RANDOM WALK IN FLUCTUATING RANDOM MEDIUM

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Abstract

A review of the results concerning random walk in a fluctuating in time random environment is given. The results concern the validity of the central limit theorem and the behaviour in time of the correlations.

1 Introduction.

The seminar will concern with random walk in fluctuating (in time) random environment.

Everybody knows the basic model of the random walk. The classical example (see Figure 1)

$$\mathcal{P}(X_t = X_{t-1} + 1) = p \qquad \mathcal{P}(X_t = X_{t-1} - 1) = q, \quad X_t \in \mathbb{Z} \tag{1.1}$$

with p and q positive numbers such that

$$p + q = 1. \tag{1.2}$$

The symmetric case is $p = q = \frac{1}{2}$. This is one of the fundamental model in the study of probability.

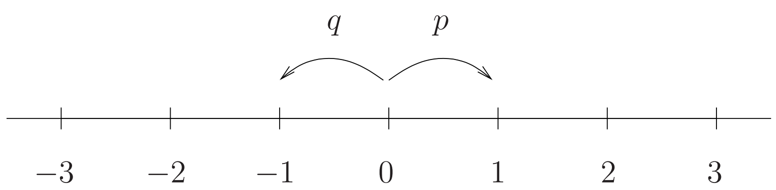

Figure 1

The model can be made more complicated in different ways. For example one can consider not only first neighbor jumps (see Figure 2), with the condition

$$q_1 + q_2 + p_1 + p_2 = 1. \tag{1.3}$$

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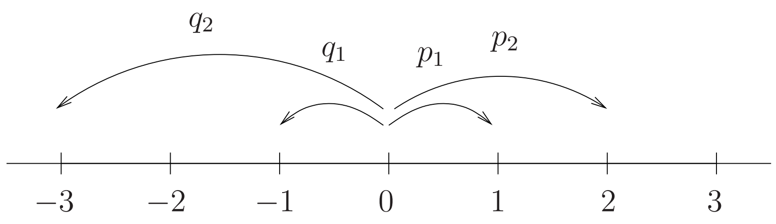

Figure 2

Another way to complicate the model is to consider the motion in higher spatial dimension d, for example in $d = 2$ (see Figure 3), with the condition $p_i \geq 0$ and $p_1 + p_2 + p_3 + p_4 = 1$. In general in $\mathbb{Z}^d$ the symmetric nearest neighbor case is

$$\mathcal{P}(X_1 = \underline{e}_j \,|X_0 = 0) = \frac{1}{2d}$$

where $\underline{e}_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the unit vector of the $j-$direction. It is also possible to have transition probabilities which allow to jump to any distance (long range).

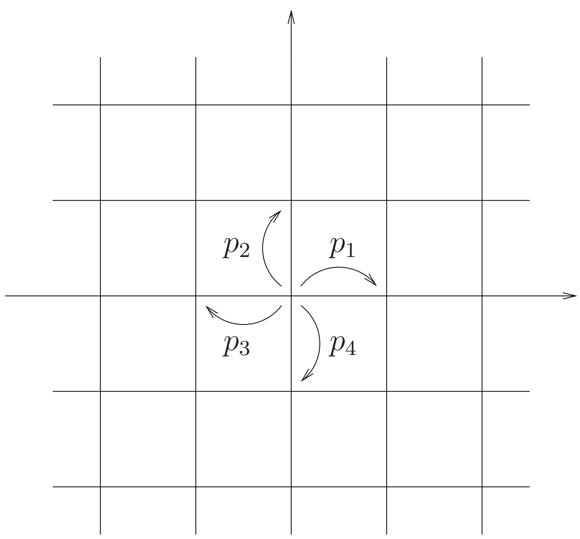

Figure 3

Another way to complicate the model is to introduce a family of positive numbers $\{p_{x,y}\}$, where $p_{x,y}$ is the probability to jump from x to y. This means that the random walk is not homogeneous in space. It is also possible consider random walks which are not homogeneous in time.

An interesting class of models is the case in which the particle jumps with a transition probabilities depending on some random field i.e. random walk in random environment. A classical example in $d = 1$ is

$$\mathcal{P}(X_t = u+1|X_{t-1} = u, \xi) = \frac{1}{2} + \frac{\epsilon}{2}\xi(u) \qquad \mathcal{P}(X_t = u-1|X_{t-1} = u, \xi) = \frac{1}{2} - \frac{\epsilon}{2}\xi(u) \quad (1.4)$$

where $X_t \in \mathbb{Z}$ and the field $\{\xi(\cdot)\}$ is a family of i.i.d. random variables defined as:

$$\xi(x) = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ \\ -1 & \text{with probability } \frac{1}{2} \end{cases} \tag{1.5}$$

These kind of models has been studied intensively by many authors. Among which Sinai, Bricmont, Kupianin, Boulthausen, Sznitzman, Zeitouni,.... Obviously there is a lot of literature on this topic.

Another class of model is the situation in which the environment evolves in time. In this talk I shall discuss this situation.

2 The model.

Let us start with some general considerations.

Let $\underline{\xi} = \{\xi(t,x),\ t \in \mathbb{Z}^+,\ x \in \mathbb{Z}^d\}$ be a collection of random variables depending on time and space (environment). Let us consider a random walk with transition probabilities depending on $\underline{\xi}$, i.e.

$$\mathcal{P}(X_t = x | X_{t-1} = y, \underline{\xi}) \equiv p(x-y; \underline{\xi}). \tag{2.1}$$

We define

$$P_0(x-y) \equiv < p(x-y;\underline{\xi}) >_{\underline{\xi}} \tag{2.2}$$

where the mean is taken with respect to the environment (in space and time) distribution. Obiously $P_0(\cdot)$ is a transition probability of an homogeneous random walk. Now we can write

$$\mathcal{P}(X_t = x | X_{t-1} = y, \underline{\xi}) = P_0(x-y) + c(x-y;\underline{\xi}). \tag{2.3}$$

The function $c(\cdot,\cdot)$ represents the infuence of the random field on the random walk. From (2.3) the following properties follow

$$\sum_{u \in \mathbb{Z}^d} c(u,\cdot) = 0 \tag{2.4}$$

$$< c(\cdot,\underline{\xi}) >_{\underline{\xi}} = 0. \tag{2.5}$$

After this short introduction we want to specify our mathematical model. In order to do this we need to specify the environment $\underline{\xi}$. People consider three different situations:

1) The field $\underline{\xi}$ is a collection of i.i.d. random variables in time and space.

2) The field $\underline{\xi}$ is an independent in space copy of a Markhov chain in time.

3) The field $\underline{\xi}$ is as in 2) except for the fact that there is a dependence in space.

We consider a particle performing a random walk in $\mathbb{Z}^d$ in interaction with an environment, which evolves in an independent way or according to a Markov rule.

¿From the discussion above we can say that given a history $\underline{\xi}$ of the environment we define the transition probabilities in the following way

$$\mathcal{P}(X_t = x | X_{t-1} = y, \underline{\xi}) = P_0(x-y) + c(x-y;\underline{\xi}), \tag{2.6}$$

where $P_0(\cdot)$ (probability distribution on $\mathbb{Z}^d$,unperturbed random walk) and $c(\cdot,\cdot)$ satisfy the following assumptions:

1) $\sum_u c(u,\underline{\xi}) = 0 \quad \forall\underline{\xi}\ , \quad < c(\cdot,\underline{\xi}) >_{\underline{\xi}} = 0$

2) $0 \le P_0(u) + c(u,\underline{\xi}) \le 1 \ \ \forall\underline{\xi} \ \ \forall u \in \mathbb{Z}^d$;

3) Finite range,i.e.: $\exists\, D > 0$ such that $P_0(u) = c(u,\underline{\xi}) = 0\ \forall u \in \mathbb{Z}^d$; with $|u| > D \ \ \forall\underline{\xi}$;

4) $P_0(\cdot)$ satisfies the hypothesis for the validity of the local limit theorem.

Cosider the quantity

$$\mathcal{P}(X_t = x|X_0 = 0, \underline{\xi}), \tag{2.7}$$

i.e. the probability to be in $x \in \mathbb{Z}^d$ at time t fixed $\underline{\xi}$. We will study the long time behaviour of

$$\mathcal{P}(X_t = x|X_0 = 0) \equiv < \mathcal{P}(X_t = x|X_0 = 0, \underline{\xi}) >_{\underline{\xi}}, \tag{2.8}$$

the so called "annealed" model; and also the long time behaviour of (2.7) the so called "quenched" model.

3 The annealed model.

Here we report briefly one of the main results for the annealed model. In order to state the result we need to specify better the function $c(\cdot,\cdot)$ and the Markov evolution of the environment. Now $\xi(t,x), x \in \mathbb{Z}^d$ is a Markov chain in time taking values in a finite set S, i.e. $\xi(t,\cdot) \in S^{\mathbb{Z}^d}$. Our model is given by the joint Markov chain:

$$(\xi(t,\cdot), X_t) \tag{3.1}$$

with infinite state space $\Omega = S^{\mathbb{Z}^d} \times \mathbb{Z}^d$ and conditionally independent transition probabilities, i.e.

$$\begin{aligned}&\mathcal{P}(X_t = x,\ \xi(t,\cdot) \in A\ |X_{t-1} = y,\ \xi(t-1,\cdot) = \bar{\xi}(\cdot)\ ) = \\ &\mathcal{P}(X_t = x\ |X_{t-1} = y,\ \xi(t-1,\cdot) = \bar{\xi}(\cdot)\ )\ \mathcal{P}(\xi(t,\cdot) \in A|X_{t-1} = y,\ \xi(t-1,\cdot) = \bar{\xi}(\cdot)\ ).\end{aligned} \tag{3.2}$$

The assumptions on the random walk transition probabilities are the same as in the previous paragraph. The only thing we need to specify is the nature of the dependence of the function $c(\cdot,\cdot)$ on the field.

$$\mathcal{P}(X_t = x\ |X_{t-1} = y,\ \xi(t-1,\cdot) = \bar{\xi}(\cdot)\ ) = P_0(x-y) + c(x-y, \bar{\xi}(y)). \tag{3.3}$$

The meaning of (3.3) is that the jump of the particle depends on the value of the environment at the starting point.

Now we come to the assumptions on the environment.

The distribution

$$\mathcal{P}(\xi(t,\cdot) \in \cdot\ |X_{t-1} = y,\ \xi(t-1,\cdot) = \bar{\xi}(\cdot)\ )$$

is a product measure for the independent (in space) variables $\{\xi(t,x),\ x \in \mathbb{Z}^d\}$ each of them distributed according to the law:

$$P(\xi(t,x)=s\,|X_{t-1}=y,\ \xi(t-1,\cdot)=\bar\xi(\cdot)\ )=\begin{cases} q_0(\bar\xi(x),s) & \text{if } x\neq y \\ \\ q_1(\bar\xi(y),s) & \text{if } x=y \end{cases} \tag{3.4}$$

where q_0 and q_1 are transition probabilities such that

$$q_1(s',s)=q_0(s',s)\ +\ \bar q(s',s)$$

where $\bar q$ verifies

$$\sum_s \bar q(s',s)=0\ \ \forall s'\in S$$

We denote

$$Q_0=((q_0(s,s')))\quad s,s'\in S\ \text{ (transfer matrix)}$$

and we assume that:

Q_0 can be diagonalized and there is a non-zero mass gap in its spectrum,i.e.

$$1\equiv\mu_0>|\mu_1|\geq|\mu_2|\geq\cdots\geq|\mu_{|S|-1}|$$

where μ_i are the eigenvalues of Q_0.

These conditions imply that the Markov chain with state space S and transition probabilities $q_0(s',s)$ is ergodic. We denote with π_0 the unique invariant measure.

Under these hypothesis we have

Theorem 3.1 *(local limit theorem)(BMP 1994).*

For fixed P_0, Q_0 and any initial distribution Π of the environment the following asymptotics holds, for $c(\cdot,\cdot)$ and $\bar q(\cdot,\cdot)$ small enough

$$P(X_t=x|X_0=u)=\frac{C}{\sqrt{(2\pi t)^d}}e^{\frac{1}{2t}(A(x-u-bt),x-u-bt)}(1+O(1))$$

where $b\in\mathbb{R}^d$ (drift) is a vector, $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{R}^d$. A is a positive definite matrix and $C=\sqrt{det\ A}$.

The asymptotics is uniform w.r.t. $x\in\mathbb{Z}^d$ if

$$|x-u-bt|<t^{\frac{1}{2}+\gamma}\quad \gamma\in(0,\frac{1}{6}).$$

The proof of Theorem 3.1 is in

[BMP 1994] C. Boldrighini, R. A. Minlos, A. Pellegrinotti: " Central limit theorem for the random walk of one or two particles in a random environment ", Advances of Soviet Mathematics, 20 : "Probability Contributions to Statistical Mechanics", R.L.Dobrushin, ed. , 21-75 (1994).

Remark 1 The matrix A in Theorem 3.1 is not equal to the matrix related to the local limit theorem for P_0.

Remark 2 The initial distribution Π can be a δ function concentrated on some configuration of the environment.

In the paper [BMP 1994] the case of two interacting particles performing a random walk on the lattice $\mathbb{Z}^d$ in a fluctuating (in time) random environment was also considered and a central limit theorem in the case of $d \geq 3$ was proved.

Other results concerning the annealed case are related to the decay of correlations. We will consider this problem in the last section.

4 The quenched model.

A natural question concerns the behaviour for large T of the quantity in (2.7),i.e.

$$\mathcal{P}(X_T = x | X_0 = 0, \underline{\xi}). \tag{4.1}$$

The simplest model to consider is the case in which the field $\underline{\xi}$ is given by a collection of i.i.d. random variables.

In this case the transition probabilities are given by

$$\mathcal{P}(X_t = x | X_{t-1} = y, \underline{\xi}) = P_0(x-y) + c(x-y, \xi(t-1, y)). \tag{4.2}$$

We denote

$$b = \sum_u u P_0(u) \qquad\qquad c_{i,j} = \sum_u (u_i - b_i)(u_j - b_j) P_0(u) \qquad\qquad C = ((c_{i,j})).$$

Let us introduce the conditional average displacement

$$E^T(\underline{\xi}) = \sum_{x \in \mathbb{Z}^d} x\, \mathcal{P}(X_T = x \mid X_0 = 0,\ \underline{\xi})$$

and, given $k = (k_1, \cdots, k_d)$, the centered normalized moments:

$$\hat{M}_T^{(k)}(\underline{\xi}) = \sum_{x \in \mathbb{Z}^d} \left(\frac{x_1 - E_1^T(\underline{\xi})}{\sqrt{T}}\right)^{k_1} \cdots \left(\frac{x_d - E_d^T(\underline{\xi})}{\sqrt{T}}\right)^{k_d} \mathcal{P}(X_T = x \mid X_0 = 0,\ \underline{\xi}).$$

Under the hypothesis for $P_0(\cdot)$ and $c(\cdot, \cdot)$ of Section 2 we have:

Theorem 4.1 *(BMP 1997,BBMP 1998) For all $d \geq 1$, if $c(\cdot, \cdot)$ is small enough ,then*

$$\lim_{T \mapsto \infty} \hat{M}_T^{(k)}(\underline{\xi}) = m_k$$

where m_k is the corresponding moment of the centered gaussian measure on $\mathbb{R}^d$ with correlation matrix C.

The proof of Theorem 4.1 is, for $d \geq 2$, in

[BMP 1997] C. Boldrighini, R. A. Minlos, A. Pellegrinotti: " Almost-sure central limit theorem for a Markov model of random walk in dynamical random environment ", Probability Theory Rel.Fields **109**, 245-273 (1997) and, for $d \geq 1$, in

[BMP 1998] M.S. Bernabei, C. Boldrighini, R. A. Minlos, A. Pellegrinotti: " Almost-sure central limit theorem for a model of random walk in fluctuating random environment ", Markov Processes Rel.Fields **4**, 381-393 (1998)

Remark 3 Theorem 4.1 implies that the C.L.T. holds for a.a. $\underline{\xi}$ with the same covariance matrix of $P_0(\cdot)$.

Remark 4 The limit gaussian distribution is not affected by the field $\underline{\xi}$. This is due to the independence of $\underline{\xi}$.

In the paper [BMP 1997] the corrections to the central limit theorem were also studied. In order to do this it is better to write the quantities in a different way.

Let f be a regular function on $\mathbb{R}^d$, then we define the quantities:

$$\mu_T^{\xi}(f) = \sum_{x\in\mathbb{Z}^d} \mathcal{P}(X_T = x \mid X_0 = 0,\ \underline{\xi}) f(\frac{x-bt}{\sqrt{T}}),$$

$$\mu_T^{0}(f) = \sum_{x\in\mathbb{Z}^d} P_0^T(x) f(\frac{x-bt}{\sqrt{T}})$$

and

$$\mu(f) = \frac{\sqrt{C}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\mathcal{A}(u)} f(u)du$$

where $\mathcal{A}(u) = \sum_{i,j} a_{ij}u_iu_j$,with $A = ((a_{ij}))$ inverse of the covariance matrix of $P_0(\cdot)$.

Theorem 4.1 can be reformulated in the following way

Theorem 4.2 *For all $d \geq 1$, if $c(\cdot,\cdot)$ is small enough then*

$$\mu_T^{\xi}(f) \mapsto \mu(f) \quad as\ \ T \mapsto \infty \qquad \forall f \in C^0 \qquad \underline{\xi} - a.e.$$

To understand the behavior of the first correction to the C.L.T. we introduce the quantity

$$\Phi_T(f|\underline{\xi}) \equiv \sqrt{T}(\mu_T^{\xi}(f) - \mu(f)),$$

then we have

Theorem 4.3 *(BMP 1997) For all $d \geq 3$, if $c(\cdot,\cdot)$ is small enough then*

$$\Phi_T(f|\underline{\xi}) \mapsto \Phi(f|\underline{\xi}) \qquad \underline{\xi} - a.e.$$

where

$$\Phi(f|\underline{\xi}) = \mu(\mathcal{E}(\underline{\xi})\cdot\nabla f) \ + \ \mu(Q_1 f)$$

where

$$\mathcal{E}(\underline{\xi}) \equiv \lim_{T\mapsto\infty} \mathcal{E}^T(\underline{\xi}) = \lim_{T\mapsto\infty} (\mathbb{E}(X_T|\underline{\xi}) - bT)$$

which also exists $\underline{\xi} - a.e.$ for $d \geq 3$.

It is also possible to look at the second correction. To do this we need to consider the quantity:

$$\Psi_T(f|\underline{\xi}) \equiv T(\mu_T^{\underline{\xi}}(f) - \mu(f) - \frac{1}{\sqrt{T}}\Phi(f|\underline{\xi})),$$

then we have

Theorem 4.4 *(BMP 1997) For all $d \geq 5$, if $c(\cdot,\cdot)$ is small enough then*

$$\Psi_T(f|\underline{\xi}) \mapsto \Psi(f|\underline{\xi}) \qquad \underline{\xi} - a.e.$$

We have an explicit expression for the functional $\Psi(f|\underline{\xi})$.
Remark 5 These results show that, in the independent case, the influence of the field $\underline{\xi}$ is in the corrections terms.
Remark 6 Theorem 4.2 is a law of large numbers. In fact from the independence in time and space of $\xi(t,x)$ and the assumption (2.5) it follows that

$$\mathbb{E}_{\underline{\xi}}(\mu_T^{\underline{\xi}}(f)) = \mu_T^0(f).$$

Then we can write

$$\mu_T^{\underline{\xi}}(f) = \mathbb{E}_{\underline{\xi}}(\mu_T^{\underline{\xi}}(f)) \ + \ I_T(f|\underline{\xi}).$$

where

$$I_T(f|\underline{\xi}) \mapsto 0 \qquad \underline{\xi} - a.e..$$

A natural question concerns the possibility to remove the smallness condition on $c(\cdot,\cdot)$.

Let us assume for our random walk the hypotheses 1),2),3) and 4) of §2 and the independence of the field $\underline{\xi}$. Moreover, introducing the Fourier transform of $P_0(\cdot)$ and $c(\cdot,\cdot)$;i.e.

$$\tilde{p}_0(\lambda) = \sum_{u\in\mathbb{Z}^d} e^{i(\lambda,u)}P_0(u) \quad \tilde{c}(\lambda,\cdot) = \sum_{u\in\mathbb{Z}^d} e^{i(\lambda,u)}c(u,\cdot), \tag{4.3}$$

we assume the following non-degeneracy condition:

$$5) \quad \int_{T^d} < |\tilde{p}_0(\lambda) + \tilde{c}(\lambda,\cdot)|^2 >_{\underline{\xi}} \frac{d\lambda}{(2\pi)^d} < 1,$$

where T^d is the d-dimensional torus. Then we have

Theorem 4.5 *(BMP 2004) For all $d \geq 1$, if assumptions 1-5 are satisfied, there is a subset $\Omega' \subset S^{\mathbb{Z}^{d+1}}$ of full measure such that $\forall\, \underline{\xi} \in \Omega'$ the random variable*

$$\frac{X_T - bT}{\sqrt{T}}$$

tends in distribution to the centered gaussian variable with correlation matrix of $P_0(\cdot)$.

The proof of Theorem 4.5 is in

[BMP 2004] C. Boldrighini, R. A. Minlos, A. Pellegrinotti: " Random walks in quenched i.i.d. space-time random environment are always a.s. diffusive", Probability Theory Rel.Fields **129**, 133-156 (2004)

Remark 7 There are no assumptions concerning the smallness of $c(\cdot,\cdot)$.

Now we consider the case in which the evolution of the environment is given by a Markov chain.

Consider an ergodic and reversible Markov chain $\zeta = \{\zeta_t : t \in \mathbb{Z}^+\}$ with finite state space S.

The state space of the space-time environment is $\hat{\Omega} = S^{\mathbb{Z}^{d+1}}$ where for each site $x \in \mathbb{Z}^d$ we have an independent (in space) copy of ζ.

Our random walk is the Markov chain $(\xi(t,\cdot), X_t)$ with transition probabilities

$$\mathcal{P}(X_t = x,\ \xi(t,\cdot) \in A\ |X_{t-1} = z,\ \xi(t-1,\cdot) = \bar{\xi}(\cdot)\ ) =$$

$$(\mathcal{P}_0(x-z)\ +\ c(x-z;\bar{\xi}(z)))P(\xi(t,\cdot) \in A|\ \xi(t-1,\cdot) = \bar{\xi}(\cdot)\ ) \tag{4.4}$$

At the hypotheses 1)-4) of §2 and at the hypotheses on the stochastic matrix Q_0 of §3 we need to add an hypothesis which relays the free random walk and the velocity of convergence to equilibrium of the Markov chain, i.e.

$$\min_{\lambda \in T^d} |\tilde{p}_0(\lambda)| > |\mu_1| \tag{4.5}$$

where μ_1 is the second eigenvalue of Q_0. Then we have:

Theorem 4.6 *(BMP 2000) For all $d \geq 3$, if $c(\cdot,\cdot)$ is small enough then*

$$\mu_T^{\xi}(f) \mapsto \bar{\mu}(f) \qquad \underline{\xi} - a.e.$$

where $\bar{\mu}$ is the gaussian measure with covariance matrix given by the gaussian distribution in Theorem 3.1.

Theorem 4.7 *(BMP 2000) For all $d \geq 3$, if $c(\cdot,\cdot)$ is small enough then*

$$\lim_{T \mapsto \infty} \sqrt{T}(\mu_T^{\xi}(f) - \bar{\mu}(f)) = \Phi(f|\underline{\xi})$$

where the convergence takes place in $L_2(\hat{\Omega}, \mathcal{P})$.

The proof of Theorem 4.6 and 4.7 is in

[BMP 2000] C. Boldrighini, R. A. Minlos, A. Pellegrinotti: " Random walks in fluctuating random environment with markov evolution". On Dobrushin's way. From probability theory to statistical physics. Amer. Math. Soc. Transl. Ser. 2 **198**, 13-35 (2000)

For the independent model there is the paper:

I. Berard: " The almost sure central limit theorem for one-dimensional nearest-neighbour random walks in a space-time random environment". Journal Appl. Prob. **41**, 83-92 (2004).

In this paper a particular model in dimension $d = 1$ is considered.

In the paper

F. Rassoul-Agha, T. Seppalainen:"An almost sure invariance principle for random walks in a space-time random environment" Probability Theory Rel.Fields **133**, 299–314 (2005)

is considered the problem of the invariance principle.

For the markovian case there is an alternative proof of the quenched C.L.T., always for $c(\cdot,\cdot)$ small and $d \geq 7$, in the paper:

A. Bandyopadhyay, O. Zeitouni:" Random walk in dynamic markovian random environment " ALEA Lat. Am. J. Probab. Math. Stat. **1**, 205-224 (2006).

Recently in the paper:

D. Dolgopyat, P. Keller, C. Liverani: " Random walk in markovian environment ". In press on Annal. Probability (2007),

under the same condition of $c(\cdot,\cdot)$ small, a C.L.T. a.e. in $\underline{\xi}$ for $d \geq 1$ is proved. In the model also a spatial interaction is considered.

5 Asymptotic of correlations.

An interesting question both from a mathematical and physical point of view is to understand the behaviour in time of correlations. Obviously this question concerns models with Markov evolution. Our random walk is as in §3. We need to add a condition relating the characteristic function of $P_0(\cdot)$ with the second eigenvalue of the transition matrix $Q_0(\cdot,\cdot)$, namely

$$\min_{\lambda \in T^d} |\tilde{p}_0(\lambda)| > |\mu_1|. \tag{5.1}$$

We take two functions $f_i(\cdot)$, $i = 1, 2$ which depend on the values of the field $\underline{\xi}$ at two points x_i, $i = 1, 2$ of the lattice $\mathbb{Z}^d$. We define, as usual,

$$< \cdot, \cdot > = < \cdot\cdot > - < \cdot >< \cdot >$$

As is to be expected the time asymptotics of correlations depends on whether $b = 0$ or $b \neq 0$ where

$$b = \sum_{x \in \mathbb{Z}^d} x \mathcal{P}(X_1 = x | X_0 = 0). \tag{5.2}$$

We have the following results

Theorem 5.1 *(BMP1 1994) If the conditions above are verified, if $c(\cdot,\cdot)$ is small and $b \neq 0$, then*

$$| < f_1(\xi(t, x_1), f_2(\xi(0, x_2) > | \leq C\theta^t \tag{5.3}$$

where $\theta \in (0, 1)$.

If a symmetric condition on random walk is added, i.e.

$$P_0(u) = P_0(-u) \qquad c(u, \cdot) = c(-u, \cdot)$$

which implies that $b = 0$, then we have

Theorem 5.2 *(BMP1 1994) If the conditions above are verified, if $c(\cdot, \cdot)$ is small and $b = 0$, then*

$$< f_1(\xi(t, x_1), f_2(\xi(0, x_2) >= \frac{C}{t^{\frac{d}{2}+1}}(1 + o(1)) \;\; t \mapsto \infty. \tag{5.4}$$

The proof of Theorems 5.1 and 5.2 is in:

[BMP1 1994] C. Boldrighini, R. A. Minlos, A. Pellegrinotti: " Interacting Random Walk in a Dynamical Random Environment. I. Decay of correlations". Ann.Inst. Henri Poincaré. Probabilités et statistiques **30**,n. 4, 519-558 (1994).

An object which plays an important role in the study of the properties of the random walk in random environment is the so called field from a point of view of the particle,i.e.

$$\eta(t, x) := \xi(t, X_t + x). \tag{5.5}$$

η_t, $t \in \mathbb{Z}_+$, contains complete information: under some general conditions of aperiodicity, if one knows a trajectory $\{\eta_t : t = 0, \dots, T\}$ of the field from a point of view of the particle, one can almost-surely recover the trajectory of the joint process $\{(\xi_t, X_t) : t = 0, \dots, T\}$ with $X_0 = 0$.

If Π_0 is an initial measure we denote by Π_t its evolution by the η_t process.

The first non-trivial problem with the process η_t is that of finding a stationary measure for it. Such a measure will not be translation invariant, but it is to be expected that far away from the origin (i.e., from the position of the random walk) it is close to the stationary measure Π of the process ξ_t.

Theorem 5.3 *(BMP2 1994) Let Π_0 be an arbitrary initial distribution on Ω, and let F be a cylinder function on Ω, i.e., a function measurable with respect to the σ-algebra generated by the variables $\eta(x) : x \in \Gamma$, for some finite set $\Gamma \subset \mathbb{Z}^\nu$. Then the following assertions hold.*

i) There are positive constants C_F and κ, C_F depending only on F and κ independent of F and of the initial measure Π_0 such that

$$|\langle F \rangle_{\Pi_t} - \langle F \rangle_{\widehat{\Pi}}| \le C_F e^{-\kappa t}. \tag{5.6}$$

ii) There are positive constants C'_F and $q \in (0, 1)$, C'_F depending only on F and q independent of F and Γ such that

$$|\langle F \rangle_{\widehat{\Pi}} - \langle F \rangle_{\Pi}| \le C'_F q^{d(\Gamma, 0)}, \tag{5.7}$$

where $d(\Gamma, 0)$ is the distance of the set Γ from the origin in $\mathbb{Z}^\nu$.

iii) The probability measures Π_t tend weakly, as $t \to \infty$, to the measure $\widehat{\Pi}$, which is stationary for the process $\eta_t : t \in \mathbb{Z}_+$.

The proof of Theorems 5.3 is in:

[BMP2 1994] C. Boldrighini, R. A. Minlos, A. Pellegrinotti: " Interacting Random Walk in a Dynamical Random Environment II. The environment from the point of view of the particle". Ann.Inst. Henri Poincaré. Probabilités et statistiques **30**,n. 4, 559-605 (1994).

Another interesting question is the following. Let $\Delta_t = X_t - X_{t-1}$ denote the increment of the random walk at time t. If $f_1,\ f_2$ are bounded function on $\mathbb{Z}^d$, we consider the correlation

$$< f_2(\Delta_t), f_1(\Delta_1) >_{\mathcal{P}_{\Pi_0,0}} \tag{5.8}$$

between the first increment and the increment at time t. We want to understand the time behaviour of (5.8).

We look at this problems in a concrete model. Let us assume that the environment is given by a local field which takes two values $\xi(t,x) = \pm 1$. Denote by $\underline{\xi} = \{\xi(t,x),\ t \in \mathbb{Z}^+,\ x \in \mathbb{Z}^d\}$ the history of the environment and by $\underline{\xi}_t = \{\xi(t,x),\ x \in \mathbb{Z}^d\}$ the configuration of the environment at time t.

For the pair $(\xi(t,\cdot), X_t)$ we take, as before, the conditional independence, i.e. for any choice of $\underline{\xi}_t$ and X_t the conditional distribution of X_{t+1} and $\underline{\xi}_{t+1}$ are independent with transition probabilities:

$$\begin{aligned} &\mathcal{P}(X_{t+1} = x |\ X_t = y,\ \underline{\xi}_t = \eta) = P_0(x-y) + ac(x-y)\eta(y) \\ &\mathcal{P}(\xi(t+1,x) = s |\ \underline{\xi}_t = \eta) = q(\eta(x), s),\ x \in \mathbb{Z}^d,\ s = \pm 1. \end{aligned} \tag{5.9}$$

Here, as before,$P_0(\cdot)$ is a non degenerated random walk on $\mathbb{Z}^d$, $a \in (0.1)$ is a fixed number such that $P_0(u) \pm ac(u) \in [0,1)\ \ \forall u \in \mathbb{Z}^d$. As before

$$\sum_{u \in \mathbb{Z}^d} c(u) = 0.$$

$Q = (q(s,s'),\ s,s' = \pm 1)$ is the transition matrix of an ergodic Markov chain, which we assume symmetric. We assume exponential decay for the transition probabilities. We take also $P_0(\cdot)$ even and $c(\cdot)$ either even or odd. Hence the Fourier transforms

$$\tilde{p}_0(\lambda) = \sum_{u \in \mathbb{Z}^d} e^{i(\lambda,u)} P_0(u), \quad \tilde{c}(\lambda) = \sum_{u \in \mathbb{Z}^d} e^{i(\lambda,u)} c(u), \quad \lambda \in T^d,$$

where T^d is a d-dimensional torus, are analytic. Obviously $\tilde{p}_0(\lambda)$ is real. We also assume $\tilde{p}_0(\lambda) > 0$ and

$$\min_{\lambda \in T^d} \tilde{p}_0(\lambda) > |\mu_1|,$$

where μ_1 is the second eigeinvalue of the matrix Q.

We have the following result:

Theorem 5.4 *(BMNP '07) Let be $d \geq 1$. Under the assumptions above the correlation (5.8) has the following asymptotics as $t \mapsto \infty$*

(i) If $a < a_0$, with $a_0 > 0$ small enough, we have

$$< f_2(\Delta_t), f_1(\Delta_1) >_{\mathcal{P}_{\Pi_0,0}} = \frac{\mu_1^t}{t^{\frac{d}{2}}}(1 + \mathcal{O}(\frac{\ln t}{t})) \quad d \text{ even}$$

$$< f_2(\Delta_t), f_1(\Delta_1) >_{\mathcal{P}_{\Pi_0,0}} = \frac{\mu_1^t}{t^{\frac{d}{2}}}(1 + \mathcal{O}(\frac{1}{t})) \quad d \text{ odd}$$

(ii) If a condition on parameters holds then there is some $a_1 > a_0$ such that for $a \in (a_0, a_1)$

$$< f_2(\Delta_t), f_1(\Delta_1) >_{\mathcal{P}_{\Pi_0,0}} = ce^{-\alpha t} \sum_{k=1}^{m} p_k(t) \cos(\theta_k t)(1 + \mathcal{O}(e^{-\delta_1 t}))$$

Here $0 < \alpha < -\ln(|\mu_1|)$, $p_k(t)$ is a polynomial of order $k = 0, \cdots, m-1$, the constants δ_1 and m depend on the transition probabilities.

The proof of Theorem 5.4 is in:

[BMNP 2007] C. Boldrighini, R. A. Minlos, F.R. Nardi, A. Pellegrinotti: " Asymptotic decay of correlations for a random walk on the lattice $\mathbb{Z}^d$ in interaction with a Markov field" In press on Moscow Mathematical Journal (2007).

The case $d = 1$ was studied in the paper:

[BMNP 2005] C. Boldrighini, R. A. Minlos, F.R. Nardi, A. Pellegrinotti: " Asymptotic decay of correlations for a random walk in interaction with a Markov field." Moscow Mathematical Journal **5**, n. 3, 507-522, (2005).

Remark 8 This result says that a transition in the asymptotic behaviour of correlation (5.8) may occour. To find a concrete model in which the behaviour *ii*) of Theorem 5.4 occors looks difficult.