\ldots n^{i_n}}$$

for $n = 1,2,\ldots$ where $i_1, i_2,\ldots$ are nonnegative integers.

Thus the distribution of Δ_n can be obtained explicitly by (3), (7) and (8).

Now let us consider the joint distribution of Δ_n and ζ_n. By Theorem 23.1 it follows that

(9) $$\underset{\sim}{P}\{\Delta_n = k,\ \zeta_n \leqq x\} = \underset{\sim}{P}\{\Delta_k = k,\ \zeta_k \leqq x\} * \underset{\sim}{P}\{\Delta_{n-k} = 0,\ \zeta_{n-k} \leqq x\}$$

for $0 \leqq k \leqq n$ and $-\infty < x < \infty$. That is, if we know the probabilities $\underset{\sim}{P}\{\Delta_n = k,\ \zeta_n \leqq x\}$ $(n = 0,1,2,\ldots)$ in two particular cases when $k = n$ and $k = 0$, then by (9) we can obtain $\underset{\sim}{P}\{\Delta_n = k,\ \zeta_n \leqq x\}$ for $k = 0,1,\ldots, n$. The following particular case has same importance in studying discrete random variables. (See E. S. Andersen [3].)

Theorem 1. We have

$$(10) \qquad \mathbf{P}\{\Delta_n = k \text{ and } \zeta_n = 0\} = \sum_{r=0}^{n-k-1} U_r V_{n-r}$$

for $k = 0,1,\ldots, n-1$ and $n = 1,2,\ldots$ where

$$(11) \qquad \sum_{n=0}^{\infty} U_n z^n = e^{C(z)}$$

and

$$(12) \qquad \sum_{n=1}^{\infty} V_n z^n = 1-e^{-C(z)}$$

and

$$(13) \qquad C(z) = \sum_{n=1}^{\infty} \frac{\mathbf{P}\{\zeta_n = 0\}}{n} z^n$$

for $|z| < 1$.

Proof. We shall provide a direct proof for this theorem. By Theorem 22.1 we have

$$(14) \qquad \begin{aligned} &\mathbf{P}\{\Delta_n = k \text{ and } \zeta_n = 0\} = \mathbf{P}\{\rho_n = k \text{ and } \zeta_n = 0\} = \\ &= \mathbf{P}\{\zeta_i < \zeta_k \text{ for } 0 \leq i < k \text{ and } \zeta_i \leq \zeta_k \text{ for } k \leq i \leq n \text{ and } \zeta_n = 0\} \end{aligned}$$

for $0 \leq k \leq n$. If $k = n$, then (14) is 0. If $0 < k < n$ and in (14) we replace the random variables $\xi_{k+1},\ldots, \xi_n$, $\xi_1,\ldots, \xi_k$ by $\xi_1, \xi_2,\ldots, \xi_n$ respectively, then $\mathbf{P}\{\Delta_n = k \text{ and } \zeta_n = 0\}$ remains unchanged. Thus we can write also that

$$(15) \qquad \mathbf{P}\{\Delta_n = k \text{ and } \zeta_n = 0\} = \mathbf{P}\{\zeta_i \leq 0 \text{ for } 0 \leq i < n-k \text{ and } \zeta_i < 0 \text{ for } n-k \leq i < n\}$$

for $0 \leq k < n$. If $0 \leq k < n$ and the event on the right-hand side of

(15) occurs, then there is an i $(0 \leqq i < n-k)$ such that $\zeta_i = 0$. Denote by r the largest such i. Then necessarily $\zeta_i < 0$ for $r < i < n$. Accordingly

$$\begin{aligned} \underset{\sim}{P}\{\Delta_n = k \text{ and } \zeta_n = 0\} &= \sum_{r=0}^{n-k-1} \underset{\sim}{P}\{\zeta_i \leqq 0 \text{ for } 0 \leqq i \leqq r,\ \zeta_r = 0, \\ &\qquad \zeta_i < 0 \text{ for } r < i < n \text{ and } \zeta_n = 0\} = \\ &= \sum_{r=0}^{n-k-1} \underset{\sim}{P}\{\zeta_i \leqq 0 \text{ for } 0 \leqq i \leqq r \text{ and } \zeta_r = 0\} \underset{\sim}{P}\{\zeta_i < 0 \text{ for} \\ &\qquad 0 < i < n-r \text{ and } \zeta_{n-r} = 0\} \end{aligned} \tag{16}$$

for $0 \leqq k < n$, or equivalently

$$\underset{\sim}{P}\{\Delta_n = k \text{ and } \zeta_n = 0\} = \sum_{r=0}^{n-k-1} \underset{\sim}{P}\{\Delta_r = 0 \text{ and } \zeta_r = 0\} \underset{\sim}{P}\{\Delta^*_{n-r-1} = 0 \text{ and } \zeta_{n-r} = 0\} \tag{17}$$

for $0 \leqq k < n$.

Let us introduce the notation

$$U_n = \underset{\sim}{P}\{\Delta_n = 0 \text{ and } \zeta_n = 0\} \tag{18}$$

for $n = 0,1,2,\ldots$ and

$$V_n = \underset{\sim}{P}\{\Delta^*_{n-1} = 0 \text{ and } \zeta_n = 0\} \tag{19}$$

for $n = 1,2,\ldots$. Then by (17)

$$\underset{\sim}{P}\{\Delta_n = k \text{ and } \zeta_n = 0\} = \sum_{r=0}^{n-k-1} U_r V_{n-r} \tag{20}$$

for $0 \leq k < n$. If we add (20) for $k = 0,1,\ldots, n-1$, then we obtain that

$$P\{\zeta_n = 0\} = \sum_{r=0}^{n-1} (n-r)U_r V_{n-r} \tag{21}$$

for $n = 1,2,\ldots$. On the other hand if we put $k = 0$ in (20) then we obtain that

$$U_n = \sum_{r=0}^{n-1} U_r V_{n-r} \tag{22}$$

for $n = 1,2,\ldots$.

Let

$$U(z) = \sum_{n=0}^{\infty} U_n z^n \tag{23}$$

and

$$V(z) = \sum_{n=1}^{\infty} V_n z^n . \tag{24}$$

These generating functions are convergent for $|z| < 1$ because evidently $U_n \leq P\{\zeta_n = 0\} \leq 1$ and $nV_n \leq P\{\zeta_n = 0\} \leq 1$ for $n = 1,2,\ldots$. If $C(z)$ is defined by (13), then by (21) and (22) we obtain that

$$C'(z) = U(z)V'(z) \tag{25}$$

and

$$U(z)-1 = U(z)V(z) \tag{26}$$

for $|z| < 1$. Accordingly

$$U'(z) = C'(z)U(z) \tag{27}$$

for $|z| < 1$, and $U(0) = 1$. Hence

$$U(z) = e^{C(z)} \tag{28}$$

and consequently by (26)

(29) $$V(z) = 1-e^{-C(z)}$$

for $|z| \leq 1$. This completes the proof of the theorem.

Finally, we shall mention a related theorem.

Theorem 2. *We have*

(30) $$\underset{\sim}{P}\{\Delta_n = k \text{ and } \zeta_{n+1} > 0\} = \sum_{j=k}^{n} \underset{\sim}{P}\{\Delta_{j+1} = j+1\}[\underset{\sim}{P}\{\Delta_{n-j} = 0\} - \underset{\sim}{P}\{\Delta_{n-j-1} = 0\}]$$

for $0 \leq k \leq n$ *and*

(31) $$\underset{\sim}{P}\{\Delta_n = k \text{ and } \zeta_{n+1} \leq 0\} = \sum_{j=k}^{n} \underset{\sim}{P}\{\Delta_{n-j} = 0\}[\underset{\sim}{P}\{\Delta_j = j\} - \underset{\sim}{P}\{\Delta_{j+1} = j+1\}]$$

for $0 \leq k \leq n$.

Proof. To prove (30) we observe that the event $\{\Delta_{n+1} \geq k+1\}$ can occur in two mutually exclusive ways, either $\{\Delta_n = k \text{ and } \zeta_{n+1} > 0\}$ occurs, or $\{\Delta_n \geq k+1\}$. Hence

(32) $$\begin{aligned} \underset{\sim}{P}\{\Delta_n = k \text{ and } \zeta_{n+1} > 0\} &= \underset{\sim}{P}\{\Delta_{n+1} \geq k+1\} - \underset{\sim}{P}\{\Delta_n \geq k+1\} = \\ &= \sum_{j=k}^{n} \underset{\sim}{P}\{\Delta_{n+1} = j+1\} - \sum_{j=k}^{n} \underset{\sim}{P}\{\Delta_n = j+1\} \end{aligned}$$

for $0 \leq k \leq n$. If we use (3), then we get (30).

To prove (31) we observe that the event $\{\Delta_n \geq k\}$ can occur in two mutually exclusive ways, either $\{\Delta_n = k \text{ and } \zeta_{n+1} \leq 0\}$ occurs, or

$\{\Delta_{n+1} \geqq k+1\}$. Hence

(33) $$\underset{\sim}{P}\{\Delta_n = k \text{ and } \zeta_{n+1} \leqq 0\} = \underset{\sim}{P}\{\Delta_n \geqq k\} - \underset{\sim}{P}\{\Delta_{n+1} \geqq k+1\} =$$

$$= \sum_{j=k}^{n} \underset{\sim}{P}\{\Delta_n = j\} - \sum_{j=k}^{n} \underset{\sim}{P}\{\Delta_{n+1} = j+1\}$$

for $0 \leqq k \leqq n$. If we use (3), then we get (31).

26. Combinatorial Methods. In some particular cases we can use special methods for finding the distribution of Δ_n, the number of positive elements, or the distribution of Δ_n^*, the number of nonnegative elements in the sequence $\xi_1, \xi_1+\xi_2, \ldots, \xi_1+\ldots+\xi_n$ for $n = 1,2,\ldots$. In what follows we shall show that if $\xi_1, \xi_2, \ldots, \xi_n$ are either mutually independent and identically distributed discrete random variables taking on the integers $-1,0,1,2,\ldots$ (or $1,0,-1,-2,\ldots$) or interchangeable discrete random variables taking on the integers $-1,0,1,2,\ldots$ (or $1,0,-1,-2,\ldots$), then we can find the distributions of Δ_n and Δ_n^* for $n = 1,2,\ldots$ by using the combinatorial methods introduced in Section 20.

Let us suppose that $\nu_1, \nu_2, \ldots, \nu_n$ are interchangeable discrete random variables taking on nonnegative integers only. Let $N_r = \nu_1+\ldots+\nu_r$ for $r = 1,2,\ldots, n$ and $N_0 = 0$. Consider the sequence $\xi_r = 1 - \nu_r$ $(r = 1,2,\ldots, n)$ and denote by Δ_n the number of positive elements in the sequence of partial sums $\zeta_r = r-N_r$ $(r = 1,2,\ldots, n)$, and denote by Δ_n^* the number of nonnegative elements in the sequence of partial sums $\zeta_r = r-N_r$ $(r = 1,2,\ldots, n)$. Our first aim is to find the distributions of Δ_n and Δ_n^* for $n = 1,2,\ldots$. (See the author [39], [40], [41].)

The following auxiliary theorem will be useful in this section.

Lemma 1. Let $k_1, k_2, \ldots, k_n$ be integers with sum $k_1+k_2+\ldots+k_n = 1$. Among the n cyclic permutations of $(k_1, k_2, \ldots, k_n)$ there is exactly one for which exactly j $(j = 1,2,\ldots, n)$ of its successive partial sums are positive.

Proof. Let $k_{j+n} = k_j$ for $j = 1,2,\ldots,$ and define $d_j = n(k_1+\ldots+k_j)-j$ for $j = 1,2,\ldots$. Then $d_{j+n} = d_j$ for $j = 1,2,\ldots$. The numbers $d_1, d_2,\ldots, d_n$ are distinct, and $d_n = 0$. We shall prove that if d_i is the r-th largest number among $d_1, d_2,\ldots d_n$, then the cyclic permutation $(k_{i+1},\ldots, k_{i+n})$ has exactly $n+1-r$ positive partial sums. This implies the theorem.

Evidently, $(k_{i+1}, k_{i+1}+ k_{i+2},\ldots, k_{i+1}+\ldots+ k_{i+n})$ has the same number of positive elements as $(d_{i+1}- d_i, d_{i+2}- d_i,\ldots, d_{i+n}- d_i)$ has nonnegative elements. For if $k_{i+1}+\ldots+ k_{i+j} > 0$, then $d_{i+j}- d_i = n(k_{i+1}+\ldots+ k_{i+j})-j\geq 0$ for $j = 1,2,\ldots,n$. Conversely, if $d_{i+j}-d_i\geq 0$, then $k_{i+1}+\ldots+ k_{i+j}>0$ for $j = 1,2,\ldots,n$. Thus $(k_{i+1}, k_{i+1}+ k_{i+2},\ldots, k_{i+1}+\ldots+ k_{i+n})$ has the same number of positive elements as $(d_1-d_i, d_2-d_i,\ldots, d_n-d_i)$ has non-negative elements. If d_i is the r-th largest number among $d_1,d_2,\ldots,d_n$, then the latter sequence contains $n+1-r$ nonnegative elements. This proves the lemma.

Lemma 1 immediately implies the following auxiliary theorem.

Lemma 2. *If $\gamma_1, \gamma_2,\ldots, \gamma_n$ are cyclically interchangeable discrete random variables taking on integral values only and if Δ_n denotes the number of positive partial sums among $\gamma_1+\ldots+ \gamma_r$ $(r = 1,2,\ldots, n)$ then*

(1) $$\mathbf{P}\{\Delta_n = j\,|\,\gamma_1+\ldots+ \gamma_n = 1\} = \frac{1}{n}$$

for $j = 1,2,\ldots, n$, provided that the conditional probability is defined.

Proof. For almost every such realization of the sequence $\gamma_1, \gamma_2, \ldots, \gamma_n$ for which $\gamma_1+\ldots+\gamma_n = 1$ we can apply Lemma 1, and thus (1) follows easily.

In the following theorems we shall assume that $\nu_1, \nu_2, \ldots, \nu_n$ are interchangeable discrete random variables taking on nonnegative integers only and n is a positive integer. We shall write $N_r = \nu_1+\ldots+\nu_r$ for $r = 1,2,\ldots, n$ and $N_0 = 0$.

Let us denote by Δ_r $(r = 1,2,\ldots, n)$ the number of positive elements in the sequence $i-N_i$ $(i = 1,2,\ldots, r)$ and by Δ_r^* $(r = 1,2,\ldots, n)$ the number of nonnegative elements in the sequence $i-N_i$ $(i = 1,2,\ldots, r)$. Let $\Delta_0 = \Delta_0^* = 1$.

We shall also use the notation

$$Q_j(r|k) = \mathrm{P}\{\Delta_r = j \mid N_r = k\} \tag{2}$$

for $0 \leqq j \leqq r \leqq n$ and $k = 0,1,2,\ldots$ and

$$Q_j^*(r|k) = \mathrm{P}\{\Delta_r^* = j \mid N_r = k\} \tag{3}$$

for $0 \leqq j \leqq r \leqq n$ and $k = 0,1,2,\ldots$ where the conditional probabilities are defined up to an equivalence. In some particular cases we can find the distributions of Δ_n and Δ_n^* by using Lemma 20.2 and Lemma 2, and in the general case by using Lemma 20.2 and Theorem 22.1.

Theorem 1. If $k = 0,1,\ldots, n-2$, then

$$(4)\qquad Q_j(n|k) = \begin{cases} 0 & \text{for } j = 0,1,\ldots,n-k-1\ , \\ \sum_{i=n-j+1}^{k+1} \frac{(n-k-1)}{i(n-i)} \underset{\sim}{P}\{N_i = i-1|N_n = k\} & \text{for } j = n-k,\ldots,n-1 \\ 1-\frac{k}{n} & \text{for } j = n\ . \end{cases}$$

Furthermore,

$$(5)\qquad Q_j(n|n-1) = \frac{1}{n} \quad \text{for } j = 1,2,\ldots, n\ ,$$

and

$$(6)\qquad Q_j(n|n) = \begin{cases} 1-\sum_{i=1}^{n-1} \frac{1}{i} \underset{\sim}{P}\{N_i = i-1|N_n = n\} & \text{for } j = 0\ , \\ \sum_{i=1}^{n-j} \frac{1}{i(n-i)} \underset{\sim}{P}\{N_i = i-1|N_n = n\} & \text{for } j = 1,2,\ldots,n-1\ . \end{cases}$$

Proof. First, we note that

$$(7)\qquad Q_n(n|k) = \underset{\sim}{P}\{\Delta_n = n|N_n = k\} = 1-\frac{k}{n}$$

for $k = 0,1,\ldots,n$. This is exactly Lemma 20.2 . Furthermore, we have

$$(8)\qquad Q_j(n|n-1) = \underset{\sim}{P}\{\Delta_n = j|N_n = n-1\} = \frac{1}{n}$$

for $j = 1,2,\ldots,n$. This follows from Lemma 2 if we apply it to the random variables $\gamma_i = 1-\nu_i$ $(i = 1,2,\ldots,n)$.

Next we prove (4) for $j = 0,1,\ldots,n-1$. If $\Delta_n = j < n$ and $N_n = k < n-1$, then there exists an r such that $N_r = r-1$. Denote by $r = i$ $(i = 1,2,\ldots, k+1)$ the greatest r with this property. Then $N_i = i-1$ and $N_r - N_i < r-i$ for $r = i+1,\ldots, n$. Thus we get

$$(9)\quad \underset{\sim}{P}\{\Delta_n = j \mid N_n = k\} = \sum_{i=1}^{k+1} \underset{\sim}{P}\{N_i = i-1 \mid N_n = k\} \cdot \underset{\sim}{P}\{\Delta_i = i+j-n \mid N_i = i-1,\ N_n = k\} \cdot$$

$$\cdot \underset{\sim}{P}\{\Delta_n - \Delta_i = n-i \mid N_i = i-1,\ N_n = k\}$$

for $j = 0,1,\ldots,n-1$ and $k = 0,1,\ldots,n-2$. Now by (8)

$$(10)\quad \underset{\sim}{P}\{\Delta_i = i+j-n \mid N_i = i-1,\ N_n = k\} = \begin{cases} \frac{1}{i} & \text{for } n-j < i \leqq n , \\ 0 & \text{otherwise,} \end{cases}$$

if we apply it to the random variables $\nu_1,\ldots,\nu_i$,and by (7)

$$(11)\quad \underset{\sim}{P}\{\Delta_n - \Delta_i = n-i \mid N_i = i-1,\ N_n = k\} = \underset{\sim}{P}\{\Delta_{n-i} = n-i \mid N_{n-i} = k-i+1\} =$$

$$= \frac{n-k-1}{n-i} \quad \text{for } i = 1,\ldots, k+1 ,$$

if we apply it to the random variables $\nu_{i+1},\ldots,\nu_n$. Thus (4) follows for $j \leqq n-1$. If $j < n-k$, then $Q_j(n|k) = 0$. If $j = n$, then (4) reduces to (7). This completes the proof of (4).

Formula (5) is identical with (8).

It remains to prove (6). If $\Delta_n = j$ where $j = 1,2,\ldots, n-1$ and $N_n = n$, then there exists an $r = 1,2,\ldots, n$ for which $N_r < r$. Denote by i the smallest r with this property. Then necessarily $N_i = i-1$, $N_r \geqq r$ for $r = 1,2,\ldots, i-1$ and $N_r < r$ holds for j indices among $r = i,\ i+1,\ldots, n$. Thus

$$(12)\quad \underset{\sim}{P}\{\Delta_n = j \mid N_n = n\} = \sum_{i=1}^{n-j} \underset{\sim}{P}\{N_i = i-1 \mid N_n = n\} \underset{\sim}{P}\{\Delta_i = 0 \mid N_i = i-1 ,\ N_n = n\} \cdot$$

$$\cdot \underset{\sim}{P}\{\Delta_n - \Delta_j = j \mid N_i = i-1,\ N_n = n\} .$$

Now by (8)

$$P\{\Delta_i = 0 | N_i = i-1,\ N_n = n\} = \frac{1}{i} \tag{13}$$

for $i = 1,2,\ldots, n-1$. If we apply Lemma 2 to the random variables $(\nu_{i+1} - 1),\ldots, (\nu_n - 1)$, then we obtain that

$$P\{\Delta_n - \Delta_j = j | N_i = i-1\ ,\ N_n = n\} = \frac{1}{(n-i)} \tag{14}$$

for $i = 1,2,\ldots, n-j$. Thus

$$P\{\Delta_n = j | N_n = n\} = \sum_{i=1}^{n-j} \frac{1}{i(n-i)} P\{N_i = i-1 | N_n = n\} \tag{15}$$

for $j = 1,2,\ldots, n-1$. If we add (15) for $j = 1,2,\ldots, n-1$, then we get

$$1 - P\{\Delta_n = 0 | N_n = n\} = \sum_{i=1}^{n-1} \frac{1}{i} P\{N_i = i-1 | N_n = n\}\ . \tag{16}$$

Formula (6) follows from (15) and (16). This completes the proof of the theorem.

Theorem 2. We have

$$P\{\Delta_n = 0\} = 1 - \sum_{i=1}^{n} \frac{1}{i} P\{N_i = i-1\} \tag{17}$$

and

$$P\{\Delta_n = j\} = \sum_{\ell=0}^{j} \left(1 - \frac{\ell}{j}\right)\left[P\{N_j = \ell\} - \sum_{i=j+1}^{n} \frac{1}{(i-j)} P\{N_j = \ell \text{ and } N_i - N_j = i-j-1\}\right] \tag{18}$$

for $j = 1,2,\ldots, n$.

<u>Proof.</u> First we shall find $\mathbf{P}\{\Delta_n > 0\}$. If $\Delta_n > 0$, then $N_r = r-1$ for some $r = 1,2,\ldots, n$. Denote by i the smallest such r . Then

$$\mathbf{P}\{\Delta_n > 0\} = \sum_{i=1}^{n} \mathbf{P}\{N_r \geq r \text{ for } r = 1,\ldots,i-1 \text{ and } N_i = i-1\} =$$

(19) $$= \sum_{i=1}^{n} \mathbf{P}\{N_i - N_r < i-r \text{ for } r = 1,\ldots,i-1 \text{ and } N_i = i-1\} =$$

$$= \sum_{i=1}^{n} \frac{1}{i} \mathbf{P}\{N_i = i-1\} ,$$

where the last equality follows from Lemma 20.2 if we apply it to the random variables ν_i , $\nu_{i-1},\ldots, \nu_1$. This proves (17) .

We note that in exactly the same way as we proved (17) we can prove the following more general formula:

(20) $$\mathbf{P}\{\Delta_n = 0 \text{ and } N_n = k\} = \mathbf{P}\{N_n = k\} - \sum_{i=1}^{n} \frac{1}{i} \mathbf{P}\{N_i = i-1 \text{ and } N_n = k\}$$

for $k = 0,1,2,\ldots$. If we add (20) for $k = 0,1,2,\ldots$, then we get (17). If $\mathbf{P}\{N_n = k\} > 0$ and if we divide (20) by $\mathbf{P}\{N_n = k\}$, then we obtain $\mathbf{P}\{\Delta_n = 0 | N_n = k\}$ for $k = 0,1,2,\ldots$. We already found this latter probability for $k \leq n$ in Theorem 1.

Next we shall prove (18). By Theorem 22.1 it follows that Δ_n and ρ_n have the same distribution. Accordingly, we can write that

(21) $$\mathbf{P}\{\Delta_n = j\} = \mathbf{P}\{i-N_i < j-N_j \text{ for } 0 \leq i < j \text{ and } i-N_i \leq j-N_j \text{ for } j \leq i \leq n\} .$$

Hence for $j = 1,2,\ldots, n$

$$
\begin{aligned}
\mathbf{P}\{\Delta_n = j\} &= \sum_{\ell=0}^{j} \mathbf{P}\{N_j - N_i < j-i \text{ for } 0 \leq i < j \mid N_j = \ell\} \cdot \\
&\quad \cdot \mathbf{P}\{N_j - N_i \leq j-i \text{ for } j \leq i \leq n \text{ and } N_j = \ell\} \\
&= \sum_{\ell=0}^{j} \left(1-\frac{\ell}{j}\right)\left[\mathbf{P}\{N_j = \ell\} - \sum_{i=j+1}^{n} \frac{1}{(i-j)} \mathbf{P}\{N_j = \ell \text{ and } N_i - N_j = i-j-1\}\right] .
\end{aligned} \tag{22}
$$

In proving (22) we took into consideration that the event $\{\Delta_n = j\}$ can occur in several mutually exclusive ways, namely $\{N_j = \ell\}$ $(\ell = 0,1,2,\ldots)$, and we applied (7) to the random variables $\nu_j, \nu_{j-1},\ldots, \nu_1$ and (20) to the random variables $\nu_{j+1},\ldots, \nu_n$. This proves (18).

In exactly the same way as we proved (18) we can prove that

$$
\begin{aligned}
\mathbf{P}\{\Delta_n = j \text{ and } N_n = k\} &= \sum_{\ell=0}^{j} \left(1-\frac{\ell}{j}\right)\Big[\mathbf{P}\{N_j = \ell, \text{ and } N_n = k\} - \\
&\quad - \sum_{i=j+1}^{n} \frac{1}{(i-j)} \mathbf{P}\{N_j = \ell,\ N_i - N_j = i-j-1 \text{ and } N_n = k\}
\end{aligned} \tag{23}
$$

for $j = 1,2,\ldots,n$ and $k = 0,1,2,\ldots$. If we add (23) for $k = 0,1,2,\ldots,$ then we obtain (18). If we divide (23) by $\mathbf{P}\{N_n = k\}$ whenever $\mathbf{P}\{N_n = k\} > 0$, then we obtain $\mathbf{P}\{\Delta_n = j \mid N_n = k\}$ for $k = 0,1,2,\ldots$. In Theorem 1 we already found this latter probability for $k \leq n$ in a somewhat simpler form.

By using the notation (2) we can obtain from (20) and (23) that

$$
Q_j(n \mid k) = \sum_{\ell=0}^{j} \mathbf{P}\{N_j = \ell \mid N_n = k\} Q_j(j \mid \ell) Q_0(n-j \mid k-\ell) \tag{24}
$$

for $j = 0,1,\ldots, n$, where

$$(25) \qquad Q_0(n|k) = 1 - \sum_{i=1}^{n} \frac{1}{i} \underset{\sim}{P}\{N_i = i-1 | N_n = k\}$$

and

$$(26) \qquad Q_n(n|k) = \begin{cases} 1 - \frac{k}{n} & \text{for } k = 0,1,\ldots, n , \\ 0 & \text{for } k > n , \end{cases}$$

for $n = 1,2,\ldots$ and $k = 0,1,2,\ldots$.

The following theorems are concerned with the distribution of Δ_n^* .

<u>Theorem 3</u>. <u>If</u> $k = 1,2,\ldots, n$,<u>then we have</u>

$$(27) \qquad Q_j^*(n|k) = \begin{cases} \sum_{i=n-j}^{k-1} \frac{(n+1-k)}{i(n-i)} \underset{\sim}{P}\{N_i = i+1 | N_n = k\} & \underline{\text{for}} \quad n-k < j < n , \\ 1 - \sum_{i=1}^{k-1} \frac{(n+1-k)}{(n-i)} \underset{\sim}{P}\{N_i = i+1 | N_n = k\} & \underline{\text{for}} \quad j = n . \end{cases}$$

<u>If</u> $k = 1,2,\ldots, n$ <u>and</u> $j = 1,2,\ldots, n-k$, <u>then</u> $Q_j^*(n|k) = 0$. <u>Furthermore, we have</u>

$$(28) \qquad Q_j^*(n|n+1) = \frac{1}{n}$$

<u>for</u> $j = 0,1,\ldots, n-1$, <u>and</u> $Q_n^*(n|n+1) = 0$.

<u>Proof</u>. We can write that

$$(29) \qquad Q_j^*(n|k) = \underset{\sim}{P}\{N_r < r+1 \text{ for } j \text{ subscripts } r = 1,2,\ldots, n | N_n = k\} .$$

By (29) we can write that

$$(30) \qquad Q_j^*(n|n+1) = \underset{\sim}{P}\{N_r > r \text{ for } n-j \text{ subscripts } r = 1,2,\ldots,n | N_n = n+1\}$$

and if we apply Lemma 2 to the random variables $\gamma_i = \nu_i - 1$ $(i = 1,2,\dots, n)$, then we obtain that $Q_j^*(n|n+1) = 1/n$ for $j = 0,1,\dots, n-1$ which proves (28).

Next we shall prove (27) for $n-k < j < n$. If $N_r < r+1$ for j subscripts $r = 1,2,\dots, n$ where $n-k < j < n$ and $N_n = k$ where $1 \leq k \leq n$, then there exists an r such that $N_r = r+1$. Let i $(i = n-j,\dots, k-1)$ be the greatest r with this property. Then $N_r < r+1$ for $j+i-n$ subscripts $r = 1,2,\dots,i$, further $N_i = i+1$ and $N_r < r+1$ for $r = i+1,\dots, n$. By (28) we have

$$\underset{\sim}{P}\{N_r < r+1 \text{ for } j+i-n \text{ subscripts } r = 1,2,\dots, i | N_i = i+1\} = \frac{1}{i} \tag{31}$$

for $n-j < i \leq n$ and by Lemma 20.2 we obtain

$$\underset{\sim}{P}\{N_r < r+1 \text{ for } r = i+1,\dots, n | N_i = i+1, N_n = k\} = \tag{32}$$

$$= \underset{\sim}{P}\{N_r - N_i < r-i \text{ for } r = i+1,\dots,n | N_i = i+1, N_n = k\} = 1 - \frac{k-i-1}{n-i}$$

for $0 \leq i \leq k-1 \leq n$, if we apply Lemma 20.2 to the random variables $\nu_{i+1},\dots, \nu_n$. Thus by the theorem of total probability we obtain that

$$Q_j^*(n|k) = \sum_{i=n-j}^{k-1} \frac{1}{i} \left(1 - \frac{k-i-1}{n-i}\right) \underset{\sim}{P}\{N_i = i+1 | N_n = k\} \tag{33}$$

for $n-k < j < n$ and $1 \leq k \leq n$ which proves (27) in this case.

It remains to prove (27) for $j = n$. We have

$$Q_n^*(n|k) = \underset{\sim}{P}\{N_r < r+1 \text{ for } r = 1,2,\dots, n | N_n = k\} = \tag{34}$$

$$= 1 - \sum_{i=1}^{k-1} \frac{(n+1-k)}{(n-i)} \underset{\sim}{P}\{N_i = i+1 | N_n = k\}$$

for $k = 1,2,\ldots, n$. It is sufficient to prove that the subtrahend on the right-hand side of (34) is the probability that $N_r \geqq r+1$ for some $r = 1,2,\ldots, n-1$ given that $N_n = k$. This event can occur in the following mutually exclusive ways: the greatest r for which $N_r \geqq r+1$ is $r = i$ $(i = 1,\ldots, k-1)$. Then $N_i = i+1$ and $N_r < r+1$ for $r = i+1,\ldots, n$, or equivalently $N_r - N_i < r-i$ for $r = i+1,\ldots, n$. By Lemma 20.2 we get

$$P\{N_r - N_i < r-i \text{ for } r = i+1,\ldots, n \mid N_i = i+1,\ N_n = k\} = 1 - \frac{k-i-1}{n-i}$$

for $0 \leqq i \leqq k-1 \leqq n$ if we apply it to the random variables $\nu_{i+1},\ldots,\nu_n$. Thus (34) follows by the theorem of total probability, and this completes the proof of the theorem.

Theorem 4. We have

$$(35)\qquad \underset{\sim}{P}\{\Delta_n^* = 0\} = \underset{\sim}{P}\{N_1 > 1\} - \sum_{i=2}^{n} \frac{1}{(i-1)} \underset{\sim}{P}\{N_1 = 0 \text{ and } N_i = i\}$$

and

$$(36)\qquad \underset{\sim}{P}\{\Delta_n^* = j\} = \sum_{\ell=0}^{j} \Big[\underset{\sim}{P}\{N_j = \ell,\ N_{j+1} > \ell+1\} - \sum_{i=1}^{j-1} \frac{(j+1-\ell)}{(j-i)} \underset{\sim}{P}\{N_i = i+1,$$

$$N_j = \ell,\ N_{j+1} > \ell+1\}\Big]$$

$$- \sum_{\ell=0}^{j} \sum_{r=j+2}^{n} \Big[\frac{1}{(r-j-1)} \underset{\sim}{P}\{N_j = \ell,\ N_{j+1} = \ell,\ N_r = \ell+r-j\} -$$

$$- \sum_{i=1}^{j-1} \frac{(j+1-\ell)}{(j-i)(r-j-1)} \underset{\sim}{P}\{N_i = i+1,\ N_j = \ell,\ N_{j+1} = \ell,\ N_r = \ell+r-j\}\Big] .$$

for $j = 0,1,\ldots, n-1$. Furthermore

(37) $$P\{\Delta_n^* = n \text{ and } N_n = k\} = P\{N_n = k\} - \sum_{i=1}^{n-1} \frac{n+1-k}{n-i} P\{N_i = i+1 \text{ and } N_n = k\}$$

for $k = 0,1,\ldots, n$.

Proof. To prove (35) we can write that

(38) $$P\{\Delta_n^* = 0\} = P\{N_r > r \text{ for } r = 1,2,\ldots,n\} = P\{N_1 > 1\} -$$
$$- P\{N_1 > 1 \text{ and } N_r \leq r \text{ for some } r = 2,\ldots, n\} .$$

To find the last probability we take into consideration that there is an $r = 2,3,\ldots, n$ such that $N_r = r$. Denote by i the smallest such r . Then

(39)
$$P\{\Delta_n^* = 0\} = P\{N_1 > 1\} - \sum_{i=2}^{n} P\{N_r > r \text{ for } r = 1,\ldots,i-1 \text{ and } N_i = i\} =$$
$$= P\{N_1 > 1\} - \sum_{i=2}^{n} \sum_{s=2}^{i} \frac{(s-1)}{(i-1)} P\{N_1 = s \text{ and } N_i = i\} = P\{N_1 > 1\} -$$
$$- \sum_{i=2}^{n} \frac{1}{(i-1)} P\{N_1 = 0 \text{ and } N_i = i\}$$

where we applied Lemma 20.2 to the random variables $\nu_i,\ldots, \nu_2$. This proves (35). We note that in exactly the same way as we proved (35) we can obtain that

(40) $$P\{\Delta_n^* = 0 \text{ and } N_n = k\} = P\{N_1 > 1 \text{ and } N_n = k\} - \sum_{i=2}^{n} \frac{1}{(i-1)} P\{N_1 = 0 ,$$
$$N_i = i \text{ and } N_n = k\}$$

for $k = 0,1,2,\ldots$. Obviously (40) is 0 if $k \leq n$.

To prove (37) we can write that

$$(41)\quad \underset{\sim}{P}\{\Delta_n^* = n \text{ and } N_n = k\} = \underset{\sim}{P}\{N_r < r+1 \text{ for } r = 1,\ldots,n \text{ and } N_n = k\} = = \underset{\sim}{P}\{N_n = k\} - \underset{\sim}{P}\{N_r \geq r+1 \text{ for some } r = 1,\ldots,n \text{ and } N_n = k\} .$$

To find the last probability we take into consideration that there is an $r = 1,2,\ldots, n-1$ such that $N_r = r+1$. Denote by i the greatest such r . Then

$$(42)\quad \begin{aligned} \underset{\sim}{P}\{\Delta_n^* = n \text{ and } N_n = k\} &= \underset{\sim}{P}\{N_n = k\} - \sum_{i=1}^{n-1} \underset{\sim}{P}\{N_i = i+1 , N_r < r+1 \\ &\text{for } i < r \leq n , N_n = k\} = \underset{\sim}{P}\{N_n = k\} - \sum_{i=1}^{n-1} \underset{\sim}{P}\{N_r - N_i < r-i \\ &\text{for } r = i+1,\ldots, n \text{ and } N_i = i+1, N_n = k\} = \\ &\underset{\sim}{P}\{N_n = k\} - \sum_{i=1}^{n-1} \frac{n+1-k}{n-i} \underset{\sim}{P}\{N_i = i+1 , N_n = k\} \end{aligned}$$

where we applied Lemma 20.2 to the random variables $\nu_{i+1},\ldots, \nu_n$. This proves (37).

Finally, we shall prove (36). By Theorem 22.1 it follows that Δ_n^* and ρ_n^* have the same distribution. Accordingly, we can write that

$$(43)\quad \underset{\sim}{P}\{\Delta_n^* = j\} = \underset{\sim}{P}\{r-N_r \leq j-N_j \text{ for } 0 \leq r \leq j \text{ and } r-N_r < j-N_j \text{ for } j < r \leq n\}.$$

The event on the right-hand side of (43) can occur in several mutually exclusive ways, namely, $N_j = \ell$ $(\ell = 0,1,2,\ldots)$. Hence for $j = 0,1,\ldots, n$ we have

$$(44)\qquad \underset{\sim}{P}\{\Delta_n^* = j\} = \sum_{\ell=0}^{j} \underset{\sim}{P}\{N_j - N_r \leq j-r \text{ for } 0 \leq r \leq j \text{ and } N_j = \ell\} \cdot \\ \cdot \underset{\sim}{P}\{N_j - N_r < j-r \text{ for } j < r < n | N_j = \ell\} .$$

In the sum the first factor can be obtained by (37) if we apply it to the random variables $\nu_j, \nu_{j-1}, \ldots, \nu_1$ and the second factor can be obtained by (35) if we apply it to the random variables $\nu_{j+1}, \ldots, \nu_n$. Thus we obtain that

$$(45)\qquad \underset{\sim}{P}\{N_j - N_r \leq j-r \text{ for } 0 \leq r \leq j \text{ and } N_j = \ell\} = \underset{\sim}{P}\{N_j = \ell\} - \\ - \sum_{i=1}^{j-1} \frac{j+1-\ell}{j-i} \underset{\sim}{P}\{N_i = i+1 \text{ and } N_j = \ell\}$$

for $\ell = 0,1,\ldots, j$ and $j = 0,1,\ldots, n$ and

$$(46)\qquad \underset{\sim}{P}\{N_j - N_r < j-r \text{ for } j < r < n | N_j = \ell\} = \underset{\sim}{P}\{N_{j+1} > \ell+1 | N_j = \ell\} - \\ - \sum_{r=j+2}^{n} \frac{1}{(r-j-1)} \underset{\sim}{P}\{N_{j+1} = \ell+1 , N_r = \ell+r-j | N_j = \ell\}$$

for $j = 0,1,\ldots, n-1$. If we multiply (45) and (46) and add for $\ell = 0,1,\ldots, j$, then we get (36) for $j = 0,1,\ldots, n-1$.

In exactly the same way as we found $\underset{\sim}{P}\{\Delta_n^* = j\}$ we can find $\underset{\sim}{P}\{\Delta_n^* = j | N_n = k\}$ for $k = 0,1,2,\ldots$ and we observe that it can be expressed as follows:

$$(47)\qquad Q_j^*(n|k) = \sum_{\ell=0}^{j} \underset{\sim}{P}\{N_j = \ell | N_n = k\} Q_j^*(j|\ell) Q_0^*(n-j|k-\ell)$$

for $j = 0,1,\ldots, n$ and $k = 0,1,2,\ldots$ where

(48) $$Q_0^*(n|k) = P\{N_1 > 1 | N_n = k\} - \sum_{i=2}^{n} \frac{1}{(i-1)} P\{N_1 = 0 \ , \ N_i = i | N_n = k\}$$

for $k = 0,1,2,\ldots$ and

(49) $$Q_n^*(n|k) = 1 - \sum_{i=1}^{n-1} \frac{(n+1-k)}{(n-i)} P\{N_i = i+1 | N_n = k\}$$

for $k = 0,1,2,\ldots, n$ and $Q_n^*(n|k) = 0$ if $k > n$.

Note. Finally, we shall be concerned with the problem mentioned at the end of Section 24 in the particular case when $\xi_i = 1 - \nu_i$ for $i = 1,2,\ldots, n$ and $\nu_1, \nu_2, \ldots, \nu_n$ are interchangeable discrete random variables taking on nonnegative integers only. Let $N_r = \nu_1 + \ldots + \nu_r$ for $r = 1,2,\ldots, n$ and $N_0 = 0$. Denote by $\Delta_n^{(c)}$ the number of elements greater than c in the sequence $r - N_r$ $(r = 1,2,\ldots, n)$.

Our next aim is to find the distribution of $\Delta_n^{(c)}$ for $c = 0, \pm 1, \pm 2, \ldots,$ that is, the probabilities

(50) $$P\{\Delta_n^{(c)} = j\} = P\{N_r < r-c \text{ for exactly } j \text{ subscripts } r = 1,2,\ldots,n\}$$

for $j = 0,1,2,\ldots, n$. Previously we considered only the particular cases $c = 0$ and $c = -1$. In the notation of Section 26 we have $\Delta_n^{(0)} = \Delta_n$ and $\Delta_n^{(-1)} = \Delta_n^*$.

Theorem 5. If $c = 0,1,\ldots, n$, then

(51) $$P\{\Delta_n^{(c)} = 0\} = 1 - \sum_{i=c+1}^{n} \frac{c+1}{i} P\{N_i = i-c-1\}$$

and if $c = 0,1,\ldots, n-1$ and $j = 1,2,\ldots, n-c$, then

$$\underset{\sim}{P}\{\Delta_n^{(c)} = j\} = \sum_{\ell=0}^{j} \left(1-\frac{\ell}{j}\right)\Big[\sum_{i=j+c}^{n} \frac{c}{(i-j)} \underset{\sim}{P}\{N_j = \ell,\ N_i - N_j = i-j-c\} -$$

(52)

$$- \sum_{i=j+c+1}^{n} \frac{c+1}{(i-j)} \underset{\sim}{P}\{N_j = \ell,\ N_i - N_j = i-j-c-1\}\Big].$$

<u>If</u> $c = 0$ <u>and</u> $i = j$, <u>then</u> $c/(i-j)$ <u>should be interpreted as</u> 1 <u>in</u> (52).

<u>Proof</u>. If $c = 0$, then Theorem 5 reduces to Theorem 2. First we shall prove (51). We have

$$\underset{\sim}{P}\{\Delta_n^{(c)} = 0\} = \underset{\sim}{P}\{N_r \geq r-c \quad \text{for} \quad r = 1,2,\ldots, n\} =$$

(53)

$$= 1-\underset{\sim}{P}\{N_r < r-c \quad \text{for some} \quad r = 1,2,\ldots, n\} .$$

If the event $\{N_r < r-c$ for some $r = 1,2,\ldots, n\}$ occurs, then there is an $r = 1,2,\ldots, n$ such that $N_r = r-c-1$. Denote by i the smallest such r . Thus we obtain that

$$\underset{\sim}{P}\{\Delta_n^{(c)} = 0\} = 1- \sum_{i=c+1}^{n} \underset{\sim}{P}\{N_i - N_r < i-r \quad \text{for} \quad r = 1,\ldots, i-1 \quad \text{and} \ N_i = i-c-1\} =$$

(54)

$$= 1- \sum_{i=c+1}^{n} \frac{c+1}{i} \underset{\sim}{P}\{N_i = i-c-1\} ,$$

where in proving the second equality we used Lemma 20.2 applied to the random variables $\nu_i, \nu_{i-1},\ldots, \nu_1$.

Next we shall prove (52) for $c = 1,2,\ldots,n-1$ and $j = 1,2,\ldots, n-c$. If $\Delta_n^{(c)} = j$, then there is an $r = 1,2,\ldots, n$ such that $N_r = r-c$. Denote by s the smallest r with this property. Then $N_r > r-c$ for

$1 \leq r < s$, $N_s = s-c$, and $N_r < r-c$ for j subscripts $r = s+1,\ldots, n$. Here the last condition may be replaced by the following one: $N_r - N_s < r-s$ for j subscripts $r = s+1,\ldots, n$. If, in addition, we replace the last condition by the following one: the first maximum in the sequence $(r-N_r) - (s-N_s)$ $(r = s,\ldots, n)$ occurs at $r = s+j$, then this does not change the probability of the event $\{\Delta_n^{(c)} = j\}$. This is a consequence of Theorem 22.2 . Now let us define $\rho(k)$ $(k = 0,1,\ldots, n)$ as the smallest $r = 0,1,\ldots, n$ (if any) for which $r-N_r = k$. According to the above reasoning we can write that

$$(55) \quad \underset{\sim}{P}\{\Delta_n^{(c)} = j\} = \sum_{s=c}^{n-j} \sum_{\ell=1}^{j} \underset{\sim}{P}\{\rho(c) = s, \rho(c+\ell)-\rho(c) = j, \rho(c+\ell+1)-\rho(c+\ell) > n-s-j\}$$

where we used that $N_{s+j} - N_s = j-\ell$ with $1 \leq \ell \leq j$. The condition $\{\rho(c+\ell+1)-\rho(c+\ell) > n-c-j\}$ should be interpreted as the complementary event of $\{\rho(c+\ell+1)-\rho(c+\ell) \leq n-c-j\}$. If we replace the random variables $\nu_1,\ldots, \nu_s, \nu_{s+1},\ldots, \nu_{s+j}$ by $\nu_{s+1},\ldots, \nu_{s+j}, \nu_1,\ldots, \nu_s$ respectively, then (55) remains unchanged and we can write that

$$\underset{\sim}{P}\{\Delta_n^{(c)} = j\} = \sum_{s=c}^{n-j} \sum_{\ell=1}^{j} \underset{\sim}{P}\{\rho(\ell) = j, \rho(\ell+c)-\rho(\ell) = s, \rho(\ell+c+1)-\rho(\ell+c) > n-s-j\} =$$

$$(56) \quad = \sum_{\ell=1}^{j} \underset{\sim}{P}\{\rho(\ell) = j, \rho(\ell+c)-\rho(\ell) \leq n-j, \rho(\ell+c+1)-\rho(\ell) > n-j\} =$$

$$= \sum_{\ell=1}^{j} [\underset{\sim}{P}\{\rho(\ell) = j, \rho(\ell+c)-\rho(\ell) \leq n-j\} - \underset{\sim}{P}\{\rho(\ell) = j, \rho(\ell+c+1)-\rho(\ell) \leq n-j\}] .$$

Now by Lemma 3 it follows that

$$(57)\quad \underset{\sim}{P}\{\rho(\ell) = j, \rho(\ell+c)-\rho(\ell) = r\} = \frac{\ell c}{jr}\underset{\sim}{P}\{N_j = j-\ell,\ N_{j+r}-N_j = r-c\}$$

for $1 \leq \ell < \ell + c \leq j + r \leq n$. If we add (57) for $r = 1,\ldots, n-j$, then we obtain the first sum on the right-hand side of (56). The second sum on the right-hand side of (56) can be obtained from the first sum by replacing c by $c+1$. Thus we get (52). This completes the proof of the theorem.

Finally, we shall prove the following theorem.

Theorem 6. *If* $c = 1,2,\ldots$ *and* $\ell = 1,2,\ldots, n+c$, *then we have*

$$\underset{\sim}{P}\{\Delta_n^{(-c)}=j \text{ and } N_n=n+c-\ell\}= \sum_{s=1}^{n-j} \frac{s}{(n-j)} \Big[\sum_{r=\ell-1}^{j} \frac{(\ell-1)}{r} \underset{\sim}{P}\{N_{n-j}=n-j-s,\ N_{n-j+r}-N_{n-j}=r-\ell+1,$$

$$(58)\qquad N_n=n+c-\ell\}- \sum_{r=\ell}^{j} \frac{\ell}{r} \underset{\sim}{P}\{N_{n-j}=n-j-s,\ N_{n-j+r}-N_{n-j}=r-\ell,\ N_n=n+c-\ell\}\Big]$$

for $j = 0,1,\ldots, n-1$ *and*

$$(59)\quad \underset{\sim}{P}\{\Delta_n^{(-c)}=n \text{ and } N_n=n+c-\ell\}= \underset{\sim}{P}\{N_n=n+c-\ell\} - \sum_{i=\ell}^{n} \frac{\ell}{i} \underset{\sim}{P}\{N_i=i-\ell \text{ and } N_n=n+c-\ell\}.$$

Furthermore, for $c = 1,2,\ldots,$ *and* $j = 0,1,2,\ldots, n-1$ *we have*

$$(60)\quad \underset{\sim}{P}\{\Delta_n^{(-c)}=j \text{ and } N_n=n+c\} = \sum_{\ell=0}^{n-j-1} \underset{\sim}{P}\{N_{n-j-1}=\ell\}Q^*_{n-j-1}(n-j-1|\ell)Q^*_0(j+1|n+c-\ell)$$

where the probabilities on the right-hand side are given by (48) *and* (49).

Proof. If $c = 0,1,2,\ldots$ and $\ell = 0,1,\ldots, n+c$, then we have

$$\underset{\sim}{P}\{\Delta_n^{(-c)}=j \text{ and } N_n=n+c-\ell\}= \underset{\sim}{P}\{N_r < r+c \text{ for } j \text{ subscripts } r = 1,2,\ldots,n \text{ and } N_n=n+c-\ell\}=$$

(61)
$$= \underset{\sim}{P}\{N_n - N_r > n-r-\ell \text{ for } j \text{ subscripts } r = 1,2,\ldots,n \text{ and } N_n = n+c-\ell\}=$$

$$= \underset{\sim}{P}\{N_i < i-\ell+1 \text{ for } n-j \text{ subscripts } i = 0,1,\ldots, n-1 \text{ and } N_n= n+c-\ell\}.$$

Accordingly,

$$(62)\qquad \underset{\sim}{P}\{\Delta_n^{(-c)} = j \text{ and } N_n = n+c-\ell\} = \underset{\sim}{P}\{\Delta_n^{(\ell-1)}= n-j \text{ and } N_n = n+c-\ell\}$$

for $c \geqq 1$ and $\ell \geqq 1$ and the right-hand side is given by a slight modification of Theorem 5.

Furthermore, we have

$$(63)\qquad \underset{\sim}{P}\{\Delta_n^{(-c)} = j \text{ and } N_n = n+c\} = \underset{\sim}{P}\{\Delta_n^{(-1)} = n-j-1 \text{ and } N_n = n+c\}$$

for $c \geqq 1$. Here $\Delta_n^{(-1)} = \Delta_n^*$ and the right-hand side can be obtained by Theorem 4 or by (47).

Throughout this section we assumed that $\nu_1, \nu_2,\ldots, \nu_n$ are interchangeable random variables taking on nonnegative integers only. If, in particular, we assume that $\nu_1, \nu_2,\ldots, \nu_n$ are mutually independent and identically distributed random variables taking on nonnegative integers only, then all the results obtained in this section can be simplified somewhat.

27. Problems

27.1. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables having a continuous and symmetric distribution. Define $\zeta_0 = 0$ and $\zeta_r = \xi_1 + \ldots + \xi_r$ for $r = 1, 2, \ldots$. Denote by Δ_n the number of positive elements in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$. Find $\mathbf{P}\{\Delta_n = j\}$ for $j = 0, 1, \ldots, n$. (See E. S. Andersen [2] and D. A. Darling [19].)

27.2. In Problem 21.4 denote by Δ_n the number of positive elements in the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$. Find $\mathbf{P}\{\Delta_n = j\}$ for $j = 0, 1, \ldots, n$.

27.4. We distribute n points at random on the interval $(0, 1)$ in such a way that independently of the others each point has a uniform distribution over $(0, 1)$. Denote by ν_r $(r = 1, 2, \ldots, n)$ the number of points in the interval $(\frac{r-1}{n}, \frac{r}{n}]$, and let $N_r = \nu_1 + \ldots + \nu_r$ for $r = 1, 2, \ldots, n$. Denote by Δ_n^* the number of subscripts $r = 1, 2, \ldots, n$ for which $N_r \leq r$. Find $\mathbf{P}\{\Delta_n^* = j\}$ for $1 \leq j \leq n$.

27.5. In Theorem 26.5 determine $\mathbf{P}\{\Delta_n^{(c)} = j\}$ for $c = 0, 1, \ldots, n-1$ and $j = 1, 2, \ldots, n-c$ by using Theorem 22.2.

27.3. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables for which $\mathbf{P}\{\xi_n = 1\} = p$ and $\mathbf{P}\{\xi_n = -1\} = q$ where $p > 0$, $q > 0$ and $p + q = 1$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1, 2, \ldots$, and $\zeta_0 = 0$. Denote by Δ_n $(n = 0, 1, 2, \ldots)$ the number of positive elements among $\zeta_0, \zeta_1, \ldots, \zeta_n$. Find $\mathbf{P}\{\Delta_n = k\}$ for $0 \leq k \leq n$.

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