CHAPTER II

MAXIMAL PARTIAL SUMS

14. The Distribution of the Maximal Partial Sum. Throughout this chapter we shall assume that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of mutually independent and identically distributed real random variables. Let us denote by $F(x)$ the distribution function of ξ_n, that is,

(1) $$F(x) = \mathbf{P}\{\xi_n \leq x\}$$

for $-\infty \leq x \leq \infty$. For such random variables the expectation

(2) $$\phi(s) = \mathbf{E}\{e^{-s\xi_n}\}$$

exists for $\mathrm{Re}(s) = 0$. The function $\phi(s)$ is the Laplace-Stieltjes transform of $F(x)$, that is,

(3) $$\phi(s) = \int_{-\infty}^{\infty} e^{-sx} dF(x)$$

for $\mathrm{Re}(s) = 0$.

Define $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. We shall say that ζ_n $(n = 0,1,2,\ldots)$ is the n-th partial sum of the random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$. Let us write

(4) $$F_n(x) = \mathbf{P}\{\zeta_n \leq x\}$$

for $n = 0,1,2,\ldots$. The distribution function $F_n(x)$ is the n-th iterated convolution of $F(x)$ with itself. Obviously

(5) $$F_0(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The distribution functions $F_n(x)$ $(n = 1,2,\ldots)$ can be obtained by the following recurrence formula

(6) $$F_n(x) = \int_{-\infty}^{\infty} F_{n-1}(x-y)dF(y)$$

for $n = 1,2,\ldots$.

The expectation

(7) $$\phi_n(s) = \underset{\sim}{E}\{e^{-s\zeta_n}\} = \int_{-\infty}^{\infty} e^{-sx}dF_n(x)$$

exists for $\operatorname{Re}(s) = 0$ and $n = 1,2,\ldots$. Obviously, we have

(8) $$\phi_n(s) = [\phi(s)]^n$$

for $n = 0,1,2,\ldots$.

Let us write also

(9) $$\zeta_n^+ = [\zeta_n]^+ = \max(0, \zeta_n)$$

for $n = 0,1,2,\ldots$ and let

(10) $$\phi_n^+(s) = \underset{\sim}{E}\{e^{-s\zeta_n^+}\} = F_n(0) + \int_{+0}^{\infty} e^{-sx}dF_n(x)$$

which exists if $\operatorname{Re}(s) \geqq 0$ and $n = 0,1,2,\ldots$. The function $\phi_n^+(s)$ is regular in the domain $\operatorname{Re}(s) > 0$ and continuous for $\operatorname{Re}(s) \geqq 0$.

In what follows we shall be interested in studying the distribution of the random variable

(11) $$\eta_n^* = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$$

for $n = 0,1,2,\ldots$. Let us define

$$\Phi_n(s) = \underset{\sim}{E}\{e^{-s\eta_n^*}\} \tag{12}$$

for $n = 0,1,2,\ldots$. The expectation (12) exists if $\mathrm{Re}(s) \geqq 0$. If we know $\Phi_n(s)$ for $\mathrm{Re}(s) \geqq 0$, then $\underset{\sim}{P}\{\eta_n^* \leqq x\}$ can be obtained by inversion.

If x is a continuity point of $\underset{\sim}{P}\{\eta_n^* \leqq x\}$, then we have

$$\underset{\sim}{P}\{\eta_n^* \leqq x\} = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{sx}}{s} \Phi_n(s)ds \tag{13}$$

where $c > 0$. If x is a discontinuity point of $\underset{\sim}{P}\{\eta_n^* \leqq x\}$, then the right-hand side of (13) is equal to $\frac{1}{2}[\underset{\sim}{P}\{\eta_n^* \leqq x\} + \underset{\sim}{P}\{\eta_n^* < x\}]$.

Our next aim is to find $\Phi_n(s)$ for $n = 0,1,2,\ldots$.

15. A Theorem of Pollaczek and Spitzer. In 1952 F. Pollaczek [47] and in 1956 F. Spitzer [54] proved the following result.

Theorem 1. *If* $\mathrm{Re}(s) \geqq 0$ *and* $|\rho| < 1$, *then*

$$\sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \exp\{\sum_{k=1}^{\infty} \frac{\rho^k}{k} \phi_k^+(s)\} . \tag{1}$$

Proof. For $n = 1,2,\ldots$ we can write that

$$\eta_n^* = \max(0,\xi_1,\xi_1+\xi_2,\ldots,\xi_1+\ldots+\xi_n) = \max(0,\xi_1+\overline{\eta}_{n-1}^*) \tag{2}$$

where $\overline{\eta}_0^*=0$ and $\overline{\eta}_{n-1}^* = \max(0,\xi_2,\xi_2+\xi_3,\ldots,\xi_2+\ldots+\xi_n)$ for $n = 2,3,\ldots$. The random variable $\overline{\eta}_{n-1}^*$ has the same distribution as η_{n-1}^* and is independent of ξ_1 . Since $\phi(s) \in \underset{\sim}{R}_0$, we can apply the results of Section 7 or Section 4. By (2) we can write that

$$\Phi_n(s) = \underset{\sim}{T}\{\phi(s)\Phi_{n-1}(s)\} \tag{3}$$

for $R(s) \geqq 0$ and $n = 1,2,\ldots$ where $\Phi_0(s) \equiv 1$. Evidently $\|\phi\| = 1$. Thus Theorem 1 follows from Theorem 7.1 or from Theorem 4.2 .

We can express $\Phi_n(s)$ $(n = 1,2,\ldots)$ explicitly with the aid of $\phi_1^+(s), \phi_2^+(s),\ldots, \phi_n^+(s)$ if we introduce the following polynomials. For $n = 1,2,\ldots$ let

$$Q_n(x_1,x_2,\ldots,x_n) = \tag{4}$$

$$\sum_{k_1+2k_2+\ldots+nk_n=n} \frac{1}{k_1!k_2!\ldots k_n!} \left(\frac{x_1}{1}\right)^{k_1}\left(\frac{x_2}{2}\right)^{k_2}\ldots\left(\frac{x_n}{n}\right)^{k_n}$$

where $k_1, k_2, \ldots, k_n$ are nonnegative integers. Write $Q_0 \equiv 1$.

<u>Theorem 2.</u> <u>We have</u>

(5) $$\Phi_n(s) = Q_n(\phi_1^+(s), \phi_2^+(s), \ldots, \phi_n^+(s))$$

<u>for</u> $\mathrm{Re}(s) \geq 0$ <u>and</u> $n = 1,2,\ldots$ <u>and</u> $\Phi_0(s) \equiv Q_0 \equiv 1$.

<u>Proof.</u> This follows from Theorem 4.3 or from Theorem 7.2 .

We can express the generating function (1) in a compact form too.

<u>Theorem 3.</u> <u>If</u> $\mathrm{Re}(s) \geq 0$ <u>and</u> $|\rho| < 1$, <u>then we have</u>

(6) $$\sum_{n=0}^{\infty} \Phi_n(s)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(s)]\}} .$$

<u>Proof.</u> If we take into consideration that $\phi(s) \in \underset{\sim}{R}$ and $\|\phi\| = 1$, then (6) follows from Theorem 4.1 . Also (6) follows from (1) if refer to Lemma 3.2 or, in particular, to formula (3.17).

The generating function (1) can also be obtained by using the method of factorization developed in Section 6.

<u>Theorem 4.</u> <u>If</u> $|\rho| < 1$ <u>and</u>

(7) $$1 - \rho\phi(s) = \Phi^+(s,\rho)\Phi^-(s,\rho)$$

<u>for</u> $\mathrm{Re}(s) = 0$ <u>where</u> $\Phi^+(s,\rho)$ <u>satisfies the requirements</u> A_1, A_2, A_3 <u>of Section</u> 6 <u>and</u> $\Phi^-(s,\rho)$ <u>satisfies the requirements</u> B_1, B_2, B_3 of <u>Section 6</u>, <u>then</u>

(8) $$\sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \frac{1}{\Phi^+(s,\rho)\Phi^-(0,\rho)}$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho| < 1$.

Proof. The theorem is a particular case of Theorem 6.2 .

By (8) we can write that

(9) $$(1-\rho) \sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \frac{\Phi^+(0,\rho)}{\Phi^+(s,\rho)}$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho| < 1$. Furthermore, we can also write that

(10) $$[1-\rho\phi(s)] \sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \frac{\Phi^-(s,\rho)}{\Phi^-(0,\rho)}$$

for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$. Formula (10) determines the generating function (1) for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$. Since the generating function (1) is a regular function of s in the domain $\mathrm{Re}(s) > 0$ and continuous for $\mathrm{Re}(s) \geqq 0$, we can extend the definition of (1) for $\mathrm{Re}(s) \geqq 0$ by analytic continuation.

Note. By using Theorem 1 we can find also the distribution of the random variable $\bar{\eta}_n = -\min(\zeta_0, \zeta_1, \ldots, \zeta_n)$ for every $n = 0,1,2,\ldots$. We can write that

(11) $$\bar{\eta}_n = \max(-\zeta_0, -\zeta_1, \ldots, -\zeta_n)$$

for $n = 0,1,2,\ldots$.

Theorem 5. We have

$$(12)\qquad \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-s\bar{\eta}_n}\}\rho^n = \frac{1}{(1-\rho)[1-\rho\phi(-s)]} \exp\{-\sum_{k=1}^{\infty} \frac{\rho^k}{k} \phi_k^+(-s)\}$$

for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$.

Proof. If we apply Theorem 1 to the random variables $-\xi_1, -\xi_2, \ldots,$ $-\xi_n, \ldots,$ then we get

$$(13)\qquad \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-s\bar{\eta}_n}\}\rho^n = \exp\{\sum_{k=1}^{\infty} \frac{\rho^k}{k} \underset{\sim}{E}\{e^{-s[-\zeta_k]^+}\}\}$$

for $\mathrm{Re}(s) \geqq 0$ and $|\rho| < 1$. If we take into consideration that

$$(14)\qquad e^{-s[-x]^+} = e^{sx} - e^{s[x]^+} + 1$$

for any s and real x , then we can write that

$$(15)\qquad \underset{\sim}{E}\{e^{-s[-\zeta_k]^+}\} = [\phi(-s)]^k - \phi_k^+(-s) + 1$$

for $\mathrm{Re}(s) = 0$ and hence we obtain (12) by (13).

The left-hand side of (12) is a regular function of s in the domain $\mathrm{Re}(s) > 0$ and continuous for $\mathrm{Re}(s) \geqq 0$. Thus the right-hand side of (12) uniquely determines (12) for $\mathrm{Re}(s) \geqq 0$ by analytical continuation.

A more general problem is to find the distribution of η_{nk} , $(n = 0,1,2,\ldots ; k = 0,1,2,\ldots)$, the k-th ordered partial sum of $\xi_1, \xi_2, \ldots, \xi_n$ if we arrange the partial sums $\zeta_0, \zeta_1, \ldots, \zeta_n$ in increasing order of magnitude. Then $\eta_n^* = \eta_{nn} = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$ and $\bar{\eta}_n = -\eta_{n0} = -\min(\zeta_0, \zeta_1, \ldots, \zeta_n)$. This problem will be studied in Chapter IV.

16. A Generalization of the Previous Results. In 1948 A. Wald [68] observed that the problem of finding $P\{\eta_n^* \leq x\}$ for $n = 0,1,2,\ldots$ can be reduced to a problem in the theory of Markov sequences. A. Wald observed that if we define a sequence of random variables $\eta_0, \eta_1, \ldots, \eta_n, \ldots$ by the recurrence formula

$$\eta_n = [\eta_{n-1} + \xi_n]^+ \tag{1}$$

for $n = 1,2,\ldots$ where $[x]^+ = \max(0, x)$ and we suppose that $\eta_0 = 0$, then η_n has the same distribution as η_n^*.

If η_0 is a nonnegative random variable and η_0 and the sequence $\{\xi_n\}$ are independent, then the random variables $\eta_0, \eta_1, \ldots, \eta_n \ldots$ form a homogeneous Markov sequence.

Now let us prove that η_n^* and η_n have the same distribution if $\eta_0 = 0$. By (1) it follows that

$$\eta_n = \max(0, \xi_n, \xi_{n-1} + \xi_n, \ldots, \xi_2 + \ldots + \xi_n, \eta_0 + \xi_1 + \ldots + \xi_n) \tag{2}$$

for $n = 1,2,,\ldots$. If in (2) we replace $\xi_n, \xi_{n-1}, \ldots, \xi_1$ by $\xi_1, \xi_2, \ldots, \xi_n$ respectively, then we obtain a new random variable which has exactly the same distribution as η_n. In the particular case when $\eta_0 = 0$, this new random variable is precisely η_n^*. This proves the statement.

Wald's observation makes it possible to solve a more general problem, namely, the problem of finding the joint distribution of η_n^* and ζ_n.

By (2) we obtain that

$$\text{(3)} \qquad \underset{\sim}{P}\{\eta_n^* \leq x, \zeta_n \leq y\} = \underset{\sim}{P}\{\eta_n \leq x\}$$

provided that $\eta_0 = [x-y]^+$.

In what follows we shall discuss the problem of finding the distribution of η_n if η_0 is a nonnegative random variable and if η_0 and the sequence $\{\xi_n\}$ are independent. This problem was solved in 1952 by F.Pollaczek [47], [48] . Pollaczek made certain restrictions on the distribution of ξ_n and he obtained the generating function of the Laplace-Stieltjes transform of η_n in the case where η_0 is a constant by solving a singular integral equation.

Let us introduce the notation

$$\text{(4)} \qquad \Omega_n(s) = \underset{\sim}{E}\{e^{-s\eta_n}\}$$

for $\mathrm{Re}(s) \geq 0$ and $n = 0,1,2,\ldots$. The Laplace-Stieltjes transform $\Omega_0(s)$ is given by the distribution of η_0 , and for $n = 1,2,\ldots$ the Laplace-Stieltjes transform $\Omega_n(s)$ can be obtained by the recurrence formula

$$\text{(5)} \qquad \Omega_n(s) = \underset{\sim}{T}\{\phi(s)\Omega_{n-1}(s)\}$$

for $\mathrm{Re}(s) \geq 0$ and $n = 1,2,\ldots$. Here $\phi(s) \in \underset{\sim}{R}_0$ and $\Omega_0(s) \in \underset{\sim}{R}_0$ and we can apply the results of Section 7.

<u>Theorem 1.</u> <u>For</u> $\mathrm{Re}(s) \geq 0$ <u>and</u> $n = 0,1,2,\ldots$ <u>we have</u>

$$(6) \qquad \Omega_n(s) = \sum_{k=0}^{n} \Phi_{n-k}(s)\underset{\sim}{T}\{\Omega_0(s)Q_k^*(s)\}$$

where $\Phi_k(s)$ $(k = 0,1,2,\ldots)$ *is given by* (14.5), $Q_0^*(s) \equiv 1$, *and*

$$(7) \qquad Q_k^*(s) = Q_k(\phi_1(s)-\phi_1^+(s),\phi_2(s)-\phi_2^+(s),\ldots,\phi_k(s)-\phi_k^+(s))$$

for $k = 1,2,\ldots,n$ *where the polynomial* $Q_k(x_1, x_2,\ldots, x_k)$ *for* $k = 1,2,\ldots$ *is defined by* (15.4) .

Proof. This theorem follows from Theorem 7.1 or from Theorem 4.2 .

We can express the generating function of $\Omega_n(s)$ in a compact form given by the following theorem.

Theorem 2. *If* $\mathrm{Re}(s) \geq 0$ *and* $|\rho| < 1$, *then*

$$(8) \qquad \sum_{n=0}^{\infty} \Omega_n(s)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(s)]\}}\underset{\sim}{T}\{\Omega_0(s)e^{-\log[1-\rho\phi(s)]+\underset{\sim}{T}\{\log[1-\rho\phi(s)]\}}\}.$$

Proof. If we take into consideration that $\phi(s) \in \underset{\sim}{R}$, $\|\phi\| = 1$ and $\Omega_0(s) \in \underset{\sim}{R}$, then (8) follows from Theorem 4.1 . Also, if we multiply (6) by ρ^n and add for $n = 0,1,2,\ldots$ and we make use of Lemma 3.2 or, in particular, formulas (3.14) and (3.17), then we obtain (8).

The generating function (8) can also be obtained by using the method of factorization developed in Section 6.

Theorem 3. *If* $|\rho| < 1$ *and*

$$(9) \qquad 1-\rho\phi(s) = \Phi^+(s,\rho)\Phi^-(s,\rho)$$

for $\mathrm{Re}(s) = 0$ *where* $\Phi^{+}(s,\rho)$ *satisfies the requirements* A_1, A_2, A_3 *of Section 6* *and* $\Phi^{-}(s,\rho)$ *satisfies the requirements* B_1, B_2, B_3 *of Section 6, then*

$$(10) \qquad \sum_{n=0}^{\infty} \Omega_n(s)\rho^n = \frac{1}{\Phi^{+}(s,\rho)} \underset{\sim}{T} \left\{ \frac{\Omega_0(s)}{\Phi^{-}(s,\rho)} \right\}$$

for $\mathrm{Re}(s) \geq 0$ *and* $|\rho| < 1$.

Proof. The theorem is a particular case of Theorem 6.2 .

By (10) we can write that

$$(11) \qquad [1-\rho\phi(s)] \sum_{n=0}^{\infty} \Omega_n(s)\rho^n = \Phi^{-}(s,\rho) \underset{\sim}{T} \left\{ \frac{\Omega_0(s)}{\Phi^{-}(s,\rho)} \right\}$$

for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$. Formula (11) determines the generating function (8) for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$. Since $\Omega_n(s)$ is regular in the domain $\mathrm{Re}(s) > 0$ and continuous for $\mathrm{Re}(s) \geq 0$, we can extend the definition of (8) for $\mathrm{Re}(s) \geq 0$ by analytic continuation.

17. Joint Distributions. Our next aim is to give a method of finding the joint distribution of η_n and ζ_n for $n = 0,1,2,\ldots$.

Let us introduce the expectation

(1) $$\Omega_n(s,v) = \underset{\sim}{E}\{e^{-s\eta_n - v\zeta_n}\}$$

for $n = 0,1,2,\ldots,$ $\operatorname{Re}(s) \geq 0$, and $\operatorname{Re}(v) = 0$. If, in particular, $\underset{\sim}{P}\{\eta_0 = 0\} = 1$, then (1) can also be expressed in the following form

(2) $$\Phi_n(s,v) = \underset{\sim}{E}\{e^{-s\eta_n^* - v\zeta_n}\}$$

for $n = 0,1,2,\ldots,$ $\operatorname{Re}(s) \geq 0$, and $\operatorname{Re}(v) = 0$.

Theorem 1. *We have*

(3) $$\sum_{n=0}^{\infty} \Omega_n(s,v)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}\underset{\sim}{T}\{\Omega_0(s)e^{-\log[1-\rho\phi(s+v)]+\underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}\}$$

for $\operatorname{Re}(s) \geq 0$, $\operatorname{Re}(v) = 0$ *and* $|\rho| < 1$.

If, in particular, $\underset{\sim}{P}\{\eta_0 = 0\} = 1$, *that is,* $\Omega_0(s) \equiv 1$, *then* (3) *reduces to*

(4) $$\sum_{n=0}^{\infty} \Phi_n(s,v)\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\}}$$

for $\operatorname{Re}(s) \geq 0$, $\operatorname{Re}(v) = 0$ *and* $|\rho| < 1$.

Here $\underset{\sim}{T}$ *operates on the variable* s , *and* v *and* ρ *are parameters.*

Proof. Since $\zeta_n = \zeta_{n-1} + \xi_n$ and $\eta_n = [\eta_{n-1} + \xi_n]^+$ for $n = 1,2,\ldots$, it follows that

$$\Omega_n(s, v) = \underset{\sim}{T}\{\phi(s+v)\Omega_{n-1}(s,v)\} \tag{5}$$

for $n = 1,2,\ldots$, $\mathrm{Re}(s) \geq 0$ and $\mathrm{Re}(v) = 0$. Here $\Omega_0(s,v) = \Omega_0(s)$. Since for $\mathrm{Re}(v) = 0$ we have $\phi(s+v) \in \underset{\sim}{R}$ and $\Omega_0(s) \in \underset{\sim}{R}$ and $\|\phi(s+v)\| = 1$, we can apply Theorem 4.1 to obtain (3) and the particular case (4).

Formula (4) was found in 1956 by F. Spitzer [54] in a somewhat different form.

The generating functions (3) and (4) can also be obtained by using the method of factorization developed in Section 6.

Theorem 2. *Let* $|\rho| < 1$ *and* $\mathrm{Re}(v) = 0$. *Let us suppose that*

$$1-\rho\phi(s+v) = \Phi^+(s,v,\rho)\Phi^-(s,v,\rho) \tag{6}$$

for $\mathrm{Re}(s) = 0$ *where* $\Phi^+(s,v,\rho)$ *as a function of* s *satisfies the requirements* A_1, A_2, A_3 *of Section* 6 *and* $\Phi^-(s,v,\rho)$ *as a function of* s *satisfies the requirements* B_1, B_2, B_3 *of Section 6*. *Then we have*

$$\sum_{n=0}^{\infty} \Omega_n(s,v)\rho^n = \frac{1}{\Phi^+(s,v,\rho)} \underset{\sim}{T}\left\{\frac{\Omega_0(s)}{\Phi^-(s,v,\rho)}\right\} \tag{7}$$

and

$$\sum_{n=0}^{\infty} \Phi_n(s,v)\rho^n = \frac{1}{\Phi^+(s,v,\rho)\Phi^-(0,v,\rho)} \tag{8}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$ *and* $|\rho| < 1$.

Proof. This theorem is a particular case of Theorem 6.2 .

By (8) we can write that

$$(9)\qquad [1-\rho\phi(v)]\sum_{n=0}^{\infty}\Phi_n(s,v)\rho^n = \frac{\Phi^+(0,v,\rho)}{\Phi^+(s,v,\rho)}$$

for $\mathrm{Re}(s) \geqq 0$, $\mathrm{Re}(v) = 0$ and $|\rho| < 1$. Furthermore, we can also write that

$$(10)\qquad [1-\rho\phi(s+v)]\sum_{n=0}^{\infty}\Phi_n(s,v)\rho^n = \frac{\Phi^-(s,v,\rho)}{\Phi^-(0,v,\rho)}$$

for $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$ and $|\rho| < 1$. This formula determines the generating function (8) for $\mathrm{Re}(s) = 0$. Since the generating function (8) is a regular function of s in the domain $\mathrm{Re}(s) > 0$ and continuous for $\mathrm{Re}(s) \geqq 0$ whenever $\mathrm{Re}(v) = 0$ and $|\rho| < 1$, we can extend the definition of (8) for $\mathrm{Re}(s) \geqq 0$ by analytic continuation.

Note. By using Theorem 1 we can find also the joint distribution of the random variables ζ_n and $\bar{\eta}_n = -\min(\zeta_0, \zeta_1, \ldots, \zeta_n) = \max(-\zeta_0, -\zeta_1, \ldots, -\zeta_n)$ for every $n = 0,1,2,\ldots$.

Theorem 3. We have

$$(11) \qquad \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-s\bar{\eta}_n - v\zeta_n}\}\rho^n = \frac{\exp\{-\sum_{k=1}^{\infty} \cdot \frac{\rho^k}{k} \phi_k^+(v, -s)\}}{[1-\rho\phi(v)][1-\rho\phi(v-s)]}$$

for $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$ and $|\rho| < 1$ where

$$(12) \qquad \phi_k^+(v,s) = \underset{\sim}{T}\{[\phi(s+v)]^k\}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$ and $k = 1,2,\ldots$ and $\underset{\sim}{T}$ operates on the variable s .

Proof. If we apply (4) to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n$ and if we replace v by $-v$, then we obtain that

$$(13) \qquad \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-s\bar{\eta}_n - v\zeta_n}\}\rho^n = e^{-\underset{\sim}{T}\{\log[1-\rho\phi(v-s)]\}}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$ and $|\rho| < 1$. Accordingly we can write that

$$(14) \qquad \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-s\bar{\eta}_n - v\zeta_n}\}\rho^n = \exp\{\sum_{k=1}^{\infty} \frac{\rho^k}{k} \underset{\sim}{E}\{e^{-s[-\zeta_k]^+ - v\zeta_k}\}\}$$

for $\mathrm{Re}(s) \geq 0$, $\mathrm{Re}(v) = 0$ and $|\rho| < 1$. If we take into consideration that

$$(15) \qquad e^{-s[-x]^+} = e^{sx} - e^{s[x]^+} + 1$$

for any s and real x , then we can write that

$$(16)\qquad \underset{\sim}{E}\{e^{-s[-\zeta_k]^+ - v\zeta_k}\} = [\phi(v-s)]^k - \phi_k^+(v,-s) + [\phi(v)]^k$$

for $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(v) = 0$. If we put (16) into (14), then we obtain (11) which was to be proved.

The left-hand side of (11) is a regular function of s in the domain $\mathrm{Re}(s) > 0$ and continuous for $\mathrm{Re}(s) \geq 0$ whenever $\mathrm{Re}(v) = 0$ and $|\rho| < 1$. Thus the right-hand side of (11) uniquely determines (11) for $\mathrm{Re}(s) \geq 0$ by analytical continuation.

Finally, we note that (11) can also be expressed in the following way

$$(17)\qquad \sum_{n=0}^{\infty} \underset{\sim}{E}\{e^{-s\bar{\eta}_n - v\zeta_n}\}\rho^n = \frac{e^{\phi(v,-s,\rho)}}{[1-\rho\phi(v)][1-\rho\phi(v-s)]}$$

for $\mathrm{Re}(s) = 0$, $\mathrm{Re}(v) = 0$ and $|\rho| < 1$ where

$$(18)\qquad \phi(v,s,\rho) = \underset{\sim}{T}\{\log[1-\rho\phi(s+v)]\} .$$

Discrete Random Variables. If, in particular, the random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are mutually independent and identically distributed discrete random variables taking on integers only, then each result which we proved in this chapter has a discrete counterpart. In the case of discrete random variables it is convenient to introduce generating functions instead of Laplace-Stieltjes transforms and to replace the transformation $\underset{\sim}{T}$ by $\underset{\sim}{\Pi}$. By using the theorems of Sections 8-12 we can easily obtain all the theorems analogous to that of Sections 15-17. A few examples for discrete random variables will be considered in the next section.

18. Examples. In what follows we shall give three examples for finding $\Omega_n(s)$ and $\Phi_n(s)$ $(n = 0,1,2,\ldots)$ in the case where $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ is a sequence of mutually independent and identically distributed random variables for which

$$E\{e^{-s\xi_n}\} = \phi(s) . \tag{1}$$

First Example. Suppose that

$$\phi(s) = \psi(s) \frac{\lambda}{\lambda - s} \tag{2}$$

for $\mathrm{Re}(s) = 0$ where $\psi(s)$ is the Laplace-Stieltjes transform of a non-negative random variable and λ is a positive constant.

By Rouché's theorem we can show that

$$\lambda - s - \lambda\rho\,\psi(s) = 0 \tag{3}$$

has exactly one root $s = \gamma(\rho)$ in the domain $\mathrm{Re}(s) \geqq 0$ if $|\rho| < 1$.

For (3) cannot have a root in the domain $|s-\lambda| \geqq \lambda$. This follows from the inequality $|\lambda\rho\psi(s)| \leqq \lambda\rho < \lambda$ if $\mathrm{Re}(s) \geqq 0$. If $|\lambda - s| = \lambda$, then $|\lambda\rho\psi(s)| < |\lambda - s|$ and by Rouché's theorem we can conclude that (3) has the same number of roots in the domain $|s-\lambda| < \lambda$ as $s-\lambda = 0$, that is exactly one root. We can apply Rouché's theorem because $\psi(s)$ is regular in the domain $\mathrm{Re}(s) > 0$ and continuous in $\mathrm{Re}(s) \geqq 0$.

Accordingly we can write that

$$1-\rho\phi(s) = \Phi^{+}(s,\rho)\,\Phi^{-}(s,\rho) \tag{4}$$

for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$ where

$$\Phi^{+}(s,\rho) = \frac{\lambda-s-\lambda\rho\phi(s)}{\gamma(\rho)-s} \tag{5}$$

for $\operatorname{Re}(s) \geq 0$ and

$$\Phi^{-}(s,\rho) = \frac{\gamma(\rho)-s}{\lambda-s} \tag{6}$$

for $\operatorname{Re}(s) \leq 0$. The functions (5) and (6) satisfy the requirements A_1, A_2, A_3 and B_1, B_2, B_3 respectively in Section 6. By Theorem 16.3 we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} \Omega_n(s)\rho^n &= \frac{\gamma(\rho)-s}{\lambda-s-\lambda\rho\psi(s)} \underset{\sim}{T}\left\{\frac{(\lambda-s)\Omega_0(s)}{\gamma(\rho)-s}\right\} = \\ &= \frac{(\lambda-s)\Omega_0(s)}{\lambda-s-\lambda\rho\psi(s)} - \frac{s[\lambda-\gamma(\rho)]\Omega_0(\gamma(\rho))}{\gamma(\rho)[\lambda-s-\lambda\rho\psi(s)]}\end{aligned} \tag{7}$$

for $\operatorname{Re}(s) \geq 0$ and $|\rho| < 1$. For obviously $1/\Phi^{-}(s,\rho) \in \underset{\sim}{R}$ if $|\rho| < 1$ (see Theorem 6.1) and by formula (5.8) we obtain that

$$\underset{\sim}{T}\left\{\frac{(\lambda-s)\Omega_0(s)}{\gamma(\rho)-s}\right\} = \frac{s}{2\pi i} \int_{C_\varepsilon^+} \frac{(\lambda-z)\Omega_0(z)}{z(s-z)[\gamma(\rho)-z]}\, dz \tag{8}$$

for $\operatorname{Re}(s) > 0$ whenever ε is a sufficiently small positive number. The integral on the right-hand side of (8) is equal to $-2\pi i$ times the sum of the residues of the integrand at the poles $z = s$ and $z = \gamma(\rho)$. Thus we obtain (7).

<u>Second Example</u>. Suppose that

$$\phi(s) = \psi(s)\alpha(-s) \tag{9}$$

for $\operatorname{Re}(s) = 0$ where $\psi(s)$ and $\alpha(s)$ are Laplace-Stieltjes transforms of nonnegative random variables and $\alpha(s)$ is a rational function of s. Then we can write that

(10) $$\alpha(s) = \frac{\pi_{m-1}(s)}{\prod_{i=1}^{m} (a_i+s)}$$

for $\text{Re}(s) \geqq 0$ where m is a positive integer, $\pi_{m-1}(s)$ is a polynomial of degree $\leqq m-1$ and $\text{Re}(a_i) > 0$ for $i = 1,2,\ldots,m$. The last statement follows from the fact that necessarily $|\alpha(s)| \leqq 1$ if $\text{Re}(s) \geqq 0$.

If $|\rho| < 1$, then the equation

(11) $$\prod_{i=1}^{m} (a_i-s) - \rho\pi_{m-1}(-s)\psi(s) = 0$$

has exactly m roots $s = \gamma_1(\rho), \gamma_2(\rho),\ldots, \gamma_m(\rho)$ in the domain $\text{Re}(s) \geqq 0$. This can be proved by using Rouché's theorem. We shall show that

(12) $$|\rho\pi_{m-1}(s)\psi(s)| < |\prod_{i=1}^{m} (a_i-s)|$$

if either $\text{Re}(s) = 0$ or $|s| \geqq R$, $\text{Re}(s) \geqq 0$ and R is large enough. If $\text{Re}(s) = 0$, then $|\rho\psi(s)\alpha(-s)| \leqq \rho < 1$ which implies (12) for $\text{Re}(s) = 0$. If $\text{Re}(s) \geqq 0$ and if we divide (12) by $|s|^m$ and let $|s| \to \infty$, then the left-hand side tends to 0, while the right-hand side tends to 1. Thus the inequality (12) holds if $\text{Re}(s) \geqq 0$, $|s| \geqq R$ and R is large enough. Accordingly, (12) cannot have a root in the region $\{s:\text{Re}(s) \geqq 0, |s| \geqq R\}$ if R large is enough. Since $\psi(s)$ is regular in the domain $\text{Re}(s) > 0$ and continuous for $\text{Re}(s) \geqq 0$, we can conclude by Rouché's theorem that (11) has the same number of roots in the domain $\{s:\text{Re}(s) > 0, |s| < R\}$ as

$\prod_{i=1}^{m} (a_i - s) = 0$. If R is large enough, then the latter equation has exactly m roots in this domain. This proves the statement.

Accordingly, we can write that

$$1-\rho\psi(s)\alpha(-s) = \Phi^{+}(s,\rho)\Phi^{-}(s,\rho) \tag{13}$$

for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$ where

$$\Phi^{+}(s,\rho) = \frac{\prod_{i=1}^{m} (a_i - s) - \rho\pi_{m-1}(-s)\psi(s)}{\prod_{i=1}^{m} (\gamma_i(\rho) - s)} \tag{14}$$

for $\mathrm{Re}(s) \geqq 0$ and

$$\Phi^{-}(s,\rho) = \prod_{i=1}^{m} \left(\frac{\gamma_i(\rho) - s}{a_i - s}\right) \tag{15}$$

for $\mathrm{Re}(s) \leqq 0$. These functions satisfy the requirements A_1, A_2, A_3 and B_1, B_2, B_3 respectively in Section 6.

By formula (15.10) we can write that

$$[1 - \rho\psi(s)\alpha(-s)] \sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \frac{\Phi^{-}(s,\rho)}{\Phi^{-}(0,\rho)} = \prod_{i=1}^{m} \left\{\left(1 - \frac{s}{\gamma_i(\rho)}\right)\left(1 - \frac{s}{a_i}\right)^{-1}\right\} \tag{16}$$

for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$. If we express (16) in the form

$$\left[\prod_{i=1}^{m} (a_i - s) - \rho\psi(s)\pi_{m-1}(-s)\right] \sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \prod_{i=1}^{m} \left\{a_i\left(1 - \frac{s}{\gamma_i(\rho)}\right)\right\}, \tag{17}$$

then (17) becomes valid for $\mathrm{Re}(s) \geqq 0$ and $|\rho| < 1$ which follows immediately by analytic continuation.

Third Example. Let us consider the previous example with the modification that

$$\phi(s) = \alpha(s)\psi(-s) \tag{18}$$

for $\mathrm{Re}(s) = 0$, that is, the sequence of random variables $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ in the previous example is replaced by the sequence $-\xi_1, -\xi_2, \ldots, -\xi_n, \ldots$. By using the results of the previous example we can write that

$$1-\rho\alpha(s)\psi(-s) = \Phi^+(s,\rho)\Phi^-(s,\rho) \tag{19}$$

for $\mathrm{Re}(s) = 0$ and $|\rho| < 1$ where now

$$\Phi^+(s,\rho) = \prod_{i=1}^{m} \left(\frac{\gamma_i(\rho)+s}{a_i+s}\right) \tag{20}$$

for $\mathrm{Re}(s) \geq 0$ and

$$\Phi^-(s,\rho) = \frac{\prod\limits_{i=1}^{m} (a_i+s)-\rho\pi_{m-1}(s)\psi(-s)}{\prod\limits_{i=1}^{m} (\gamma_i(\rho)+s)} \tag{21}$$

for $\mathrm{Re}(s) \leq 0$. These functions satisfy the requirements A_1, A_2, A_3 and B_1, B_2, B_3 respectively of Section 6.

By formula (15.9) we can write that

$$(1-\rho) \sum_{n=0}^{\infty} \Phi_n(s)\rho^n = \frac{\Phi^+(0,\rho)}{\Phi^+(s,\rho)} = \prod_{i=1}^{m} \{(1+ \frac{s}{\gamma_i(\rho)})(1+ \frac{s}{a_i})^{-1}\} \tag{22}$$

for $\mathrm{Re}(s) \geq 0$ and $|\rho| < 1$.

Next we shall give two examples for finding the generating function of the maximal partial sum of discrete random variables.

Fourth Example. Let us assume that $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ are mutually independent and identically distributed random variables taking on integers only. Write $\zeta_n = \xi_1 + \ldots + \xi_n$ for $n = 1,2,\ldots$ and $\zeta_0 = 0$. Our aim is to find the generating function of $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$. Let us write

$$(23) \qquad u_n(s) = E\{s^{\eta_n}\}$$

for $n = 0,1,2,\ldots,$ and $|s| \leq 1$.

In what follows we suppose that

$$(24) \qquad E\{s^{\xi_n}\} = a(s)b(\tfrac{1}{s})$$

for $|s| = 1$ where $a(s)$ and $b(s)$ are generating functions of non-negative discrete random variables and $b(s)$ is a rational function of s. Then we can write that

$$(25) \qquad b(s) = \frac{\pi_{m-1}(s)}{\prod_{r=1}^{m} (1-\beta_r s)}$$

for $|s| \leq 1$ where $\pi_{m-1}(s)$ is a polynomial of degree $\leq m-1$. Since $|b(s)| \leq 1$ for $|s| \leq 1$, it follows that $|\beta_r| < 1$ for $r = 1,2,\ldots, m$.

In this case we have $u_0(s) \equiv 1$ and $u_n(s) = \Pi\{u_{n-1}(s)a(s)b(\frac{1}{s})\}$ for $n = 1,2,\ldots,$ and $|s| = 1$. If for $\mathrm{Re}(s) = 0$ and for $|\rho| < 1$ we have

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taking on integers only

(26) $$1-\rho a(s)b(\tfrac{1}{s}) = g^{+}(s,\rho)g^{-}(s,\rho)$$

where $g^{+}(s,\rho)$ and $g^{-}(s,\rho)$ satisfy the requirements (a_1), (a_2) and (b_1), (b_2), (b_3) respectively in Section 12, then by Theorem 12.2 we obtain that

(27) $$\sum_{n=0}^{\infty} u_n(s)\rho^n = \frac{1}{g^{+}(s,\rho)g^{-}(1,\rho)}$$

for $|s| \leqq 1$ and $|\rho| < 1$.

If $|\rho| < 1$, then

(28) $$|\rho s^m \pi_{m-1}(\tfrac{1}{s})a(s)| < |\prod_{r=1}^{m}(s-\beta_r)|$$

for $|s| = 1$ and hence by Rouché's theorem we can conclude that

(29) $$\prod_{r=1}^{m}(s-\beta_r)-\rho s^m \pi_{m-1}(\tfrac{1}{s})a(s) = 0$$

has exactly m roots $s = \delta_r(\rho)$ $(r = 1,2,\ldots, m)$ in the unit circle $|s| < 1$. Thus we can easily see that in (26) we can choose

(30) $$g^{+}(s,\rho) = \frac{\prod_{r=1}^{m}(s-\beta_r)-\rho s^m \pi_{m-1}(\tfrac{1}{s})a(s)}{\prod_{r=1}^{m}(s-\delta_r(\rho))}$$

for $|s| \leqq 1$ and

(31) $$g^{-}(s,\rho) = \prod_{r=1}^{m}\left(\frac{s-\delta_r(\rho)}{s-\beta_r}\right)$$

for $|s| \geqq 1$. Finally, by (27) we obtain that

$$(32) \quad [\prod_{r=1}^{m}(s-\beta_r)-\rho s^m \pi_{m-1}(\frac{1}{s})a(s)] \sum_{n=0}^{\infty} u_n(s)\rho^n = \prod_{r=1}^{m}\{\frac{(1-\beta_r)(s-\delta_r(\rho))}{1-\delta_r(\rho)}\}$$

for $|s| \leqq 1$ and $|\rho| < 1$. The distribution of η_n is uniquely determined by $u_n(s)$.

Fifth Example. Let us consider the previous example with the modification that

$$(33) \quad E\{s^{\xi_n}\} = a(\frac{1}{s})b(s)$$

for $|s| = 1$, that is, the sequence of random variables $\xi_1, \xi_2, \dots, \xi_n, \dots$ in the previous example is replaced by the sequence $-\xi_1, -\xi_2, \dots, -\xi_n, \dots$. By using the results of the previous example we can write that

$$(34) \quad 1-\rho a(\frac{1}{s})b(s) = g^+(s,\rho)g^-(s,\rho)$$

for $|s| = 1$ and $|\rho| < 1$ where now

$$(35) \quad g^+(s,\rho) = \prod_{r=1}^{m}(\frac{s-\delta_r(\rho)}{1-\beta_r s})$$

for $|s| \leqq 1$ and

$$(36) \quad g^-(s,\rho) = \frac{\prod_{r=1}^{m}(1-\beta_r s)-\rho\pi_{m-1}(s)a(\frac{1}{s})}{\prod_{r=1}^{m}(s-\delta_r(\rho))}$$

for $|s| \geqq 1$. These functions satisfy the requirements (a_1), (a_2) and

(b_1), (b_2), (b_3) respectively in Section 12.

Finally, by (27) we obtain that

$$(37) \qquad (1-\rho) \sum_{n=0}^{\infty} u_n(s)\rho^n = \prod_{r=1}^{m} \left\{ \left(\frac{1-\beta_r s}{1-\beta_r} \right) \left(\frac{1-\delta_r(\rho)}{s-\delta_r(\rho)} \right) \right\}$$

for $|s| \leqq 1$ and $|\rho| < 1$.

19. The Method of Ladder Indices. In this section we shall present another method for finding the distribution of the maximal partial sum of mutually independent and identically distributed real random variables. This method is called the method of ladder indices and is due to W. Feller [19].

First we shall formulate a simple combinatorial theorem, then we shall deduce several consequences of this theorem and finally we shall provide a new proof for the formula of Pollaczek and Spitzer which has already been proved in Section 15.

Let $x_1, x_2, \ldots, x_n$ be n real numbers. Consider their partial sums $s_0 = 0$, $s_k = x_1 + \ldots + x_k$ $(k = 1, 2, \ldots, n)$. We say that i $(i = 1, 2, \ldots, n)$ is a ladder index of $(s_0, s_1, \ldots, s_n)$ if $s_i > s_0$, $s_i > s_0, \ldots, s_i > s_{i-1}$.

Consider the n cyclic permutations of $(x_1, x_2, \ldots, x_n)$: $C_0 = (x_1, \ldots, x_n)$, $C_1 = (x_2, \ldots, x_1), \ldots, C_{n-1} = (x_n, \ldots, x_{n-1})$. Denote by $s_k^{(\nu)}$ $(k = 0, 1, \ldots, n)$ the partial sums in the cyclic permutation C_ν, that is,

$$(1) \qquad s_k^{(\nu)} = \begin{cases} s_{\nu+k} - s_\nu & \text{for } k = 1, 2, \ldots, n-\nu, \\ s_n - s_\nu + s_{k-n+\nu} & \text{for } k = n-\nu+1, \ldots, n. \end{cases}$$

Theorem 1. Let $s_n > 0$. Let us consider all those cyclic permutations among $C_0, C_1, \ldots, C_{n-1}$ in which n is a ladder index.

If the number of such cyclic permutations is r *, then* $r \geqq 1$ *, and each such permutation has exactly* r *ladder indices.*

Proof. First we shall prove that $r \geqq 1$. Choose ν such that $s_\nu > s_1, \ldots, s_\nu > s_{\nu-1}$, $s_\nu \geqq s_{\nu+1}, \ldots, s_\nu \geqq s_n$. Then in C_ν the partial sum $s_n^{(\nu)} = s_n$ is maximal and so n is a ladder index in C_ν .

Without loss of generality we may assume that n is a ladder index in C_0 , that is, in the original arrangement of the n elements $x_1, x_2, \ldots, x_n$. Then we have $s_n > s_i$ for $i = 0, 1, \ldots, n-1$. Now n is a ladder index in C_ν if and only if $s_\nu > s_0 = 0$, $s_\nu > s_1, \ldots, s_\nu > s_{\nu-1}$. For $\wedge$ $s_k^{(\nu)} = s_n - s_\nu + s_{k-n+\nu} < s_n$ for $k = n-\nu+1, \ldots, n-1$ and $s_k^{(\nu)} = s_{\nu+k} - s_\nu < s_{\nu+k} < s_n$ for $k = 1, 2, \ldots, n-\nu$, and the converse is also true.

That is n is a ladder index in C_ν if and only if ν is a ladder index in the original arrangement C_0 . Thus the number of permutations $C_0, C_1, \ldots, C_{n-1}$ in which n is a ladder index is equal to the number of ladder indices in C_0 . Hence the theorem follows.

For example, let $x_1 = -1$, $x_2 = 1$, $x_3 = 2$, $x_4 = 1$. Then $C_0 = (-1, 1, 2, 1)$, $C_1 = (1, 2, 1, -1)$, $C_2 = (2, 1, -1, 1)$, $C_3 = (1, -1, 1, 2)$ and the partial sums in C_0 are $(0, -1, 0, \underline{2}, \underline{3})$, in C_1 are $(0, 1, 3, 4, 3)$, in C_2 are $(0, 2, 3, 2, 3)$, and in C_3 are $(0, \underline{1}, 0, 1, \underline{3})$. There are two cyclic permutations C_0 and C_3 in which 3 is a ladder index and both C_0 and C_3 contain exactly 2 ladder indices which are underlined in the above examples.

$\wedge$ these conditions imply that

We note that an analogous theorem can be formulated for the so-called weak ladder indices. We say that i $(i = 1,2,\ldots, n)$ is a <u>weak ladder index</u> of $(s_0, s_1,\ldots, s_n)$ if $s_i \geqq s_0,\ldots, s_i \geqq s_{i-1}$.

<u>Theorem 2. Let</u> $s_n \geqq 0$. <u>Let us consider all those cyclic permutations among</u> $C_0, C_1,\ldots, C_n$ <u>in which</u> n <u>is a weak ladder index. If the number of such cyclic permutations is</u> r , <u>then</u> $r \geqq 1$, <u>and each such permutation has exactly</u> r <u>ladder indices</u>.

<u>Proof</u>. We can prove this theorem in exactly the same way as we proved the previous theorem.

Now let us assume that $\xi_1, \xi_2,\ldots, \xi_n,\ldots$ is a sequence of mutually independent and identically distributed real random variables. Define $\zeta_0 = 0$ and $\zeta_n = \xi_1+\ldots+ \xi_n$ for $n \geqq 1$. Denote by p_n $(n = 1,2,\ldots)$ the probability that the first ladder index is n in the sequence $\zeta_0, \zeta_1,\ldots, \zeta_n,\ldots$, that is,

(2) $$p_n = \mathbf{P}\{\zeta_1 \leqq 0 , \zeta_2 \leqq 0,\ldots, \zeta_{n-1} \leqq 0 , \zeta_n > 0\} .$$

Let

(3) $$\pi(z) = \sum_{n=1}^{\infty} p_n z^n$$

for $|z| \leqq 1$. Denote by $p_n^{(r)}$ $(n = r, r+1,\ldots ; r = 1,2,\ldots)$ the probability that the r-th ladder index is n in the sequence $\zeta_0, \zeta_1,\ldots, \zeta_n,\ldots$. Then we have

(4) $$p_n^{(r)} = \sum_{j=r-1}^{n-1} p_j^{(r-1)} p_{n-j}$$

for $r = 2,3,\ldots$ and $n = r, r+1,\ldots$ where $p_n^{(1)} = p_n$. It follows from

(4) that

$$(5)\qquad \sum_{n=r}^{\infty} p_n^{(r)} z^n = [\pi(z)]^r$$

for $r = 1,2,\ldots$ and $|z| \leqq 1$.

<u>Theorem 3.</u> <u>If</u> $|z| < 1$, <u>then</u>

$$(6)\qquad \pi(z) = 1 - e^{-\sum_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n > 0\}} .$$

<u>Proof.</u> Let $C_k^{(n)} = (\xi_1^{(k)},\ldots, \xi_n^{(k)})$ $(k = 1,2,\ldots, n)$ be the n cyclic permutations of $(\xi_1,\ldots, \xi_n)$. For each $C_k^{(n)}$ let us define the partial sums as $\zeta_0^{(k)}, \zeta_1^{(k)},\ldots, \zeta_n^{(k)}$. Fix an integer r $(r = 1,2,\ldots, n)$ and define $\chi_k = 1$ if n is the r-th ladder index of $C_k^{(n)}$ and $\chi_k = 0$ otherwise. We have $\underset{\sim}{P}\{\chi_k = 1\} = p_n^{(r)}$. On the other hand by Theorem 1

$$(7)\qquad \underset{\sim}{P}\{\chi_k = 1\} = \underset{\sim}{E}\{\chi_k\} = \frac{1}{n}\underset{\sim}{E}\{\chi_1+\ldots+ \chi_n\} = \frac{r}{n}\underset{\sim}{P}\{\chi_1+\ldots+ \chi_n = r\} .$$

Hence

$$(8)\qquad \sum_{r=1}^{n} \frac{p_n^{(r)}}{r} = \frac{1}{n}\underset{\sim}{P}\{\zeta_n > 0\}$$

for $n = 1,2,\ldots$. Let us multiply (8) by z^n and add for $n = 1,2,\ldots$. Then we obtain that

$$(9)\qquad \begin{aligned} \sum_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n > 0\} &= \sum_{n=1}^{\infty} \sum_{r=1}^{n} \frac{p_n^{(r)} z^n}{r} = \sum_{r=1}^{\infty} \frac{1}{r} \sum_{n=r}^{\infty} p_n^{(r)} z^n = \\ &= \sum_{r=1}^{\infty} \frac{[\pi(z)]^r}{r} = \log \frac{1}{1-\pi(z)} \end{aligned}$$

for $|z| < 1$. This completes the proof of the theorem.

In what follows we shall mention a few corollaries of Theorem 3.

We have

$$(10) \qquad \sum_{n=0}^{\infty} \underset{\sim}{P}\{\zeta_r \leq 0 \text{ for } 0 \leq r \leq n\} z^n = e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n \leq 0\}}$$

for $|z| < 1$. For (10) can also be expressed as

$$(11) \qquad \begin{aligned} 1 + \sum_{n=1}^{\infty} (1-p_1-\dots-p_n) z^n &= \frac{1}{1-z} - \sum_{n=1}^{\infty} (p_1+\dots+p_n) z^n = \\ = \frac{1-\pi(z)}{1-z} &= e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n > 0\} + \sum_{n=1}^{\infty} \frac{z^n}{n}} \end{aligned}$$

for $|z| < 1$. This proves (10).

We have also

$$(12) \qquad 1 + \sum_{n=1}^{\infty} \underset{\sim}{P}\{\zeta_j < \zeta_n \text{ for } j = 0,1,\dots,n-1\} z^n = e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n > 0\}}$$

for $|z| < 1$. For

$$(13) \qquad \underset{\sim}{P}\{\zeta_j < \zeta_n \text{ for } j = 0,1,\dots,n-1\} = \sum_{r=1}^{n} p_n^{(r)}$$

and hence (12) can also be expressed as

$$(14) \qquad \begin{aligned} 1 + \sum_{n=1}^{\infty} \sum_{r=1}^{n} p_n^{(r)} z^n &= 1 + \sum_{r=1}^{\infty} \sum_{n=r}^{\infty} p_n^{(r)} z^n = \\ &= 1 + \sum_{r=1}^{\infty} [\pi(z)]^r = \frac{1}{1-\pi(z)} \end{aligned}$$

for $|z| < 1$. This proves (12).

Finally we note that

$$(15)\qquad 1 + \sum_{n=1}^{\infty} \mathbf{P}\{\zeta_j \leq \zeta_n \text{ for } j = 0,1,\ldots,n\} z^n = e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{P}\{\zeta_n \geq 0\}}$$

for $|z| < 1$. This follows immediately from (10) if we apply it to the random variables $-\xi_1, -\xi_2, \ldots, -\xi_n, \ldots$.

Denote by $\tau_1 = \rho_1, \tau_2 = \rho_1 + \rho_2, \ldots, \tau_k = \rho_1 + \ldots + \rho_k, \ldots$ the successive ladder indices in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$. It is easy to see that $\{\rho_k\}$ is a sequence of mutually independent and identically distributed random variables for which

$$(16)\qquad \mathbf{P}\{\rho_k = n\} = p_n$$

for $n = 1,2,\ldots$ and $k = 1,2,\ldots$.

Furthermore, $\zeta_{\rho_1}, \zeta_{\rho_1+\rho_2} - \zeta_{\rho_1}, \ldots, \zeta_{\rho_1+\ldots+\rho_k} - \zeta_{\rho_1+\ldots+\rho_{k-1}}, \ldots$ are also mutually independent and identically distributed random variables. Next we shall be interested in finding the expectation

$$(17)\qquad \mathbf{E}\{e^{-s\zeta_{\rho_1}} z^{\rho_1}\} = \sum_{n=1}^{\infty} \mathbf{E}\{e^{-s\zeta_n} \delta(\rho_1 = n)\} z^n$$

for $\mathrm{Re}(s) \geq 0$ and $|z| \leq 1$. Here $\delta(\rho_1 = n)$ is the indicator variable of the event $\{\rho_1 = n\}$, that is, $\delta(\rho_1 = n) = 1$ if $\rho_1 = n$ and $\delta(\rho_1 = n) = 0$ if $\rho_1 \neq n$. Knowing (17), the joint distribution of ζ_{ρ_1} and ρ_1 can be obtained by inversion. If $z = 1$ in (17), then we obtain the

Laplace-Stieltjes transform $\underset{\sim}{E}\{e^{-s\zeta_{\rho_1}}\}$ for $\mathrm{Re}(s) \geqq 0$, which determines the distribution of ζ_{ρ_1}. If $s = 0$ in (17), then we obtain the generating function $\pi(z) = \underset{\sim}{E}\{z^{\rho_1}\}$ for $|z| \leqq 1$, which determines the distribution of ρ_1. The following result has been found by G. Baxter [4].

Theorem 4. If $|z| < 1$ and $\mathrm{Re}(s) \geqq 0$, then

$$(18) \qquad \underset{\sim}{E}\{e^{-s\zeta_{\rho_1}} z^{\rho_1}\} = 1 - e^{-\sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^{\infty} e^{-sx} d_x \underset{\sim}{P}\{0 < \zeta_n \leqq x\}} .$$

Proof. Let I be a subinterval of $(0, \infty)$. Denote by $p_n^{(r)}(I)$ the probability that the r-th ladder index is n and $\zeta_n \in I$. In exactly the same way as we proved (8), we can prove that

$$(19) \qquad \sum_{r=1}^{n} \frac{p_n^{(r)}(I)}{r} = \frac{1}{n} \underset{\sim}{P}\{\zeta_n \in I\}$$

for $n = 1,2,\ldots$. For if we add the condition $\zeta_n \in I$ to the conditions in (7), then each equation remains valid. By (19) it follows that

$$(20) \qquad \sum_{r=1}^{\infty} \frac{1}{r} \left(\sum_{n=r}^{\infty} p_n^{(r)}(I) z^n \right) = \sum_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n \in I\}$$

for $|z| < 1$.

Now let us suppose that, in particular, $I = (0, x]$ where $0 \leqq x < \infty$ and in this case let us use the notation

$$(21) \qquad G_n^{(r)}(x) = p_n^{(r)}(I) = \underset{\sim}{P}\,\{\text{the r-th ladder index is } n \text{ and } \zeta_n \leqq x\}$$

for $x \geqq 0$ and $1 \leqq r \leqq n$. In particular, we shall write $G_n^{(1)}(x) = G_n(x)$, that is,

$$(22) \qquad G_n(x) = p_n^{(1)}(I) = \underset{\sim}{P}\{\zeta_1 \leqq 0, \ldots, \zeta_{n-1} \leqq 0 \ , \ 0 < \zeta_n \leqq x\}$$

for $n = 1,2,\ldots$ and $x \geqq 0$.

Evidently we have

$$(23) \qquad G_n^{(r)}(x) = \sum_{j=1}^{n-1} \int_0^x G_{n-j}^{(r-1)}(x-y) dG_j(y)$$

for $r = 2,3,\ldots$ and $n = r, r+1, \ldots$. Let

$$(24) \qquad \gamma_n^{(r)}(s) = \int_0^\infty e^{-sx} dG_n^{(r)}(x)$$

and

$$(25) \qquad \gamma_n(s) = \int_0^\infty e^{-sx} dG_n(x)$$

for $Re(s) \geqq 0$ and $1 \leqq r \leqq n$. By (23) we obtain that

$$(26) \qquad \sum_{n=r}^\infty \gamma_n^{(r)}(s) z^n = [\sum_{n=1}^\infty \gamma_n(s) z^n]^r$$

for $r = 1,2,\ldots$ and $Re(s) \geqq 0$ and $|z| < 1$.

By (19)

$$(27) \qquad \sum_{r=1}^n \frac{1}{r} G_n^{(r)}(x) = \frac{1}{n} \underset{\sim}{P}\{0 < \zeta_n \leqq x\}$$

for $x \geqq 0$ and $n = 1,2,\ldots$. Thus by (26) and (27) we get

$$(28) \qquad \sum_{n=1}^\infty \gamma_n(s) z^n = 1 - e^{-\sum_{n=1}^\infty \frac{z^n}{n} \int_0^\infty e^{-sx} d_x \underset{\sim}{P}\{0<\zeta_n \leqq x\}}$$

for $\operatorname{Re}(s) \geq 0$ and $|z| < 1$. This completes the proof of Theorem 4.

We note that

(29) $$1 + \sum_{n=1}^{\infty} z^n \int_0^{\infty} e^{-sx} d_x \underset{\sim}{P}\{\zeta_j < \zeta_n \leq x \text{ for } j = 0,1,\ldots, n-1\} =$$

$$= e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \int_0^{\infty} e^{-sx} d_x \underset{\sim}{P}\{0 < \zeta_n \leq x\}}$$

for $\operatorname{Re}(s) \geq 0$ and $|z| < 1$. This follows from the following relation

(30) $$\underset{\sim}{P}\{\zeta_j < \zeta_n \leq x \text{ for } j = 0,1,\ldots, n-1\} = \sum_{r=1}^{n} G_n^{(r)}(x)$$

for $x \geq 0$ and $n = 1,2,\ldots$. By (30) we have

(31) $$1 + \sum_{n=1}^{\infty} z^n \int_0^{\infty} e^{-sx} d_x \underset{\sim}{P}\{\zeta_j < \zeta_n \leq x \text{ for } j = 0,1,\ldots, n-1\} =$$

$$= \frac{1}{1 - \sum_{n=1}^{\infty} \gamma_n(s) z^n}$$

for $\operatorname{Re}(s) \geq 0$ and $|z| < 1$ which is exactly (29).

Now we are in the position to provide another proof for the theorem of Pollaczek and Spitzer (Theorem 15.1).

<u>Theorem 5</u>. <u>Let</u> $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$ <u>and</u>

(32) $$\Phi_n(s) = \underset{\sim}{E}\{e^{-s\eta_n}\}$$

<u>for</u> $\operatorname{Re}(s) \geq 0$ <u>and</u> $n = 0,1,2,\ldots$. <u>If</u> $\operatorname{Re}(s) \geq 0$, <u>and</u> $|z| < 1$, <u>then we have</u>

$$(33)\quad \sum_{n=0}^{\infty} \Phi_n(s)z^n = e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{-0}^{\infty} e^{-sx} d_x \underset{\sim}{P}\{\zeta_n \leq x\}} =$$

$$= \frac{1}{1-z}\, e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{0}^{\infty} (e^{-sx}-1) d_x \underset{\sim}{P}\{\zeta_n \leq x\}} .$$

Proof. We can write that

$$(34)\quad \underset{\sim}{P}\{\eta_n \leq x\} = \sum_{j=0}^{n} \underset{\sim}{P}\{\zeta_i < \zeta_j \text{ for } 0 \leq i \leq j,\ \zeta_i \leq \zeta_j \text{ for } j \leq i \leq n \text{ and } \zeta_j \leq x\} =$$

$$= \sum_{j=0}^{n} \underset{\sim}{P}\{\zeta_i < \zeta_j \leq x \text{ for } 0 \leq i \leq j\}\, \underset{\sim}{P}\{\zeta_r \leq 0 \text{ for } r = 0,1,\ldots, n-j\}$$

for $n = 1,2,\ldots$ and $x \geq 0$. For the event $\{\eta_n \leq x\}$ can occur in several mutually exclusive ways. In the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n$ the first maximal element is ζ_j and $\zeta_j \leq x$. Obviously $\underset{\sim}{P}\{\zeta_i \leq \zeta_j$ for $j \leq i \leq n\} = \underset{\sim}{P}\{\zeta_r \leq 0$ for $r = 0,1,\ldots, n-j\}$. If we form the Laplace-Stieltjes transform of (34), multiply it by z^n, and add for $n = 0,1,2,\ldots$ then we obtain the product of the following two expressions.

The first expression is

$$(35)\quad 1 + \sum_{j=1}^{\infty} z^j \int_{0}^{\infty} e^{-sx} d_x \underset{\sim}{P}\{\zeta_i < \zeta_j \leq x \text{ for } 0 \leq i \leq j\} = \frac{1}{1 - \sum_{n=1}^{\infty} \gamma_n(s) z^n} =$$

$$= e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{0}^{\infty} e^{-sx} d_x \underset{\sim}{P}\{0<\zeta_n \leq x\}}$$

which is exactly (29), and the second expression is

$$1 + \sum_{n=1}^{\infty} z^n \underset{\sim}{P}\{\zeta_r \leqq 0 \ \text{ for } \ r = 0,1,\ldots, n\} = \frac{1-\pi(z)}{1-z} =$$

(36)

$$= e^{\sum_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n \leqq 0\}} = \frac{e^{-\sum_{n=1}^{\infty} \frac{z^n}{n} \underset{\sim}{P}\{\zeta_n > 0\}}}{1-z}$$

which is exactly (10). This completes the proof of (33).

20. <u>Combinatorial Methods.</u> In some particular cases we can use special methods for finding the distribution of

(1) $$\eta_n = \max(0,\ \xi_1,\ \xi_1+\xi_2,\ldots,\ \xi_1+\xi_2+\ldots+\xi_n)$$

for $n = 1,2,\ldots$. In what follows we shall show that if $\xi_1, \xi_2,\ldots, \xi_n$ are either mutually independent and identically distributed discrete random variables taking on the integers $-1,0,1,2,\ldots$ (or $1,0,-1,-2,\ldots$), or interchangeable discrete random variables taking on the integers $-1,0,1,2$ (or $1,0,-1,-2,\ldots$), then we can find the distribution of (1) in a very simple way by using the following auxiliary theorem.

<u>Lemma 1.</u> <u>Let</u> $k_1, k_2,\ldots, k_n$ <u>be nonnegative integers with sum</u> $k_1+k_2+\ldots+k_n = k \leq n$. <u>Among the</u> n <u>cyclic permutations of</u> $(k_1, k_2,\ldots,k_n)$ <u>there are exactly</u> $n-k$ <u>for which the sum of the first</u> r <u>elements is less than</u> r <u>for all</u> $r = 1,2,\ldots, n$.

<u>Proof.</u> Let $k_{r+n} = k_r$ for $r = 1,2,\ldots,$ and set $\sigma_r = k_1+\ldots+k_r$ for $r = 1,2,\ldots$ and $\sigma_0 = 0$. Define

(2) $$\delta_r = \begin{cases} 1 & \text{if } i-\sigma_i > r-\sigma_r \text{ for } r < i \leq r+n , \\ 0 & \text{otherwise,} \end{cases}$$

and

(3) $$\psi_r = \min\{i-\sigma_i \quad \text{for} \quad r < i \leq r+n\}$$

for $r = 0,1,\ldots$. Evidently $\delta_r = \psi_{r+1}-\psi_r$ for $r = 0,1,\ldots$. Since $\sigma_{r+n} = \sigma_r+\sigma_n$ for $r = 0,1,\ldots,$ we have $\delta_{r+n} = \delta_r$ and $\psi_{r+n} = \psi_r+n-k$

for $r = 0,1,\ldots$. By using the above notation, we can state that among the n cyclic permutations of $(k_1, k_2,\ldots, k_n)$ there are exactly

(4) $$\sum_{r=1}^{n} \delta_r = \psi_{n+1} - \psi_1 = n - k$$

permutations for which the sum of the first r elements is less than r for $r = 1,2,\ldots, n$. This completes the proof of Lemma 1.

A Corollary. It follows immediately from Lemma 1 that among the $n!$ permutations of $(k_1, k_2,\ldots, k_n)$ there are exactly $(n-1)!(n-k)$ for which the r-th partial sum is less than r for all $r = 1,2,\ldots, n$.

It might be interesting to mention briefly the historical background of Lemma 1. If we assume that each k_i $(i = 1,2,\ldots, n)$ is either 0 or 2 , then the above corollary of Lemma 1 reduces to the classical ballot theorem which was first formulated in 1887 by J. Bertrand [5] and proved in the same year by D. André [2]. It should be noted, however, that this particular case can also be deduced from a result of duration of plays which was found in 1708 by A. De Moivre [14 p. 262] and in a different version in 1718 also by A. De Moivre [15 p. 121]. A. De Moivre did not give proofs of his results. Proofs for De Moivre's result were given only in 1773 by P. S. Laplace [39 pp. 188-193] and in 1776 by J. L. Lagrange [38 pp. 230-238]. See also W. A. Whitworth [70], [71].

If we assume that each k_i $(i = 1, 2, \ldots, n)$ is either 0 or $\mu+1$ where μ is a positive integer, then the above mentioned corollary reduces to a generalization of the classical ballot theorem, which was formulated in 1887 by É. Barbier [3] and proved in 1924 by A. Aeppli [1].

See also A. Dvoretzky and Th. Motzkin [17], H. D. Grossman [25], S. G. Mohanty and T. V. Narayana [46], and the author [57], [58].

Now we shall prove the corollary of Lemma 1 in a slightly more general form which we shall use in what follows.

Lemma 2. *Let* $\nu_1\ \nu_2,\ldots,\ \nu_n$ *be interchangeable random variables taking on nonnegative integers. Set* $N_r = \nu_1 + \nu_2 + \ldots + \nu_r$ *for* $r = 1,2,\ldots,n$. *Then we have*

$$(5)\qquad P\{N_r < r \text{ for } r = 1,2,\ldots,n \mid N_n = k\} = \begin{cases} 1 - \frac{k}{n} & \text{if } k = 0,1,\ldots,n, \\ 0 & \text{otherwise,} \end{cases}$$

where the conditional probability is defined up to an equivalence.

Proof. We can easily deduce Lemma 2 from Lemma 1; however, in what follows we shall give a separate proof. We can prove (5) easily by mathematical induction. If $n = 1$, then (5) is evidently true. Suppose that (5) is true when n is replaced by $n - 1$ $(n = 2,3,\ldots)$. We shall prove that it is true for n too. Hence by mathematical induction it follows that (5) is true for all $n = 1,2,\ldots$. If $k \geq n$, then (5) is obviously true. Let $k < n$. By assumption

$$(6)\qquad P\{N_r < r \text{ for } r = 1,2,\ldots,n-1 \mid N_{n-1} = j\} = \begin{cases} 1 - \frac{j}{n-1} & \text{if } 0 \leq j \leq n-1, \\ 0 & \text{if } j \geq n-1. \end{cases}$$

Thus by the theorem of total probability

$$\mathop{P}\{N_r < r \text{ for } r = 1,2,\ldots, n \mid N_n = k\} = \sum_{j=0}^{n-1} \left(1 - \frac{j}{n-1}\right) \mathop{P}\{N_{n-1} = j \mid N_n = k\} =$$

(7)

$$= 1 - \frac{1}{n-1} \mathop{E}\{N_{n-1} \mid N_n = k\} = 1 - \frac{1}{(n-1)} \frac{(n-1)k}{n} = 1 - \frac{k}{n}$$

for $k = 0,1,\ldots, n-1$. For $\mathop{E}\{N_{n-1} \mid N_n = k\} = (n-1)k/n$.

It follows immediately from (5) that

(8) $$\mathop{P}\{N_r < r \text{ for } r = 1,2,\ldots, n\} = \mathop{E}\{[1 - \frac{N_n}{n}]^+\}$$

where $[x]^+ = \max(0, x)$. We note that (5) and (8) remain valid under the slightly weaker assumption that $\nu_1, \nu_2, \ldots, \nu_n$ are cyclically interchangeable random variables taking on nonnegative integers only.

It will be convenient to express Lemma 2 in the following equivalent way.

Lemma 3. *Let $\nu_1\ \nu_2, \ldots, \nu_n$ be interchangeable random variables taking on nonnegative integers. Set $N_r = \nu_1 + \ldots + \nu_r$ for $r = 1,2,\ldots, n$ and $N_0 = 0$. Define $\rho(k)$ $(k = 0,1,\ldots, n)$ as the smallest $r = 0,1,\ldots,n$ for which $r - N_r = k$ if such an r exists. We have*

(9) $$\mathop{P}\{\rho(k) = j\} = \frac{k}{j} \mathop{P}\{N_j = j - k\}$$

for $1 \leq k \leq j \leq n$, and $\mathop{P}\{\rho(0) = 0\} = 1$.

Proof. We can interpret $\rho(k)$ as the first passage time of $r - N_r$ $(r = 0,1,\ldots, n)$ through k (if any). Obviously $\mathop{P}\{\rho(0) = 0\} = 1$. For $1 \leq k \leq j \leq n$ we can write that

$$\begin{aligned}(10)\qquad \mathbf{P}\{\rho(k) = j\} &= \mathbf{P}\{r-N_r < k \text{ for } 1 \leq r < j \text{ and } j - N_j = k\} = \\ &= \mathbf{P}\{N_j - N_r < j-r \text{ for } 1 \leq r < j \text{ and } j - N_j = k\} = \\ &= \mathbf{P}\{N_i < i \text{ for } 1 \leq i < j \text{ and } N_j = j-k\} = \frac{k}{j}\mathbf{P}\{N_j = j-k\}\end{aligned}$$

where the last equality follows from Lemma 2.

An Identity. We have the following obvious relation for $1 \leq s < k \leq j \leq n$

$$(11)\qquad \sum_{i=1}^{j-1} \mathbf{P}\{\rho(s) = i,\ \rho(k)-\rho(s) = j-i\} = \mathbf{P}\{\rho(k) = j\}\ .$$

If we take into consideration that $\rho(k)-\rho(s)$ has the same distribution as $\rho(k-s)$, then (11) can also be expressed as follows:

$$(12)\qquad \sum_{i=1}^{j-1} \frac{s(k-s)}{i(j-i)} \mathbf{P}\{N_i = i-s,\ N_j = j-k\} = \frac{k}{j}\mathbf{P}\{N_j = j - k\}\ .$$

Interchangeable random variables. By using Lemma 2 we can easily find the distribution of (2) if $\xi_1, \xi_2, \ldots, \xi_n$ are interchangeable random variables which can be expressed either as $\xi_i = \nu_i - 1$ $(i = 1,2,\ldots,n)$ or as $\xi_i = 1-\nu_i$ $(i = 1,2,\ldots, n)$ where $\nu_1, \nu_2, \ldots, \nu_n$ are interchangeable discrete random variables taking on nonnegative integers only.

Theorem 1. *Let* $\nu_1, \nu_2, \ldots, \nu_n$ *be interchangeable random variables taking on nonnegative integers only. Let* $N_r = \nu_1 + \nu_2 + \ldots + \nu_r$ *for* $r = 1,2,\ldots,n$ *and* $N_0 = 0$. *We have*

$$(13)\qquad \begin{aligned}\mathbf{P}\{\max_{1\leq r\leq n} (N_r - r) < k\} &= \mathbf{P}\{N_n < n + k\} - \\ &- \sum_{j=1}^{n-1} \sum_{\ell=0}^{n-j} \left(1 - \frac{\ell}{n-j}\right)\mathbf{P}\{N_j = j+k\ ,\ N_n = j+k+\ell\}\end{aligned}$$

for $k = 0, \pm 1, \pm 2, \ldots$. *If* $k < 0$, *then both sides of* (13) *are* 0 .

Proof. We shall prove a slightly more general formula from which (13) follows. If $i = 1,2,\dots, n-1$, and $k = 0, \pm 1, \pm 2, \dots$, then

$$\begin{aligned} &P\{N_r < r+k \text{ for } r = 1,2,\dots, n \text{ and } N_n \leqq n+k-i\} = \\ (14) \quad &= P\{N_n \leqq n+k-i\} - \sum_{j=1}^{n-i} \sum_{\ell=0}^{n-i-j} \left(1 - \frac{\ell}{n-j}\right) P\{N_j = j+k,\ N_n = j+k+\ell\}. \end{aligned}$$

It is sufficient to prove that the subtrahend on the right-hand side of (14) is the probability that $N_r \geqq r+k$ for some $r = 1,2,\dots, n-1$ and $N_n \leqq n+k-i$. This event can occur in the following mutually exclusive ways: the greatest r for which $N_r \geqq r+k$ is $r = j$ ($j = 1,2,\dots, n-i$). Then $N_j = j+k$ and $N_r < r+k$ for $r = j+1,\dots, n$, or equivalently, $N_r - N_j < r-j$ for $r = j+1,\dots, n$. By Lemma 2

$$(15) \quad P\{N_r - N_j < r-j \text{ for } r = j+1,\dots,n \mid N_j = j+k,\ N_n = j+k+\ell\} = 1 - \frac{\ell}{n-j}$$

if $0 \leqq \ell \leqq n-j$ and if the left-hand side is defined. If we multiply (15) by $P\{N_j = j+k,\ N_n = j+k+\ell\}$ and add for all (j, ℓ) satisfying $1 \leqq j \leqq$ $\leqq j + \ell \leqq n-i$, then we obtain the subtrahend on the right-hand side of (14). If $i = 1$ in (14), then we obtain (13) which was to be proved.

If, in particular, $k = 0$, then by Lemma 1 we can write also that

$$(16) \quad P\{N_r < r \text{ for } r = 1,2,\dots,n \text{ and } N_n \leqq n-i\} = \sum_{j=1}^{n-i} \left(1 - \frac{j}{n}\right) P\{N_n = j\}$$

for $i = 0,1,\dots, n-1$.

Theorem 2. *Let $\nu_1, \nu_2, \dots, \nu_n$ be interchangeable random variables taking on nonnegative integers only. Let $N_r = \nu_1 + \nu_2 + \dots + \nu_r$ for $r = 1,2,\dots, n$ and $N_0 = 0$. We have*

$$(17)\qquad \underset{\sim}{P}\{\max_{1\leq r\leq n}(r-N_r)<k\} = 1-\sum_{j=k}^{n}\frac{k}{j}\underset{\sim}{P}\{N_j=j-k\}$$

for $k = 1,2,\ldots$.

Proof. We shall find the probability of the complementary event of $\{\max_{1\leq r\leq n}(r-N_r)<k\}$, that is, the probability that $N_r \leq r-k$ for some $r = 1,2,\ldots, n$. This latter event can occur in the following mutually exclusive ways: the smallest r such that $N_r = r-k$ is $r=j$ $(j=k,\ldots,n)$. Then $N_j=j-k$ and $N_r > r-k$ for $r=1,\ldots,j-1$, or equivalently, $N_j-N_r < j-r$ for $r=1,\ldots,j-1$. By Lemma 1

$$(18)\qquad \underset{\sim}{P}\{N_j-N_r<j-r \text{ for } r=1,\ldots,j-1 \mid N_j=j-k\} = \frac{k}{j}$$

for $0 < k \leq j$ where the conditional probability is defined up to an equivalence. If we multiply (18) by $\underset{\sim}{P}\{N_j = j-k\}$ and add for $k \leq j \leq n$, then we get the probability of the complimentary event. This proves (17).

In a similar way as (17) we can prove the following more general formula

$$(19)\qquad \begin{aligned} &\underset{\sim}{P}\{r-N_r<k \text{ for } r=1,2,\ldots,n \text{ and } n-N_n<k-i\} = \\ &= \underset{\sim}{P}\{N_n>n+i-k\} - \sum_{j=k}^{n}\frac{k}{j}\underset{\sim}{P}\{N_j=j-k\,,\ N_n>n+i-k\}\end{aligned}$$

for $n = 1,2,\ldots$, $k = 1,2,\ldots$ and $i = 0, \pm 1, \pm 2,\ldots$.

Independent Random Variables. If we suppose, in particular, that $\nu_1, \nu_2, \ldots, \nu_n$ are mutually independent and identically distributed random variables taking on nonnegative integers only, then Theorem 1 and Theorem 2 can be expressed in somewhat simpler forms.

As previously, let us write $N_r = \nu_1 + \nu_2 + \ldots + \nu_r$ for $r = 1,2,\ldots, n$ and $N_0 = 0$. Furthermore, let us introduce the notation

(20) $$P_{ik}(n) = \underset{\sim}{P}\{N_r - r < k \text{ for } r = 1,2,\ldots, n \text{ and } N_n - n < k-i\}$$

for $n = 1,2,\ldots$, $i = 0, \pm 1, \pm 2, \ldots$ and $k = 0, \pm 1, \pm 2, \ldots$. Let $P_{ik}(0) = 1$ if $k \geqq i$ and $P_{ik}(0) = 0$ if $k < i$. Obviously $P_{ik}(n) = 0$ if $k < 0$. We note also that $P_{0k}(n) = P_{1k}(n)$ if $n \geqq 1$.

Let us introduce also the notation

(21) $$Q_{ik}(n) = \underset{\sim}{P}\{r - N_r < k \text{ for } r = 0,1,\ldots,n \text{ and } n - N_n < k-i\}$$

for $n = 1,2,\ldots$, $i = 0, \pm 1, \pm 2, \ldots$ and $k = 1,2,\ldots$. Let $Q_{ik}(0) = 1$ if $k \geqq i$ and $Q_{ik}(0) = 0$ if $k < i$. Obviously $Q_{ik}(n) = Q_{0k}(n)$ if $i < 0$.

In case of independent random variables Theorem 1 or more generally formula (14) reduces to the following one.

Theorem 3. If $\nu_1, \nu_2, \ldots, \nu_n$ are mutually independent and identically distributed discrete random variables taking on nonnegative integers only, then we have

(22) $$P_{ik}(n) = \underset{\sim}{P}\{N_n \leqq n+k-i\} - \sum_{j=1}^{n-1} P_{i0}(n-j)\underset{\sim}{P}\{N_j = j+k\}$$

for $n = 1,2,\ldots$, $i = 0,1,2,\ldots$ and $k = 0, \pm 1, \pm 2, \ldots$, and

(23) $$P_{i0}(n) = \sum_{j=0}^{n-i} (1-\frac{j}{n})\underset{\sim}{P}\{N_n = j\}$$

<u>for</u> $n = 1,2,\ldots$ <u>and</u> $i = 0,1,2,\ldots$. <u>We have</u> $P_{i0}(0) = 1$ <u>for</u> $i = 0,1,2,\ldots$.

<u>Proof</u>. If we take into consideration that in (14)

(24) $$\underset{\sim}{P}\{N_j = j+k\ ,\ N_n = j+k+\ell\} = \underset{\sim}{P}\{N_j = j+k\}\underset{\sim}{P}\{N_{n-j} = \ell\}$$

and if we use (16), then we obtain (22) for $i \geqq 1$. If we define $P_{i0}(0) = 1$ for $i \geqq 0$, then we can easily see that (22) remains valid for $i = 0$ too. Formula (23) is exactly (16).

In case of independent random variables Theorem 2 or more generally formula (19) reduces to the following one.

<u>Theorem 4</u>. <u>If</u> $\nu_1, \nu_2, \ldots, \nu_n$ <u>are mutually independent and identically distributed discrete random variables taking on nonnegative integers only, then we have</u>

(25) $$Q_{ik}(n) = \underset{\sim}{P}\{N_n > n+i-k\} - \sum_{j=k}^{n} \frac{k}{j}\underset{\sim}{P}\{N_j = j-k\}\underset{\sim}{P}\{N_{n-j} > n-j+i\}$$

<u>for</u> $n = 1,2,\ldots, k = 1,2,\ldots$ <u>and</u> $i = 0, \pm 1, \pm 2,\ldots$.

<u>Proof</u>. Since in this case

(26) $$\underset{\sim}{P}\{N_j = j-k\ ,\ N_n > n+i-k\} = \underset{\sim}{P}\{N_j = j-k\}\underset{\sim}{P}\{N_{n-j} > n-j+i\}\ ,$$

we obtain (25) by (19).

An Infinite Sequence of Independent Random Variables. Now let us suppose that $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ is an infinite sequence of mutually independent and identically distributed discrete random variables taking on nonnegative integers only. In this case we can define $P_{ik}(n)$ and $Q_{ik}(n)$ for every $n = 0,1,2,\ldots$, and our next aim is to find the generating functions

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{ik}(n) z^n w^k \tag{27}$$

and

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} Q_{ik}(n) z^n w^i \tag{28}$$

for $|z| < 1$ and $|w| < 1$.

We shall introduce the notation

$$\underset{\sim}{P}\{\nu_n = j\} = h_j \tag{29}$$

for $j = 0,1,2,\ldots$ and

$$\underset{\sim}{E}\{z^{\nu_n}\} = h(z) = \sum_{j=0}^{\infty} h_j z^j \tag{30}$$

for $|z| \leq 1$. The generating function $h(z)$ is regular in the circle $|z| < 1$, and continuous in $|z| \leq 1$. Obviously, $|h(z)| \leq 1$ for $|z| \leq 1$. By (30) we can write that

$$\underset{\sim}{E}\{z^{N_k}\} = [h(z)]^k \tag{31}$$

for $k = 0,1,2,\ldots$.

We shall need the following auxiliary theorem.

Lemma 4. *If* $|z| < 1$, *then the equation*

$$w = z\, h(w) \tag{32}$$

has exactly one root $w = \delta(z)$ *in the unit circle* $|w| < 1$, *and*

$$[\delta(z)]^k = \sum_{n=k}^{\infty} \frac{k}{n} P\{\underset{\sim}{N}_n = n-k\} z^n \tag{33}$$

for $k = 1,2,\ldots,$ *and* $|z| < 1$.

Proof. If $|w| = 1$, then $|z\, h(w)| \leq |z| < 1$ and thus by Rouché's theorem it follows that (32) has the same number of roots in the domain $|w| < 1$ as the equation $w = 0$, that is, exactly one root. We shall denote this root by $\delta(z)$.

If $f(w)$ is a regular function of w in the domain $|w| < 1$,then by Lagrange's expansion we obtain that

$$f[\delta(z)] = f(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \left[\frac{d^{n-1} f'(x)[h(x)]^n}{dx^{n-1}}\right]_{x=0} \tag{34}$$

for $|z| < 1$. If we apply (34) to the function $f(x) = x^k$ $(k = 1,2,\ldots)$, then we obtain (33).

Furthermore, we note that

$$\sum_{k=-\infty}^{\infty} w^k \sum_{j=1}^{\infty} P\{\underset{\sim}{N}_j = j+k\} z^j = \frac{zh(w)}{w-zh(w)} \tag{35}$$

for $|z\, h(w)| < |w| \leq 1$. This can be seen as follows.

By (31) we obtain that

$$\sum_{k=-\infty}^{\infty} \mathbf{P}\{N_j = j+k\}w^k = \left[\frac{h(w)}{w}\right]^j \tag{36}$$

for $j = 0,1,2,\ldots$ and $0 < |w| \leq 1$. If we multiply (36) by z^j and add for $j = 1,2,\ldots$, then we get (35).

<u>Theorem 5</u>. <u>If $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ is a sequence of mutually independent and identically distributed discrete random variables taking on nonnegative integers only, then we have</u>

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} P_{ik}(n)z^n w^k = z\,\frac{[1-\delta(z)]h(w)w^i - (1-w)h(w)[\delta(z)]^i}{(1-w)[1-\delta(z)][w-z\,h(w)]} \tag{37}$$

<u>for</u> $|z| < 1$, $|w| < 1$ <u>and</u> $i = 0,1,2,\ldots$.

<u>Proof</u>. Since $P_{ik}(n) = 0$ if $k < 0$, we can extend the second summation in (37) to $-\infty < k < \infty$ without changing the sum. Then by (22) and (36) we obtain that

$$\sum_{k=-\infty}^{\infty} P_{ik}(n)w^k = \frac{w^i}{1-w}\left[\frac{h(w)}{w}\right]^n - \sum_{j=1}^{n-i} P_{i0}(n-j)\left[\frac{h(w)}{w}\right]^j \tag{38}$$

for $0 < |w| < 1$, $n = 1,2,\ldots$ and $i = 0,1,2,\ldots$. If $n = 0$, then (38) is equal to $w^i/(1-w)$ for $|w| < 1$.

By (23) it follows that

$$\sum_{n=i}^{\infty} P_{i0}(n)z^n = \frac{[\delta(z)]^i}{1-\delta(z)} \tag{39}$$

for $|z| < 1$ and $i = 0,1,2,\ldots$. This can be proved as follows. By (23) and (33) we have

$$\sum_{n=i}^{\infty} P_{i0}(n)z^n = \sum_{n=i}^{\infty} z^n \sum_{j=0}^{n-i} (1-\frac{j}{n})P\{N_n = j\} = \sum_{n=i}^{\infty} z^n \sum_{j=i}^{n} \frac{j}{n} P\{N_n = n-j\} =$$

(40)

$$= \sum_{j=i}^{\infty} \sum_{n=j}^{\infty} \frac{j}{n} P\{N_n = n-j\}z^n = \sum_{j=i}^{\infty} [\delta(z)]^j = \frac{[\delta(z)]^i}{1-\delta(z)}$$

for $|z| < 1$ and $i = 0,1,2,\ldots$. If $i = 0$,then (40) remains true because $P_{00}(n) = P_{10}(n)$ if $n \geq 1$ and $P_{00}(0) = 1$.

Since $P_{i0}(n) = 0$ if $0 \leq n < i$, it follows from (38) and (40) that

$$(41) \quad \sum_{n=i}^{\infty} \sum_{k=-\infty}^{\infty} P_{ik}(n)z^n w^k = \frac{w^i z\, h(w)}{(1-w)[w-zh(w)]} - \left(\frac{[\delta(z)]^i}{1-\delta(z)}\right)\left(\frac{z\, h(w)}{w-zh(w)}\right)$$

for $|z\, h(w)| < |w| < 1$. By analytical continuation we can extend the definition of the right-hand side of (41) to the domain $|z| < 1$, $|w| < 1$ and thus we obtain (37).

Theorem 6. If $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ is a sequence of mutually independent and identically distributed discrete random variables taking on nonnegative integers only, then we have

$$(42) \quad \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} Q_{ik}(n)z^n w^i = \frac{z-[\delta(z)]^k}{(1-w)(1-z)} - \frac{zw^k h(w)-w[\delta(z)]^k}{(1-w)[w-zh(w)]}$$

for $|z| < 1$, $|w| < 1$ and $k = 1,2,\ldots$.

Proof. By Theorem 2 we have

$$(43) \qquad Q_{0k}(n) = 1 - \sum_{j=k}^{n} \frac{k}{j} P\{N_j = j-k\}$$

for $1 \leq k \leq n$. Hence by (33) we obtain that

$$(44) \qquad \sum_{n=1}^{\infty} Q_{0k}(n)z^n = \frac{z-[\delta(z)]^k}{1-z}$$

for $|z| < 1$. This proves (42) for $w = 0$.

By (25) and (43) it follows that

$$(45) \qquad Q_{0k}(n) - Q_{ik}(n) = \underset{\sim}{P}\{N_n \leqq n+i-k\} - \sum_{j=k}^{n} \frac{k}{j} \underset{\sim}{P}\{N_j = j-k\} \underset{\sim}{P}\{N_{n-j} \leqq n-j+i\}$$

for $n = 1,2,\ldots$, $k = 1,2,\ldots$ and $i = 0, \pm 1, \pm 2, \ldots$. If $i \leqq 0$, then (45) is 0 because by (21) we have $Q_{ik}(n) = Q_{0k}(n)$ for $i \leqq 0$.

If we take into consideration that

$$(46) \qquad \sum_{i=-\infty}^{\infty} \underset{\sim}{P}\{N_n \leqq n+i-k\} w^i = \frac{w^{k-n}[h(w)]^n}{1-w}$$

for $0 < |w| < 1$, then by (43) we obtain that

$$(47) \qquad \sum_{i=0}^{\infty} [Q_{0k}(n) - Q_{ik}(n)] w^i = \frac{w^{k-n}[h(w)]^n}{1-w} - \frac{1}{1-w} \sum_{j=k}^{n} \frac{k}{j} \underset{\sim}{P}\{N_j = j-k\} \left[\frac{h(w)}{w}\right]^{n-j}$$

for $0 < |w| < 1$. If we multiply (47) by z^n, add for $n = 1,2,\ldots,$ and use (33) and (44), then we obtain that

$$(48) \qquad \frac{z-[\delta(z)]^k}{(1-w)(1-z)} - \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} Q_{ik}(n) z^n w^i = \frac{zw^k h(w) - w[\delta(z)]^k}{(1-w)[w-zh(w)]}$$

for $0 < |w| < 1$ and $|zh(w)| < |w|$. By analytical continuation we can extend the definition of the right-hand side of (48) for $|w| < 1$ and $|z| < 1$, and thus we obtain (42).

The Use of Markov Chains. Finally, we note that Theorem 5 and Theorem 6 can also be proved by using the theory of Markov chains.

First, we observe that if we define a sequence of random variables η_n $(n = 0,1,\ldots)$ by the recurrence formula

(49) $$\eta_n = [\eta_{n-1}-1]^+ + \nu_n$$

for $n = 1,2,\ldots,$ then

(50) $$\underset{\sim}{P}\{\eta_n \leq k \mid \eta_0 = i\} = \underset{\sim}{P}\{N_r < r+k \text{ for } r = 1,2,\ldots, n \text{ and } N_n \leq n+k-i\}$$

where $N_r = \nu_1 + \ldots + \nu_r$ for $r = 1,2,\ldots, n$ and $N_0 = 0$.

Accordingly, if $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ is a sequence of mutually independent and identically distributed discrete random variables taking on nonnegative integers only, then by (20) we can write that

(51) $$P_{ik}(n) = \underset{\sim}{P}\{\eta_n \leq k \mid \eta_0 = i\} .$$

If η_0 is a discrete random variable taking on nonnegative integers only and if η_0 and the sequence $\{\nu_n\}$ are independent, then the sequence of random variables $\{\eta_n\}$ forms a homogeneous Markov chain with state space $I = \{0,1,2,\ldots\}$ and transition probabilities

(52) $$P_{ik} = \begin{cases} h_k & \text{if } i = 0 \text{ and } k \geq 0 , \\ h_{k-i+1} & \text{if } i \geq 1 \text{ and } k \geq i-1 , \\ 0 & \text{if } i \geq 1 \text{ and } k < i-1 \end{cases}$$

where we used the notation (29).

If we denote by $p_{ik}^{(n)}$ $(n = 0,1,2,\ldots)$ the n-step transition probabilities, that is, $p_{ik}^{(n)} = \mathbf{P}\{\eta_n = k \mid \eta_0 = i\}$, then we have

$$P_{ik}(n) = \sum_{j=0}^{k} p_{ij}^{(n)} \tag{53}$$

for $n \geq 0$, $i \geq 0$ and $k \geq 0$.

<u>Theorem 7.</u> <u>We have</u>

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p_{ik}^{(n)} z^n w^k = \frac{[1-\delta(z)]w^{i+1} - z(1-w)h(w)[\delta(z)]^i}{[1-\delta(z)][w-zh(w)]} \tag{54}$$

<u>for</u> $|z| < 1$ <u>and</u> $|w| \leq 1$ <u>where</u> $\delta(z)$ <u>is defined in Lemma 4.</u>

<u>Proof.</u> If $h_0 = 0$ or $z = 0$, then $\delta(z) = 0$ and (54) is obviously true. Let us suppose that $h_0 > 0$ and $z \neq 0$. In this case $\delta(z) \neq 0$. Let

$$U_{ni}(w) = \mathbf{E}\{w^{\eta_n} \mid \eta_0 = i\} = \sum_{k=0}^{\infty} p_{ik}^{(n)} w^k \tag{55}$$

for $|w| \leq 1$. By (49) we have

$$U_{ni}(w) = h(w)\left[\frac{U_{n-1,i}(w) - p_{i0}^{(n-1)}}{w} + p_{i0}^{(n-1)}\right] \tag{56}$$

for $|w| \leq 1$ and clearly $U_{0i}(w) = w^i$. Hence

$$\sum_{n=0}^{\infty} U_{ni}(w) z^n = \frac{w^{i+1} - z(1-w)h(w) \sum_{n=0}^{\infty} p_{i0}^{(n)} z^n}{w-zh(w)} \tag{57}$$

for $|w| \leq 1$ and $|z| < 1$. If $|z| < 1$, then the left-hand side of (57) is a bounded function of w in the circle $|w| \leq 1$. Obviously the absolute value of (57) is $\leq 1/(1-|z|)$ if $|w| \leq 1$. If $|z| < 1$, then the denominator of the right-hand side of (57) has exactly one root $w = \delta(z)$ in the unit circle $|w| < 1$. This must be a root of the numerator too. Thus it follows that

$$(58) \qquad \sum_{n=0}^{\infty} p_{io}^{(n)} z^n = \frac{[\delta(z)]^i}{[1-\delta(z)]}$$

for $|z| < 1$. Putting (58) into (57) we obtain that

$$(59) \qquad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p_{ik}^{(n)} z^n w^k = \frac{[1-\delta(z)]w^{i+1} - z(1-w)h(w)[\delta(z)]^i}{[1-\delta(z)][w-zh(w)]}$$

for $|z| < 1$ and $|w| \leq 1$. This proves (54).

By (53) and (54) we can obtain (37). If we subtract w^i from (59) and multiply the difference by $1/(1-w)$, then we obtain (37).

Second, we observe that if we define a sequence of random variables $\bar{\eta}_n$ $(n = 0,1,2,\ldots)$ by the recurrence formula

$$(60) \qquad \bar{\eta}_n = [\bar{\eta}_{n-1} + 1 - \nu_n]^+$$

for $n = 1,2,\ldots,$ then

$$(61) \qquad \underset{\sim}{P}\{\bar{\eta}_n < k \mid \bar{\eta}_0 = i\} = \underset{\sim}{P}\{r - N_r < k \text{ for } r = 0,1,\ldots,n \text{ and } n - N_n < k - i\}$$

where $N_r = \nu_1 + \ldots + \nu_r$ for $r = 1,2,\ldots,n$ and $N_0 = 0$.

Accordingly, if $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ is a sequence of mutually independent and identically distributed discrete random variables taking on nonnegative integers only, then by (21) we can write that

$$Q_{ik}(n) = \underset{\sim}{P}\{\bar{\eta}_n < k \mid \bar{\eta}_0 = i\} \tag{62}$$

for $i = 0,1,\ldots,\ k = 1,2,\ldots$ and $n = 0,1,2,\ldots$.

If $\bar{\eta}_0$ is a discrete random variable taking on nonnegative integers only and if $\bar{\eta}_0$ and the sequence $\{\nu_n\}$ are independent, then the sequence of random variables $\{\bar{\eta}_n\}$ forms a homogeneous Markov chain with state space $I = \{0,1,2,\ldots,\}$ and transition probabilities

$$q_{ik} = \begin{cases} 1 - (h_0+\ldots+h_i) & \text{if } k = 0 , \\ h_{i+1-k} & \text{if } k = 1,\ldots, i+1, \\ 0 & \text{if } k > i+1 , \end{cases} \tag{63}$$

where we used the notation (29).

If we denote by $q_{ik}^{(n)}$ $(n = 0,1,2,\ldots)$ the n-step transition probabilities, that is, $q_{ik}^{(n)} = \underset{\sim}{P}\{\bar{\eta}_n = k \mid \bar{\eta}_0 = i\}$, then we have

$$Q_{ik}(n) = \sum_{j=0}^{k-1} q_{ij}^{(n)} \tag{64}$$

for $n \geq 0$, $i \geq 0$ and $k \geq 1$.

<u>Theorem 8</u>. <u>We have</u>

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q_{ik}^{(n)} z^n w^i = \frac{(1-w)(1-z)w^{k+1} + z[w-h(w)][1-\delta(z)][\delta(z)]^k}{(1-w)(1-z)[w-zh(w)]} \tag{65}$$

<u>for</u> $|z| < 1$ <u>and</u> $|w| < 1$, <u>where</u> $\delta(z)$ <u>is defined in Lemma 4</u>.

<u>Proof</u>. If $h_0 = 0$ or $z = 0$,then $\delta(z) = 0$ <u>and</u> (65) is

obviously true. In what follows we assume that $h_0 > 0$ and $z \neq 0$, in which case $\delta(z) \neq 0$.

Let us introduce the generating function

$$(66) \qquad V_{nk}(w) = \sum_{i=0}^{\infty} q_{ik}^{(n)} w^i$$

for $|w| < 1$. If we take into consideration that

$$(67) \qquad q_{ik}^{(n)} = \sum_{j=0}^{i+1} q_{ij} q_{jk}^{(n-1)}$$

for $n = 1,2,\ldots$, $i = 0,1,2,\ldots$ and $k = 0,1,2,\ldots$, then we obtain that

$$(68) \qquad w\, V_{nk}(w) - h(w) V_{n-1,k}(w) = \frac{w-h(w)}{1-w}\, q_{0k}^{(n-1)}$$

for $n = 1,2,\ldots$ and $|w| < 1$, and clearly $V_{0k}(w) = w^k$. From (68) it follows that

$$(69) \qquad \sum_{n=0}^{\infty} V_{nk}(w) z^n = \frac{(1-w)w^{k+1} + z[w-h(w)] \sum_{n=0}^{\infty} q_{0k}^{(n)} z^n}{(1-w)[w-zh(w)]}$$

for $|w| < 1$ and $|z| < 1$. If $|z| < 1$, then the left-hand side of (69) is a bounded function of w in the circle $|w| < 1-\varepsilon$ where ε is an arbitrary small positive number. Obviously the absolute value of (69) is $\leq 1/(1-|z|)(1-|w|)$. If $|z| < 1$, then the denominator of the right-hand side of (69) has exactly one root $w = \delta(z)$ in the unit circle $|w| < 1$. This must be a root of the numerator too. Thus it follows that

$$\sum_{n=0}^{\infty} q_{0k}^{(n)} z^n = \frac{[1-\delta(z)][\delta(z)]^k}{(1-z)} \tag{70}$$

for $|z| < 1$. Putting (70) into (69) we obtain that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} q_{ik}^{(n)} z^n w^i = \frac{(1-w)(1-z)w^{k+1} + z[w-h(w)][1-\delta(z)][\delta(z)]^k}{(1-w)(1-z)[w-zh(w)]} \tag{71}$$

for $|z| < 1$ and $|w| < 1$ which proves (65).

By (64) and (65) we obtain that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} Q_{ik}^{(n)} z^n w^i = \frac{(1-z)w(1-w^k) + z[w-h(w)]\{1-[\delta(z)]^k\}}{(1-w)(1-z)[w-zh(w)]} \tag{72}$$

for $|z| < 1$, $|w| < 1$ and $k = 1,2,\ldots$. If we subtract $(1-w^k)/(1-w)$ from (72), then we obtain (42).

21. PROBLEMS

21.1. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed real random variables having a continuous and symmetric distribution. Denote by ν_n the number of ladder indices among $1, 2, \ldots, n$. Prove that

$$P\{\nu_n = k\} = \binom{2n-k}{n} \frac{1}{2^{2n-k}}$$

for $k = 0, 1, \ldots, n$.

21.2. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent and identically distributed random variables for which $P\{\xi_n = 1\} = p$ and $P\{\xi_n = -1\} = q$ where $p > 0$, $q > 0$ and $p+q = 1$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1, 2, \ldots$ and $\zeta_0 = 0$. Denote by τ_k $(k = 1, 2, \ldots)$ the k-th ladder index in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$. Find the distribution of τ_k.

21.4. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be mutually independent random variables having the same stable distribution function $R_\alpha(x)$ for which

$$\phi_\alpha(\omega) = \int_{-\infty}^{\infty} e^{i\omega x}\, dR_\alpha(x)$$

is determined by

$$\log \phi_\alpha(\omega) = -c|\omega|^\alpha (1 - i\beta \operatorname{sgn} \omega \tan \frac{\alpha\pi}{2})$$

where $c > 0$, $0 < \alpha \leqq 2$, $\alpha \neq 1$, $-1 \leqq \beta \leqq 1$. Let $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1, 2, \ldots$, and $\zeta_0 = 0$. Denote by τ_k the k-th ladder index in the sequence $\zeta_0, \zeta_1, \ldots, \zeta_n, \ldots$. Find the distribution of τ_k for $k = 1, 2, \ldots$.

21.3. In Problem 21.2 write $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$ for $n = 1, 2, \ldots$. Determine $P\{\eta_n \geq k\}$ for $k = 1, 2, \ldots$.

21.5. Let $\nu_1, \nu_2, \ldots, \nu_n$ be interchangeable random variables taking on nonnegative integers only. Set $N_r = \nu_1 + \nu_2 + \ldots + \nu_r$ for $r = 1, 2, \ldots, n$ and $N_0 = 0$. Prove that

$$E\{\max_{0 \leq r \leq n} (N_r - r)\} = \sum_{j=1}^{n} \frac{1}{j} E\{[N_j - j]^+\} .$$

21.6. Let $\nu_1, \nu_2, \ldots, \nu_n$ be interchangeable random variables taking on nonnegative integers only. Set $N_r = \nu_1 + \nu_2 + \ldots + \nu_r$ for $r = 1, 2, \ldots, n$ and $N_0 = 0$. Prove that

$$E\{\max_{0 \leq r \leq n} (r - N_r)\} = \sum_{j=1}^{n} \frac{1}{j} E\{[j - N_j]^+\} .$$

21.7. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of mutually independent and identically distributed real random variables. Set $\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n = 1, 2, \ldots$ and $\zeta_0 = 0$. Find the expectation of $\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n)$ for $n = 1, 2, \ldots$.

21.8. Let $\xi_n = \chi_n - \theta_n$ for $n = 1, 2, \ldots$ where $\{\chi_n\}$ and $\{\theta_n\}$ are independent sequences of mutually independent nonnegative random variables. Let us suppose, in particular, that $P\{\theta_n \leq x\} = 1 - e^{-\lambda x}$ for $x \geq 0$ where λ is a positive constant. Find the distribution function of the random variable $\eta_n = \max(0, \xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \ldots + \xi_n)$.

21.9. Let $\xi_n = \chi_n - \theta_n$ for $n = 1, 2, \ldots$ where $\{\chi_n\}$ and $\{\theta_n\}$ are independent sequences of mutually independent and identically distributed nonnegative random variables. Let us suppose, in particular, that

$$P\{\theta_n \leq x\} = \begin{cases} 1 - \sum_{j=0}^{m-1} e^{-\lambda x} \frac{(\lambda x)^j}{j!} & \text{for } x \geq 0 , \\ 0 & \text{for } x < 0 , \end{cases}$$

where λ is a positive constant and m is a positive integer. Find the distribution function of the random variable $\eta_n = \max(0, \xi_1, \xi_1 + \xi_2, \ldots, \xi_1 + \ldots + \xi_n)$.

21.10. A box contains n cards marked $k_1, k_2, \ldots, k_n$ where $k_1, k_2, \ldots, k_n$ are nonnegative integers with sum $k_1 + k_2 + \ldots + k_n = k$. We draw all the n cards without replacement from the box. Let us suppose that all the $n!$ results are equally probable. Find the probability that for every $r = 1, 2, \ldots, n$ the sum of the first r numbers drawn is less than r. (See the Corollary to Lemma 20.1.)

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